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METRICS WITH VALUES IN ENDOMORPHISMS OF A FIBRE BUNDLE

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Introduction

The essential features of an inner product on a (finite-dimensional) real vector space V are symmetry and positivity. Hence in § 1 the notion of inner product is generalized to linear maps $\Phi: V \otimes V \to \mathrm{Hom}(W, W)$, where W is a real finite-dimensional inner product space and Φ has the properties: (i) for all $u, v \in V$, $\Phi(u, v) = \Phi(v, u)^*$; (ii) for all $v \neq 0$, $\Phi(v, v)$ is a positive transformation. Such a map is called a *Hom* (W)-*valued inner product*; the set of Hom (W)-valued inner products on V may be identified with a subset of $\mathrm{Hom}(V, V) \otimes \mathrm{Hom}(W, W)$. Real inner products on R^n, hermitian inner products on C^n $(= R^{2n})$, and certain Clifford structures are examples of Hom (W)-valued inner products.

There are natural actions of $GL(V) \otimes GL(W)$ and $SO(V) \otimes SO(W)$ on the set of Hom (W)-valued inner products; these actions are investigated and their isotropy groups computed for some special cases. Generically these groups are trivial.

If M is a smooth oriented manifold, the notion of Riemannian metric on M is generalized in § 2 to the smooth assignment of a Hom-valued inner product on the tangent space, and the space W at each point may be taken as the fibre at the point of a vector bundle $\mathscr{W}$ on M. Thus a *Hom* $(\mathscr{W})$-*valued metric of type* Φ is a reduction of $B(M) \otimes B(\mathscr{W})$ to a principal bundle $B_{\underline{G}}$, where $\underline{G}$ is the isotropy group of Φ under the GL-action.

Hom $(\mathscr{W})$-*valued connexions* and *covariant differentiation* are defined in § 3. Compatibility and symmetry conditions for a connexion are given; there exist symmetric connexions compatible with a given metric, and the self-adjoint parts of all such connexions are equal. There are two ways of computing second covariant derivatives; generally these methods produce unequal results. There is at most one symmetric compatible connexion for which the second derivatives are equal; if such a connexion exists it is said to be *invariant*. Invariance is a generalized kähler condition.

In § 4 a *torsion* form is introduced using G-structures. Vanishing of torsion is interpreted as an integrability condition in several cases. An invariant com-

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patible symmetric connexion is shown to have torsion zero. *Hom* ($\mathscr{W}$)*-valued curvature* is defined and constant curvature analyzed for invariant metrics. A structural equation is then derived.

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1. Hom (W)-valued inner products

Let V be a real n-dimensional vector space, and W real m-dimensional inner product space. A *Hom* (W)*-valued inner product* on V is a linear map $\Phi: V \otimes V \to W^* \otimes W$ satisfying:

(i) for all $v, v' \in V$, $\Phi(v, v') = \Phi(v', v)^*$,

(ii) for all $v \neq 0$, $\Phi(v, v)$ is positive.

Φ induces an inner product g on V by $g(v, v') = \dfrac{1}{m} \operatorname{tr} \Phi(v, v')$. Denote by $\hat{g}$ the map $V \to V^*: v \to g(v, \cdot)$ and by $\tilde{g}$ the map $\hat{g}^{-1}: V^* \to V$. Also define $\tilde{\Phi}: V \to L(V^*, W^* \otimes W)$ by $\tilde{\Phi}(v)\psi = \Phi(v, \tilde{g}\psi)$. Let $\operatorname{tr} \Phi: W \to W$ be $\sum_{i=1}^{n} \Phi(v_i, v_i)$, where $\{v_i\}$ is any g-orthonormal basis for V. Clearly this definition is independent of choice of basis.

Any inner product structure $\langle\, ,\, \rangle$ on W induces from Φ a map $V \otimes V \to (W \otimes W)^*$ and its adjoint $W \otimes W \to (V \otimes V)^*$. Identifying V with V^* via g then gives $\Phi^*: W \otimes W \to V^* \otimes V$, which in turn induces a new inner product $\langle\, ,\, \rangle^\sim$ on W: $\langle w, w' \rangle^\sim = \dfrac{1}{n} \operatorname{tr} \Phi^*(w, w')$. Φ is *proper* if $\langle\, ,\, \rangle = \langle\, ,\, \rangle^\sim$.

Proposition. *A necessary and sufficient condition that Φ be proper is that* $\operatorname{tr} \Phi = n \operatorname{id}_W$.

