J. DIFFERENTIAL GEOMETRY
11 (1976) 91-101

AN ASYMPTOTIC FORMULA OF GELFAND AND GANGOLLI FOR THE SPECTRUM OF $\Gamma\backslash G$

NOLAN R. WALLACH

1. Introduction

In [6], Gelfand outlined a proof of an asymptotic formula for the distribution of multiplicities of spherical principal series in $L^2(\Gamma\backslash G)$, where G is a connected semi-simple Lie group with finite center and Γ is a discrete subgroup of G so that $\Gamma\backslash G$ is compact (see Corollary 1.3 for a formulation of this formula). As pointed out by Gangolli [3] the formula of Gelfand is marginally wrong and the proof of the formula (even in the case $G = SL(2, \boldsymbol{R})$) has a gap. In Gangolli [3] a method using the heat equation was used to prove the (corrected) Gelfand formula for G complex semi-simple. Also Gangolli and Warner have in an as yet unpublished manuscript proved the Gelfand formula if Γ has no noncentral elements of finite order. In this paper we use the asymptotic expansion of the fundamental solution of the heat equation to prove a general asymptotic formula which we now describe.

Let G and Γ be as above. Let K be a maximal connected compact subgroup of G. Let $\hat{G}$ (resp. $\hat{K}$) denote the set of equivalence classes of irreducible unitary representations of G (resp. K). If $\tau \in \hat{K}$, let d_τ be the dimension of any element of the class τ. If $\omega \in \hat{G}$, and $\tau \in \hat{K}$, then let $[\tau : \omega|_K]$ denote the multiplicity of τ in ω looked at as a direct sum of irreducible representations of K (i.e., $\omega = \sum [\tau : \omega|_K]\tau$). If $\omega \in \hat{G}$, let λ_ω be the value of the Casimir operator of G on any element of the class ω. Let $Z(G)$ be the center of G and let $Z(\Gamma) = Z(G) \cap \Gamma$. Let $\hat{K}_\Gamma$ be the subset of $\hat{K}$ consisting of those τ such that $Z(\Gamma)$ acts trivially on any element of the class τ. Let Π_Γ denote the right regular representation of G on $L^2(\Gamma\backslash G)$. Then $\Pi_\Gamma = \sum_{\omega\in\hat{G}} n_\Gamma(\omega)\omega$, $n_\Gamma(\omega) \in Z$, $n_\Gamma(\omega) \geq 0$. Our main result is

Theorem 1.1. *There is a constant C_G depending only on G so that if $\tau \in \hat{K}_\Gamma$ and if $[Z(\Gamma)]$ is the number of elements in $Z(\Gamma)$, then*

$$\sum_{\omega\in\hat{G}} n_\Gamma(\omega)[\tau : \omega|_K]e^{t\lambda_\omega} = C_G d_\tau \frac{[Z(\Gamma)]}{(4\pi t)^{d/2}} \operatorname{vol}(\Gamma\backslash G) + o(t^{-d/2}) \qquad \text{as } t \to 0\,, \quad t > 0\,,$$

where $\operatorname{vol}(\Gamma\backslash G)$ *is the volume of* $\Gamma\backslash G$ *relative to a fixed choice of Haar*

Received June 28, 1974.

measure on G, *and* $d = \dim G/K = \dim G - \dim K$.

It should be pointed out that if τ is the class of the trivial representation of K, 1, then $[1 : \omega|_K] = 0$ or 1 for $\omega \in \hat{G}$.

Using the Gärding inequality we give a simple proof of the following result of Gangolli-Warner [5] (for $\tau = 1$), Harish-Chandra (unpublished) in general.

Theorem 1.2. *If* $\tau \in \hat{K}$, *then*

$$\sum [\tau : \omega|_K] n_\Gamma(\omega)(1 + |\lambda_\omega|)^{-d/2-\varepsilon} < \infty$$

for all $\varepsilon > 0$, $d = \dim (G/K)$ *as before.*

Of course, if $\tau \notin \hat{K}_\Gamma$ then $[\tau : \omega|_K] = 0$ when $n_\Gamma(\omega) \neq 0$. Hence Theorem 1.2 has interest only in the case $\tau \in \hat{K}_\Gamma$.

The above theorem combined with Theorem 1.1 and a Tauberian argument (see Gangolli [3], [4]) implies the Gelfand conjecture for split rank G equal to one. In this case the result has already been proved by Eaton [1].

2. The equivariant heat equation

Let M be a compact, connected manifold, and let G be a finite group acting effectively on M by diffeomorphisms (that is, if $gx = x$ for all $x \in M$, then g is the identity element of G). We include the following well-known result for completeness.

Lemma 2.1. *If* $g \in G$, $g \neq e$ (e: *the identity of* G) *and* $M_g = \{x \in M \mid gx = x\}$, *then* M_g *has measure zero in* M (*see the proof for the meaning of this*).

Proof. Let $\langle\ ,\ \rangle$ be a Riemannian structure on M so that G acts by isometries. Let $p_0 \in M_g$. Let Exp_{p_0} be the exponential map of $(M, \langle\ \ \rangle)$ (see Helgason [8]), and let $r > 0$ be so small that if $B_{p_0}(r) = \{x \in T(M)_{p_0} \mid \langle x, x\rangle < r^2\}$, then $\mathrm{Exp}_{p_0} : B_{p_0}(r) \to U = \mathrm{Exp}_{p_0}(B_{p_0}(r))$ is a diffeomorphism. If $g \in G - \{e\}$ and $x \in T(M)_{p_0}$, then $g \cdot \mathrm{Exp}_{p_0}(x) = \mathrm{Exp}_{p_0}(g_{*p_0}(x))$ (g_{*p_0} is the differential of the action of g at p_0). Thus, if $\langle x, x\rangle < r^2$ and $g \cdot \mathrm{Exp}_{p_0}(x) = \mathrm{Exp}_{p_0}(x)$, then $g_{*p_0}(x) = x$. Now g_{*p_0} preserves $\langle\ ,\ \rangle$ at p_0. Hence, if $V_{p_0} = \{x \in T(M)_{p_0} \mid g_{*p_0}x = x\}$, then $T(M)_{p_0} = V_{p_0} \oplus V_{p_0}^\perp$ and, by the above, $\mathrm{Exp}_{p_0}(V_{p_0}) = U \cap M_g$. If $V_{p_0} = T(M)_{p_0}$, then $g \cdot \mathrm{Exp}_0(x) = \mathrm{Exp}_0(x)$ for all $X \in T(M)_{p_0}$. Since $\mathrm{Exp}_0(T(M)_{p_0}) = M$, g is the identity, and therefore $\dim V_{p_0} < \dim T(M)_{p_0}$. Thus $\mathrm{Exp}_{p_0}(V_{p_0})$ is a submanifold of U of dimension less than n. Hence $U \cap M_g$ has measure zero relative to any coordinate system. Since M_g can be covered by a finite number of such U, the result follows.

Corollary 2.2. *Let* $\mathring{M} = \{x \in M \mid gx \neq x \text{ for any } g \neq e\}$. *Then* $M - \mathring{M}$ *has measure zero in* M.

Proof. $M - \mathring{M} = U_{g \neq e} M_g$.

Let $E \xrightarrow{p} M$ be a C^∞ Hermitian G-vector bundle over M. That is, E is a complex vector bundle over M. If $E_x = p^{-1}(x)$, then there is $\langle\ ,\ \rangle_x$ an inner product on E_x varying smoothly with x, and G acts on E by diffeomorphisms such that $gE_x \subset E_{g\cdot x}$ and $g\colon E_x \to E_{g\cdot x}$ is a linear isometry of the fibres.

Let $C^\infty(M; E)$ denote the space of C^∞ cross-sections of E, and let $(g \cdot f)(x) = gf(g^{-1}x)$ for $g \in G$, $f \in C^\infty(M, E)$. Suppose that there is an elliptic operator D: $C^\infty(M; E) \to C^\infty(M; E)$ so that the following hold:

(1) $D(g \cdot f) = g \cdot (Df)$.

(2) If $\xi \in T(M)^*_x$, then $\sigma(D)(\xi) = -\langle \xi\ \xi \rangle I$,

where $T(M)^*$ is the cotangent bundle of M, and $\sigma(D)$ is the top order symbol of D, and $\langle\ ,\ \rangle$ is a Riemannian structure on M.

(3) If μ_0 is the Riemannian measure on M corresponding to $\langle\ ,\ \rangle$, then for $f_i \in C^\infty(M; E)$, $i = 1, 2$, defining $\int_M \langle f_1(x), f_2(x)\rangle d\mu_0(x) = (f_1, f_2)$ we assume $(Df_1, f_2) = (f_1, Df_2)$ and $(Df, f) \geq 0$ for $f \in C^\infty(M; E)$.

Actually results similar to the ones we shall derive are true under *very* much less stringent conditions than (1), (2), (3).

Let $\tilde{E} \to \boldsymbol{R} \times M$ be the pull-back bundle $p_2^*E = \{(t, v) \mid t \in \boldsymbol{R}, v \in E\}$, $I \times p\colon p_2^*E \to \boldsymbol{R} \times M$ the projection, and $L = \partial/\partial t + D$ the evolution operator associated with D.

Let $C^\infty(M; E)_\lambda = \{f \in C^\infty(M; E) \mid Df = \lambda f\}$ for $x \in \boldsymbol{R}$. If $C^\infty(M; E)_\lambda \neq (0)$, $\lambda \in \boldsymbol{R}$, then $\lambda \geq 0$. Gärding's inequality (see Palais et. al. [10], F. Warner [3] or Greenfield and Wallach [7]) implies

Lemma 2.3. $\sum_{\lambda\neq 0} \dim C^\infty(M; E)_\lambda \lambda^{-d/2-\varepsilon} < \infty$ *for all* $\varepsilon > 0$, $d = \dim M$.

If $\phi, f, g \in C^\infty(M; E)$, then define

$$\int_M (f \hat{\otimes} g)(x, y)\phi(y)dy = \int_M \langle g(y), \phi(y)\rangle d\mu_0(y) f(x) .$$

Let $E \hat{\otimes} E \to M \times M$ be the exterior tensor product of E with itself. If $h \in C^\infty(E \hat{\otimes} E)$, then $\int_M h(x, y)\phi(y)d\mu_0(y)$ makes sense for $\phi \in C^\infty(E)$.