Proof. Let $v_1, \cdots, v_n$ be a g-orthonormal basis for V. We have $\langle w, w' \rangle^\sim = \dfrac{1}{n} \operatorname{tr} (\Phi^*(w, w')) = \dfrac{1}{n} \sum_{i=1}^{n} g(\Phi^*(w, w')v_i, v_i) = \dfrac{1}{n} \sum_i \langle \Phi(v_i, v_i)w, w' \rangle = \dfrac{1}{n}\langle (\operatorname{tr} \Phi)w, w' \rangle$. q.e.d.

There is a natural action of $GL_V \otimes_R GL_W$ on the set of Hom (W)-valued inner products on V, given by

$$((R, S)\Phi)(v, v') = S^*\Phi(Rv, Rv')S$$

for $(R, S) \in GL_V \otimes GL_W$. (The action is of $GL_V \otimes_R GL_W$ instead just of $GL_V \times GL_W$ since, by the linearity of Φ, $(R, tS)\Phi = (tR, S)\Phi = t^2(R, S)\Phi$ for $(R, S) \in GL_V \times GL_W, t \in R$.) This action restricts to an action of $SO_V \otimes SO_W$. Let $\underline{G}$ and G denote the identity components of the isotropy of the GL and SO actions for some Φ, respectively.

Example 1. If $\dim W = 1$, then a Hom (W)-valued inner product on V is just a real-valued inner product on V. In this case $\underline{G} = G = SO_V$.

1.1. Proposition. *If $n \geq 2$, $m \geq 2$, then for generic Φ, $\underline{G}$ is trivial.*

Proof. Suppose there is a 1-parameter family of solutions (R_t, S_t) to the equation $(R_t, S_t)\Phi = \Phi$ ($|t| < \varepsilon$) such that $(R_0, S_0) = I$. We may assume that $\frac{d}{dt}(R_t, S_t)|_{t=0} \neq 0$. Choosing coordinates in V and writing $\Phi = (\phi_{ij})$, $R = (R_{ij})$ ($\phi_{ij} \in \mathrm{Hom}(W, W)$, $R_{ij} \in R$), we have from $(R_t, S_t)\Phi = \Phi$ the system of equations

$$S_t^{-1*}\phi_{ij}S_t^{-1} = \sum_{\alpha,\beta} \phi_{\alpha\beta}R_{\alpha i,t}R_{\beta j,t} \qquad \text{for all } i, j ,$$

whence

$$0 = \lim_{t\to 0} \frac{1}{t}\left(S_t^{*-1}\phi_{ij}S_t^{-1} - \sum_{\alpha,\beta} \phi_{\alpha\beta}R_{\alpha i,t}R_{\beta j,t}\right) ,$$

i.e.,

$$0 = \phi_{ij}S_t^{-1\prime}(0) + S_t^{*-1\prime}(0)\phi_{ij} - \sum_{\beta} \phi_{i\beta}R'_{\beta j,t}(0) - \sum_{\alpha} R'_{\alpha i,t}(0)\phi_{\alpha j} .$$

This is a system of $\frac{1}{2}nm(nm + 1)$ linearly independent homogeneous linear equations in $n^2 + m^2 - 1$ variables (the -1 occurs because GL_V and GL_W are tensored together over R). Thus there are no nonzero values for $\frac{d}{dt}(R_t, S_t)|_{t=0}$, a contradiction, if $\frac{1}{2}mn(mn + 1) \geq n^2 + m^2 - 1$. But $n \geq 2$ and $m \geq 2$ imply

$$\begin{aligned} m &> \sqrt{\frac{13}{2}} > \frac{-n}{2(n^2-2)} + \frac{\sqrt{13}}{2} \\ &> \frac{-n}{2(n^2-2)} + \sqrt{2 + \frac{2}{n^2-2} + \frac{n^2}{4(n^2-2)^2}} \\ &= \frac{-n + \sqrt{n^2 - 4(n^2-2)(2-2n^2)}}{2(n^2-2)} , \end{aligned}$$

whence $(n^2 - 2)m^2 + nm + (2 - 2n^2) > 0$ by the quadratic formula, i.e., $\frac{1}{2}mn(mn + 1) > m^2 + n^2 - 1$.

Notation. $G_V = G \cap (SO_V \oplus \{I\})$, $G_W = G \cap (\{I\} \oplus SO_W)$, $G' = G/G_V + G_W$.