For $\lambda \in \boldsymbol{R}$ and $\lambda \geq 0$, let $\phi_{\lambda,1}, \cdots, \phi_{\lambda,n_\lambda}$ be an orthonormal basis of $C^\infty(M; E)_\lambda$ ($\dim C^\infty(M; E)_\lambda = n_\lambda < \infty$ by the elliptic regularity theorem). Then Lemma 2.3 implies that

$$\sum_\lambda e^{-\lambda t}\left(\sum_{i=1}^{n_\lambda} \phi_{\lambda,i}(x) \hat{\otimes} \phi_{\lambda,i}(y)\right) = K(t, x, y)$$

defines a C^∞ cross-section of

$$P_2^*(E \hat{\otimes} E)|_{(0,\infty)\times M\times M}\ , \qquad (P_2(t, x, y) = (x, y)) .$$

It is well known and easily proved that if $\phi \in C^\infty(M; E)$, then the unique solution to the Cauchy problem:

(i) $Lf = 0$,

(ii) $\lim_{\substack{t\to 0\\ t>0}} f(t, x) = \phi(x)$

is given by

$$f(t, x) = \int_M K(t, x, y)\phi(y)d\mu_0(y) .$$

Set $I_G^\infty(E)$ equal to the space of all $f \in C^\infty(M; E)$ such that $g \cdot f = f$ for $g \in G$. If $\phi \in I_G^\infty(E)$, then the uniqueness above implies that if $Lf = 0$ and $\lim_{\substack{t\to 0\\ t>0}} f(t, x) = \phi(x)$, then $g \cdot f(t, g^{-1} \cdot x) = f(t, x)$ for $g \in G$.

Let $C^\infty(M; E)_\lambda^0 = C^\infty(M; E)_\lambda \cap I_G^\infty(E)$. Then we may assume that $\phi_{\lambda,1}, \cdots, \phi_{\lambda,m_\lambda}$ form an orthonormal basis of $C^\infty(M; E)_\lambda^0$. Let

$$K_G(t, x, y) = \sum_\lambda e^{-\lambda t} \sum_{i=1}^{m_\lambda} \phi_{\lambda,i}(x) \hat{\otimes} \phi_{\lambda,i}(y) .$$

Let $(g \cdot f)(t, x) = gf(t, g^{-1} \cdot x)$ for $f \in C^\infty(\boldsymbol{R} \times M; \tilde{E})$ and $g \in G$. Let $I_G^\infty(\tilde{E})$ be the f in $C^\infty((0, \infty) \times M; \tilde{E})$ such that $g \cdot f = f$ for $g \in G$.

Clearly, if $(K(t)\phi)(x) = \int_M K(t, x, y)\phi(y)dy$, $t > 0$, then $K(t): I_G^\infty(E) \to I_G^\infty(\tilde{E})$. If $(K_G(t)\phi) = \int_M K_G(t, x, y)\phi(y)dy$ for $t > 0$, then $K_G(t): C^\infty(M; E) \to I_G^\infty(\tilde{E})$.

If $v \in E_x$ and $w \in E_y$, then set $(g \otimes 1)(v \hat{\otimes} w) = gv \hat{\otimes} w$, $(1 \otimes g)(v \hat{\otimes} w) = v \hat{\otimes} gw$. $(g \otimes h)(v \hat{\otimes} w) = gv \hat{\otimes} hw$, $g, h \in G$. Hence $G \times G$ acts on $E \hat{\otimes} E$. Clearly

$$K_G(t, x, y) = \frac{1}{[G]} \sum_{g \in G} (g \otimes 1)K(t, g^{-1}x, y) ,$$

where $[G]$ is the number of elements in G.

We also look at $x \to K(t, x, x)$ and $x \to K_G(t, x, x)$ as a C^∞ cross-section of Hom (E, E). Let I be the identity cross-section. The next result is classical, so we will only sketch its proof.

Lemma 2.4. (a) $K(t, x, x) = (4\pi t)^{-d/2}I_x + O(t^{-(d-1)/2})$ *as* $t \to 0$, $t > 0$.

(b) *Let* ρ *be the Riemannian metric corresponding to* $\langle\ ,\ \rangle$ *on* M. *Then there are constants* $C > 0$, $h > 0$ *so that*

$$\|K(t, x, y)\| \le Ct^{-d/2} \exp(-h\rho(x, y)^2/t) .$$

Here the norm is relative to the tensor product Hermitian structure on $E \hat{\otimes} E$.

Proof (outline). Let $\varepsilon > 0$ be such that

(a) $\mathrm{Exp}_p: B_p(\varepsilon) \to B(p; \varepsilon) = \{x \in M \mid \rho(x, p) < \varepsilon\}$ is a diffeomorphism for $p \in M$.

(b) $E|_{B(p;\varepsilon)}$ is a trivial bundle for $p \in M$.

Let $p_1, \cdots, p_N \in M$ be such that if $U_i = B(p_i; \varepsilon/2)$, $U_1 \cup \cdots \cup U_N = M$. Let $W_i = B(p_i; \varepsilon)$. Let $\{x_1^i, \cdots, x_d^i\}$ be a corresponding system of normal coordinates on W_i, and $\Psi_i = (x_1^i, \cdots, x_d^i)$ the corresponding chart ($\Psi_i(W_i) = \{(x_1, \cdots, x_d) \mid \sum x_i^2 < \varepsilon^2\}$). Let $\Psi_i: E|_{W_i} \to W_i \times C^m$ be a vector bundle isomorphism, and let $\phi_1, \cdots, \phi_N$ be a partition of unity for M, supp $\phi_i \subset U_i$.

Let $\xi_i \in C^\infty(M)$, $0 \leq \xi_i(x) \leq 1$, $x \in M$, $\operatorname{supp} \xi_i \subset U_i$, $\xi_i(x) = 1$ for $x \in \operatorname{supp} \phi_i$.

If $f \in C^\infty(M; E)$, then $F_i = \Psi \circ f \circ \Psi_i^{-1}: \Psi_i(W_i) \to \Psi_i(W_i) \times C^m$, $F_i(x) = (x, f_i(x))$. $\Psi_i \circ Df \circ \Psi_i^{-1} = (x, D_i f_i(x))$ where

$$D_i = -\sum a_{kl}^i \frac{\partial^2}{\partial x_k \partial x_l} + \sum b_k^i \frac{\partial}{\partial x_k} + C^i \ ,$$

where $(a_{kl}^i(x))$ is a positive definite matrix $b_k^i, C^i \in C^\infty(\Psi_i(W_i), \operatorname{End}(C^n))$. Let $(a^{i,kl}(x)) = (a_{kl}^i(x))^{-1}$, and set

$$Z_i(t, x, y) = (4\pi t)^{-d/2} \exp\left(-\frac{1}{4t} \sum_{k,l} a^{i,k,l}(y)(x_k - y_k)(x_l - y_l)\right)$$

for $t > 0$.

Define for $f \in C^\infty(M; E)$,

$$(Z(t)f)(x) = \sum_{i=1}^N \xi_i(x) \Psi_i^{-1}\left(x, \int_{V_i} \phi_i(y) Z_i(t, \Psi_i(x), \Psi_i(y)) f_i(y) d\mu_0(y)\right) .$$

Then it is easily seen (see Friedman [2, Theorem 1, p. 4]) that

$$\lim_{\substack{t \to 0 \\ t > 0}} (Z(t)f)(x) = f(x)$$

for $x \in M$. It is also clear that $Z(t)$ has a C^∞ kernel $Z(t, x, y)$. That is, $(Z(t)f)(x) = \int_M Z(t, x, y) f(y) d\mu_0(y)$ where $Z(t, x, y) \in E_x \hat{\otimes} E_y$.

If $f \in C^\infty((0, \infty) \times M; \tilde{E})$, $g \in C^\infty(M; E)$ define $L(f \hat{\otimes} g) = Lf \hat{\otimes} g$. Arguing as in Friedman [2, Chapter 1, § 4] we define

$$\Phi_1(t, x, y) = -LZ(t, x, y) \ .$$

Supposing that Φ_ν has been defined, set

$$\Phi_{\nu+1}(t, x, y) = -\int_0^t \int_M LZ(t\sigma, x, \xi) \Phi_\nu(\sigma, \xi, y) d\mu_0(\xi) d\sigma \ .$$

Then the above arguments of Friedman imply that if $\Phi(t, x, y) = \sum_{\nu=1}^\infty \Phi_\nu(t, x, y)$, then Φ converges uniformly and absolutely on compact subsets of $(0, \infty) \times M \times M$ to a C^∞ cross-section of $C^\infty((0, \infty) \times M \times M; P_2^*(E \hat{\otimes} E))$. Furthermore we have that there are $C > 0, h > 0$ so that

(a) $\|Z(t, x, y)\| \leq Ct^{-d/2} \exp\left(-\frac{h}{t} \rho(x, y)^2\right)$,

(b) $\|\Phi(t, x, y)\| \leq Ct^{-(d+1)/2} \exp\left(-\frac{h}{t} \rho(x, y)^2\right)$,

(c) $\|LZ(t, x, y)\| \leq Ct^{-(d+1)/2} \exp\left(-\frac{h}{t}\rho(x, y)^2\right)$

for $0 < t \leq T < \infty, x, y \in M$.

Also arguing as in [2, Theorem 8, p. 19] we see

$$K(t, x, y) = Z(t, x, y) + \int_0^t \int_M Z(t - \sigma, x, \xi)\Phi(\sigma, \xi, y)d\mu_0(\xi)d\sigma .$$

Using [2, Lemma 3, p. 15] we see that if

$$V(t, x, y) = \int_0^t \int_M Z(t - \sigma, x, \xi)\Phi(\sigma, \xi, y)d\mu_0(\xi)d\sigma ,$$

then

$$\|V(t, x, y)\| \leq Ct^{-(d+1)/2} \exp\left(-\frac{h}{t}\rho(x, y)^2\right)$$

for $0 < t \leq T$.

The lemma now follows from the fact that $Z(t, x, y)$ obviously satisfies (1), (2) of the lemma.

Lemma 2.5. *Let for* $\lambda \in \boldsymbol{R}$, $m_\lambda = \dim C^\infty(M; E)_\lambda^0 = \dim \{f \in C^\infty(M; E) \mid Df = \lambda f, g \cdot f = f$ *for all* $g \in G\}$. *Let* $\operatorname{vol}(M) = \int_M d\mu_0(x)$. *Let m be the fibre dimension of E. If* $d = \dim M$, *then*

$$\sum_\lambda m_\lambda e^{-\lambda t} = \frac{m}{[G]} \frac{\operatorname{vol}(M)}{(4\pi t)^{d/2}} + o(t^{-d/2})$$

as $t \to 0$, $t > 0$.