The proposition shows that unlike the situation when dim V or dim W is 1, for dimensions > 1 generically $G = \{I\}$. Observe also that there is no reason to expect generally that $\underline{G} = G$, i.e., that (as in the one-dimensional cases) every isotropy element will be orthogonal. Since G is the maximal compact subgroup of $\underline{G}$, of course $\underline{G}$ retracts onto G. In addition there are other relations between $\underline{G}$ and G:

1.2. Proposition. *If* $(R, S) \in \underline{G}$ *we may assume that* $R \in SL_V$; *then* $S \in SL_W$. *Thus there is a natural inclusion* $\underline{G} \to SL_V \oplus SL_W$. *Moreover,* $\underline{G} \cap (SL_V \oplus SO_W) = \underline{G} \cap (SO_V \oplus SL_W) = G$.

Proof. $\left(\frac{R}{\det R}, (\det R)S\right) = (R, S)$ as an element of $\underline{G} \subset GL_V \otimes_R GL_W$, so clearly we may assume that $\det R = 1$. Now for all $v \in V$, $\det(\Phi(v, v)) = \det(S^*\Phi(Rv, Rv)S) = (\det S)^2 (\det \Phi(Rv, Rv))$, so $\frac{\det \Phi(Rv, Rv)}{\det \Phi(v, v)} = (\det S)^2$ is independent of v. Hence $\det \Phi(R^N v, R^N v) = (\det S)^{2N} \det \Phi(v, v)$, which is possible for all v, N only if $(\det S)^2 = 1$. Therefore $\det S = 1$ since (R, S) is in the identity component of the isotropy. Finally, if R (respectively, S) is orthogonal, conjugation of Φ by R (resp., S) leaves invariant the positive transformation $\operatorname{tr} \Phi \in \operatorname{Hom}(W, W)$ (resp., $\operatorname{tr} \Phi^* \in \operatorname{Hom}(V, V)$). Conjugation of $\operatorname{tr} \Phi$ by S (resp., $\operatorname{tr} \Phi^*$ by R) must therefore also leave it invariant. Therefore $S \in SO_W$ (resp., $R \in SO_V$).

1.3. Corollary. $\underline{G} \cap (GL_V \otimes \{I\}) = G_V$, $\underline{G} \cap (\{I\} \otimes GL_W) = G_W$.

Observe that $\Phi: V \otimes V \to W^* \otimes W$ induces (identifying W with W^* via the inner product $\langle\, ,\rangle$) a symmetric bilinear form $(V \otimes W) \otimes (V \otimes W) \to R$. In general this form need not be positive. (The lack of necessary positivity in higher dimensions than 1 is why not to expect that generally $\underline{G} = G$. When $\dim W = 1$, the form is, in effect, Φ itself, and therefore is positive.)

1.4. Proposition. *If the induced bilinear form on $V \otimes W$ is positive, then $\underline{G} = G$.*

Proof. A positive bilinear form is preserved under conjugation only by orthogonal transformations, and (R, S) is orthogonal as an element of $GL_{V \otimes W}$ if and only if $R \in O_V, S \in O_W$.

Example 2. Suppose $\dim W = 2$, $J_W : W \to W$ is $\langle\, ,\rangle$-orthogonal with $J_W^2 = -I$, $\dim V = 2k$. Suppose V_r and V_i are complementary k-dimensional subspaces of V with projections π_r, π_i. Suppose Φ is a $\operatorname{Hom}(W)$-valued inner product satisfying

$$\begin{aligned}\Phi(u, v) = {} & (g(\pi_r u, \pi_r v) + g(\pi_i u, \pi_i v))I \\ & + (g(\pi_i u, \pi_r v) - g(\pi_r u, \pi_i v))J_W\end{aligned}$$

for some positive symmetric real-valued bilinear form g on V. Then V is given an almost-complex structure J_V defined by $\tilde{\Phi}(J_V v) = J_W \tilde{\Phi}(v)$, Φ is an hermitian metric on V, and $g = \frac{1}{2} \operatorname{tr} \Phi$. In this case $G_V = U(k) \subset SO(2k)$, $G_W = SO(2) = S^1$, $G' = 0$.