Proof. If $f, g \in C^\infty(M; E)$, define $\operatorname{tr}(f(x) \otimes g(x)) = \langle f(x), g(x) \rangle$. Then clearly

$$\sum_\lambda m_\lambda e^{-\lambda t} = \int_M \operatorname{tr}(K_G(t, x, x))d\mu_0(x) .$$

Now

$$K_G(t, x, y) = \frac{1}{[G]}K(t, x, y) + \frac{1}{[G]} \sum_{g \neq e} (g \otimes 1) \cdot K(t, g^{-1} \cdot x, y) .$$

Thus Lemma 2.4 will imply the lemma if we can show that if $g \neq e$ then

$$\int_M \|(g \otimes 1)K(t, g^{-1}x, x)\| d\mu_0(x) = o(t^{-d/2})$$

as $t \to 0$, $t > 0$.

Let now $g \in G - \{e\}$ be fixed and $\varepsilon > 0$ be given. Let U be open in M so that $U \supset M_{g^{-1}}$ (see Lemma 2.1) and $\int_U d\mu_0(x) < \frac{1}{2}\varepsilon CV$, C and V to be determined. Let

$$J(t) = \int_M \|(g \otimes 1)K(t, g^{-1}x, x)\| d\mu_0(x) = \int_M \|K(t, g^{-1}x, x)\| d\mu_0(x) \ .$$

Then

$$J(t) = \int_{M-U} \|K(t, g^{-1}x, x)\| d\mu_0(x) + \int_U \|K(t, g^{-1}x, x)\| d\mu_0(x) \ .$$

Now

$$\|K(t, g^{-1}x, x)\| \leq Ct^{-d/2} \exp\left(-\frac{h}{t}\rho(g^{-1}x, x)\right) \leq Ct^{-d/2}V \ ,$$

$$V = \max_{\substack{x,y \in M \\ t \leq 1}} \exp\left(-\frac{h}{t}\rho(x, y)\right) \ .$$

Thus

$$t^{d/2}J(t) \leq \int_{M-U} \|K(t, g^{-1}x, x)\| d\mu_0(x) + \tfrac{1}{2}\varepsilon \ .$$

Now $M - U$ is compact and $M - U \subset M - M_{g^{-1}}$. Hence there is $\delta > 0$ so that if $x \in M - U$ then $\rho(g^{-1}x, x) \geq \delta$. Applying Lemma 2.4 again we find that $t^{d/2}J(t) \leq \frac{1}{2}\varepsilon + C \operatorname{vol}(M)e^{-\delta^2 h/t}$ if $t \leq 1$. Take $\mu > 0$ so that $e^{-\delta^2 h/t} < \frac{1}{2}\varepsilon C \operatorname{vol}(M)$ if $0 < t < \mu$. Then $t^{d/2}J(t) < \varepsilon$ for $0 < t < \mu$. q.e.d.

In the next section we apply these results to $\Gamma \backslash G$.

3. Applications to $\Gamma \backslash G$

Let G be a semi-simple Lie group with finite center and such that G has no connected, compact, normal subgroups. Let $K \subset G$ be a maximal connected, compact subgroup. Let $X = G/K$. Let $\mathfrak{g}$ be the Lie algebra of G, and B the Killing form of $\mathfrak{g}$. Let $\mathfrak{k} \subset \mathfrak{g}$ be the Lie algebra of K, and $\mathfrak{p}$ the orthogonal compliment to $\mathfrak{k}$ in relative to B. Then it is well known that $B|_{\mathfrak{p}\times\mathfrak{p}}$ is positive definite. We put the G-invariant Riemannian structure $\langle\ ,\ \rangle$ on X; this corresponds to making $\Pi_{*_e}\colon \mathfrak{p} \to T(X)_{ek}$($\Pi\colon G \to G/K$ is the natural map, and Π_{*_e} is its differential at $e \in G$) an isometry of $B|_{\mathfrak{p}\times\mathfrak{p}}$ and $\langle\ ,\ \rangle_{ek}$.

Let now (τ, V) be an irreducible unitary representation of K. We form the G-hermitian vector bundle over X, $G \underset{\tau\otimes I}{\times} (V \otimes V^*) = \boldsymbol{V}$ where $G \underset{\tau\otimes I}{\times} (V \otimes V^*)$ is the associated bundle to the principal bundle $K \to G \xrightarrow{\Pi} X$ (cf. Kobayashi-Nomizu [9] or Wallach [12]). Then $\boldsymbol{V}$ is completely described as follows:

(1) If g is in G, then g induces a linear map $V_x \to V_{gx}$ which we denote $v \to g \cdot v$. The corresponding action of G on V is C^∞.

(2) The representation of K on V_{ek} given by $v \to k \cdot v$, $v \in V_{ek}$, is equivalent to $(\tau \otimes I, V \otimes V^*)$ as a unitary representation.

If $f \in C^\infty(X; V)$, let $(g \cdot f)(x) = gf(g^{-1} \cdot x)$. Then $g \cdot f \in C^\infty(X; V)$ for $f \in C^\infty(X; V)$. Let $X_1, \cdots, X_n$ be a basis of $\mathfrak{g}$, and let $Y_1, \cdots, Y_n$ be such that $B(X_i, Y_j) = \delta_{ij}$. Then defining $(X \cdot f)(x) = \frac{d}{dt}(\exp tX \cdot f(\exp(-tX) \cdot x)|_{t=0}$ for $X \in \mathfrak{g}$ and $f \in C^\infty(X; V)$ we set

$$\Omega_V f = \sum_{i=1}^{n} X_i Y_i \cdot f .$$

Thus $\Omega_V g \cdot f = g \Omega_V f, g \in G$.

A simple computation shows that if $\xi \in T(X)^*_{ek}$, then $\sigma(\Omega_V)(\xi) = \langle \xi, \xi \rangle I$. Define a G-invariant connection on V by $(\nabla_u f)(ek) = (X \cdot f)(ek)$ for $u \in T(G/K)_{ek}$, $u = \Pi_{*_e}(X), X \in \mathfrak{p}$. The corresponding connection on V satisfies

$$X \cdot \langle \Psi, \eta \rangle = \langle \nabla_X \Psi, \eta \rangle + \langle \Psi, \nabla_X \eta \rangle .$$

Let ∇^2 be the connection Laplacian on V corresponding to the connection ∇ and the Riemannian structure on X.

Lemma 3.1. *Let $\Omega_K = -\sum Y_i^2$ where $Y_1, \cdots, Y_k$ form a basis of $\mathfrak{k}$ so that $B(Y_i, Y_j) = -\delta_{ij}$. Let λ_τ be defined by $\tau(\Omega_K) = \lambda_\tau I$ (Schur's lemma implies this makes sense). If $f \in C^\infty(X; V)$, then*

$$\Omega_V f = \nabla^2 f + \lambda_\tau f .$$

Proof. If $f \in C^\infty(X; V)$, define $\tilde{f}(g) = g^{-1} \cdot f(gk)$. Then $\tilde{f}: G \to V_{ek}$ and $\tilde{f}(gk) = k^{-1}\tilde{f}(g)$ for $k \in K$, $g \in G$. Let $(L_g \phi)(x) = \phi(g^{-1}x)$ for $\phi: G \to V_{ek}$, where ϕ is of class C^∞, and $g, x \in G$. We note that if $A(f) = \tilde{f}$ for $f \in C^\infty(X; V)$ and we define $B(\phi)(gk) = g \cdot \phi(g)$ for $\phi: G \to V_{ek}$, then $\phi(gk) = k^{-1} \cdot \phi(g)$, $k \in K$, $g \in G$. Thus $B(\phi) \in C^\infty(X; V)$ and $AB(\phi) = \phi$, $BA(f) = f$.

Let $(R_X \phi)(g) = \frac{d}{dt}\phi(g \exp tX)|_{t=0}$ for $X \in \mathfrak{g}$ and $\phi: G \to V_{ek}$, ϕ being of class C^∞. Then a direct computation shows that if $X_1, \cdots, X_p$ form an orthonormal basis of $\mathfrak{p}$ relative to $B|_{\mathfrak{p}\times\mathfrak{p}}$, then $A(\nabla^2 f) = \sum_{i=1}^{p} R_{X_i}^2 A(f)$. Also

$$\begin{aligned} A(\Omega_V f) &= \sum_{i=1}^{p} R_{X_i}^2 A(f) - \sum_{i=1}^{p} R_{X_i}^2 A(f) \\ &= \sum_{i=1}^{p} R_{X_i}^2 A(f) + \tau(\Omega_K)(A(f)) = A(\nabla^2 f) + \lambda_\tau A(f) . \end{aligned}$$

Applying B gives the result.

Let now $\Gamma \subset G$ be a discrete subgroup so that $\Gamma\backslash G$ is compact and $g\Gamma g^{-1} \cap K = \{e\}$ for all $g \in G$. Then Γ acts freely and properly discontinuously on X and V. We may thus form $E = \Gamma\backslash V \to \Gamma\backslash X = M$.

Since Γ acts by isometries on X, we may "push" the Riemannian structure and volume element on X down to M. The Hermitian structure on V induces a Hermitian structure on E. Finally Ω_V and ∇^2 are G-invariant operators on V, and thus the induced second order elliptic operators on E. We still have $\Omega_V = \nabla^2 + \lambda_\tau I$.

Set $D = -(\Omega_V - \lambda_\tau I) = -\nabla^2$. Then $(Df, f) \geq 0$, $D = D^*$ and $\sigma(D, \xi) = -\langle \xi, \xi \rangle I$. Thus D satisfies (1), (2), (3) of § 2.

Let $\boldsymbol{f}(g)(k) = f(gk)$ for $f \in C^\infty(\Gamma\backslash G)$. Then $\boldsymbol{f}: \Gamma\backslash G \to C^\infty(K)$. Let $C_\tau^\infty(K)$ be the subspace of $C^\infty(K)$ spanned by the matrix entries of (τ, V). Let χ_τ be the character of (τ, V). Define $f_\tau(g) = \int_K \chi_\tau(e)\overline{\chi_\tau(k)} f(gk)dk$ for $f \in C^\infty(\Gamma\backslash G)$. Then $f_\tau: \Gamma\backslash G \to C_\tau^\infty(K)$ and $\boldsymbol{f}_\tau(gu)(k) = \boldsymbol{f}_\tau(g)(uk)$. Let $C_\tau^\infty(\Gamma\backslash G) = \{f \in C^\infty(\Gamma\backslash G) \mid f_\tau = f\}$. Let $(\mu(k)\phi)(x) = \phi(k^{-1}x)$ for $\phi \in C_\tau^\infty(K)$, and $k, x \in K$. We therefore see that if $f \in C_\tau^\infty(\Gamma\backslash G)$, then $\boldsymbol{f}: \Gamma\backslash G \to C_\tau^\infty(K)$ and $\boldsymbol{f}(gu) = \mu(u)^{-1}\boldsymbol{f}(x)$ for $x, u \in K$.

Let Π_Γ be the right regular representation of G on $L^2(\Gamma\backslash G)$. That is, if $\phi \in L^2(\Gamma\backslash G)$ then $(\pi_\Gamma(x)\phi)(\Gamma g) = \phi(\Gamma g x)$ for $g, x \in G$. Then it is well known that $\pi_\Gamma = \sum_{\omega \in \hat{G}} n_\Gamma(\omega)\omega$. $\hat{G}$ is the set of all equivalence classes of irreducible unitary representations of G.

If $\lambda \in \boldsymbol{R}$, let $\hat{G}_\lambda = \{\omega \in G \mid \pi_\omega(\Omega) = -\lambda I$ for every π_ω in the class $\omega\}$.

Lemma 3.2. *Set* $C^\infty(M; E)_\lambda = \{\phi \in C^\infty(M; E) \mid D\phi = \lambda\phi\}$. *Then*

$$\dim C^\infty(M; E)_\lambda = \sum_{\omega \in \hat{G}_{\lambda - \lambda_\tau}} n_\Gamma(\omega) \cdot [\tau : \omega|_K] d_\tau\ ,$$

$d_\tau = \dim V = \chi_\tau(e)$.

Proof. E can be looked upon as the set of equivalence classes of pairs $(x, v), x \in \Gamma\backslash G,\ v \in V \otimes V^*$ with $(xk, (\tau(k) \otimes I)^{-1}v) \equiv (x, v)$ for $k \in K$. Let $[x, v]$ denote the equivalence class of (x, v). Let $C^\infty(\Gamma\backslash G; \tau)$ denote the space of all $\phi: \Gamma\backslash G \to V \otimes V^*$, $\phi \in C^\infty$ and $\phi(xk) = (\tau(k)^{-1} \otimes I)\phi(x)$. Define $B(\phi)(x) = [x, \phi(x)]$ for $\phi \in C^\infty(\Gamma\backslash G; \tau)$. Then B defines a bijection of $C^\infty(\Gamma\backslash G; \tau)$ and $C^\infty(M; E)$. Now as a representation of $K, (\mu, C_\tau^\infty(K))$ is equivalent to $(\tau \otimes I, V \otimes V^*)$. Thus we have $B^{-1}: C^\infty(M; E) \to C_\tau^\infty(\Gamma\backslash G)$. B^{-1} is bijective and extends to a bounded bijective operator on the appropriate L^2-completions. But then $B^{-1}(C_\tau^\infty(M; E)_\lambda) = \{f \in C_\tau^\infty(\Gamma\backslash G) \mid \Omega f = -(\lambda - \lambda_\tau)f\}$. If $f \in C_\tau^\infty(\Gamma\backslash G)$, then $f = \sum f_\omega$, $f_\omega \in n_\Gamma(\omega)H_\omega, (\pi_\omega, H_\omega) \in \omega$. Thus $\Omega f = \sum \lambda_\omega f_\omega$, and the result now follows.

Suppose now that $\Gamma_1 \subset G$ is an arbitrary discrete subgroup so that $\Gamma_1\backslash G$ is compact. Then there is a normal subgroup Γ of Γ_1 so that Γ acts freely and properly discontinuously on X, and if $H = \Gamma_1\backslash\Gamma$ then H is a finite group of isometries of $\Gamma\backslash X$ (cf. Raghunathan [11]).

Now $E \to M = \Gamma\backslash X$ is an H-vector bundle, since E is the associated bundle to $\Gamma\backslash G \to \Gamma\backslash X$ and H acts on the left on $\Gamma\backslash G$. Let $Z(\Gamma_1) = \Gamma_1 \cap Z(G)$, where $Z(G)$ is the center of G. We note that since $Z(G) \subset K$, $Z(\Gamma_1) \subset K$. Also, if $z \in Z(G)$ then $\tau(z) = \xi_\tau(z)I$, $\xi_\tau \colon Z(G) \to T^1$ being a character. Thus, if $\gamma \in Z(\Gamma_1)$ and $h = \gamma\Gamma$, then $h \cdot v = \xi_\tau(\gamma)v$ for $v \in E$. We therefore see that $C^\infty(M; E)^0_\lambda = \{f \in C^\infty(M; E)_\lambda \mid h \cdot f = f, h \in H\} \neq 0$ only if $\tau|_{Z(\Gamma_1)} = I$.

We assume that $\tau|_{Z(\Gamma_1)} = I$. Arguing as above we find

Lemma 3.3. $\dim C^\infty(M; E)^0_\lambda = \sum_{\omega \in \hat{G}_{\lambda - \lambda_\tau}} n_{\Gamma_1}(\omega)[\tau : \omega|_K]d_\tau$, *where* $\Pi_{\Gamma_1} = \sum n_{\Gamma_1}(\omega)\omega$, *and* Π_{Γ_1} *is the right regular representation of* G *on* $L^2(\Gamma_1\backslash G)$.

Now H does not necessarily act effectively on $\Gamma\backslash X$. Let $H_0 = \{h \in H \mid h\Gamma x = \Gamma x$ for all $x \in X\}$. Then, as is easily seen, H_0 is the image of $Z(\Gamma_1)$ in H. Since $Z(\Gamma_1) \cap \Gamma = (e)$, we see that $[H_0] = [Z(\Gamma_1)]$. Finally E is an H/H_0 vector bundle if and only if H_0 acts trivially on the fibres of E, that is, if and only if $\tau \in \hat{K}_{\Gamma_1}$ (see the introduction for the definition of $\hat{K}_{\Gamma_1}$).

Combining the above observations with Lemma 3.3 and Lemma 2.5 we see

$$(1) \qquad e^{\lambda_\tau t} \sum_{\omega \in \hat{G}} e^{\lambda_\omega t} n_{\Gamma_1}(\omega) d_\tau [\tau : \omega|_K] = \frac{[Z(\Gamma_1)]}{[\Gamma_1\backslash\Gamma]} t^{-d/2} \operatorname{vol}(M) d_\tau^2 + o(t^{-d/2}) \quad \text{as } t \to 0, \quad t > 0 .$$

Normalize Haar measure dg on G so that if $X_1, \cdots, X_n$ form a basis of $\mathfrak{g}$ so that $-B(X_i, \theta X_j) = \delta_{ij}$ $(\theta|_{\mathfrak{k}} = I, \theta|_{\mathfrak{p}} = -I)$, then $dg(X_1, \cdots, X_n) = 1$. Let C_G^{-1} be the volume of K relative to the Riemannian volume element on K corresponding to the inner product $-B|_{\mathfrak{k} \times \mathfrak{k}}$. Then

$$\operatorname{vol}(\Gamma_1\backslash G) = [\Gamma_1/\Gamma]^{-1} \cdot \operatorname{vol}(\Gamma\backslash G) = [\Gamma_1/\Gamma]^{-1} C_G^{-1} \operatorname{vol}(\Gamma\backslash X) .$$

Hence $C_G \operatorname{vol}(\Gamma_1\backslash G) = [\Gamma_1/\Gamma]^{-1} \cdot \operatorname{vol}(\Gamma\backslash X)$. These observations combined with (1) above prove

Theorem 3.4. *There is a constant* C_G *depending only on* G *so that if* Γ *is a discrete subgroup of* G *with* $\Gamma\backslash G$ *compact and if* $\tau \in \hat{K}_\Gamma$, *then*

$$\sum_{\omega \in \hat{G}} n_\Gamma(\omega)[\pi : \omega|_K] e^{t\lambda_\omega} = C_G d_\tau \frac{[Z(\Gamma)]}{(4\pi t)^{d/2}} \operatorname{vol}(\Gamma\backslash G) + o(t^{-d/2}) , \quad \textit{as } t \to 0 , \quad t > 0 .$$

We also note that Lemma 2.3 combined with Lemmas 3.2 and 3.3 immediately imply Theorem 1.2 of the introduction.

References

[1] T. Eaton, Thesis, University of Washington, Seattle, 1973.

[2] A. Friedman, *Partial differential equations of parabolic type,* Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

[3] R. Gangolli, *Asymptotic behavior of spectra of compact quotients of certain symmetric spaces,* Acta. Math. **121** (1968) 151–192.
[4] ——, *Spectra of discrete subgroups*, Proc. Sympos. Pure Math. Vol. 26, Amer. Math. Soc., 1973, 431–436.
[5] R. Gangolli & G. Warner, *On Selberg's trace formula*, Japan. J. Math., to appear.
[6] I'. M. Gelfand, *Automorphic forms and the theory of representations*, Proc. Internal. Conf. Math. (Stockholm, 1962), Inst. Mittag-Leffler, Djursholm, 1963, 74–85.
[7] S. Greenfield & N. R. Wallach, *Remarks on global hypoellipticity*, Trans. Amer. Math. Soc. **183** (1973) 153–164.
[8] S. Helgason, *Differential geometry and symmetric spaces,* Academic Press, New York, 1962.
[9] S. Kobayashi & K. Nomizu, *Foundations of differential geometry*. I, Interscience, New York, 1962.
[10] R. Palais et al., *Seminar on the Atiyah-Singer index theorem*, Annals of Math. Studies, No. 57, Princeton University Press, Princeton, 1965.
[11] M. Raghunathan, *Discrete subgroups of Lie groups*, Springer, Berlin, 1972.
[12] N. Wallach, *Harmonic analysis on homogeneous spaces*, Marcel Dekker, New York, 1973.
[13] F. Warner, *Foundations of differential geometry and Lie groups*, Scott, Foresman and Co., Glenview, Illinois, 1971.

RUTGERS UNIVERSITY