Example 2 bis (cf. [4, p. 22]). A *complex*-bilinear map $F: C^n \otimes C^n \to C^m$ is *V-hermitian* if for all $u, v \in C^n$, $F(u, v) = \overline{F(v, u)}$, $F(u, u) \in \overline{V}$ (where V is a given convex cone, not containing an entire line, in R^M; for instance the cone of positive transformations in $GL(m, C)$ $(m^2 = M)$), $F(u, u) = 0$ only if $u = 0$. By suitably embedding C^M in $GL(m, C)$ we obtain the complex analog of a $\operatorname{Hom}(GL(m, C))$-valued inner product on C^n (i.e., a map $\Phi: U \otimes U \to W^* \otimes W$ such that $\Phi(u, u') = \overline{\Phi(u', u)^*}$, $\Phi(u + tu', u'') = \Phi(u', u'') + t\Phi(u', u'')$ for $t \in C$, and $\Phi(u, u)$ is positive for $u \neq 0$).

Example 3. Suppose V is of even dimension $2k$, and $\alpha: C(V) \to \mathrm{Hom}\,(W)$ represents the Clifford algebra of V as endomorphisms of a vector space of dimension 2^k over R or 2^{k-1} over $H =$ quaternions. Then $\Phi(u, v) = \alpha(u)\alpha(v)^*$ defines a $\mathrm{Hom}\,(W)$-valued inner product on V, which may be considered a $C(V)$-valued inner product. Here $G_V = Sp_V$ and $G_W = SO_W$.

Example 4. Dim $V = 2$, dim $W = 2k$. There are three possibilities: (i) $G = I \oplus G_W$; (ii) $G = S^1 \oplus G_W, G_V = S^1$; (iii) $G = S^1 \oplus G_W, G_V = 0$. (The only other seeming possibility would be that $G_V = S^1$ and $G' = S^1$, but this cannot happen. For suppose $G_V = S^1$, $(R, S) \in G$. Then $(R, I) \in G$ also; hence so is $(R, S)(R, I)^{-1} = (I, S)$, and $(I, S) \in G_W$. Therefore $G' = G/G_V \oplus G_W$ is trivial.) Let $J: V \to V$ be g-orthogonal with $J^2 = -I$. A necessary and sufficient condition for (ii) is that there exist a positive linear map $A: W \to W$ with tr $A = 2k$, and a skew-adjoint map $B: W \to W$, such that for any $v, v' \in V$, $\Phi(v, v') = g(v, v')A + g(Jv, v')B$. In fact, this condition is clearly sufficient. On the other hand, if $G_V = S^1$, we must have: for all v such that $|v| = 1$, $\Phi(v, v) = A$ for some fixed positive A, and $\Phi(Jv, v) = B$ for some fixed B. But then $B = \Phi(J^2v, Jv) = -\Phi(v, Jv) = -B^*$, so B is skew-adjoint. Finally, for all $v, v' \in V$, write $v' = av + bJv$; then $\Phi(v, v') = |v|^2\Phi(v/|v|, av/|v| + bJv/|v|) = a|v|^2A - b|v|^2B = g(v, v')A + g(Jv, v')B$.

Necessary and sufficient conditions for (iii) are: there exists $e^M \in SO_W$ such that: (a) for all $\theta \in R$, $e^{-M\theta}(\mathrm{tr}\,\Phi)e^{M\theta} = \mathrm{tr}\,\Phi$; (b) for any nonzero $v \in V$

$$e^{-M\theta}B'e^{M\theta} = B' ,$$
$$e^{-\pi M/2}B''e^{\pi M/2} = -B'' ,$$
$$\Phi(v, v) = \tfrac{1}{2}(\mathrm{tr}\,\Phi + e^{-\pi M/4}B''e^{\pi M/4} - e^{\pi M/4}B''e^{-\pi M/4}) ,$$
$$\Phi(Jv, Jv) = \tfrac{1}{2}(\mathrm{tr}\,\Phi - e^{-\pi M/4}B''e^{\pi M/4} + e^{\pi M/4}B''e^{\pi M/4}) ,$$

where we have written $\Phi(Jv, v) = B = B' + B''$ with B'' (respectively, B') self-adjoint (resp. skew-adjoint). For suppose $(e^{i\theta}, e^{M\theta})$ fixes Φ, θ generic. Since by hypothesis $G_V = 0$, we have $G = G_W + S^1$, where S^1 is given by $\{(e^{i\theta}, e^{M\theta}) : \theta \in R\}$. Since conjugation by $e^{i\theta}$ does not alter tr Φ, it follows that $e^{-M\theta}(\mathrm{tr}\,\Phi)e^{M\theta} = \mathrm{tr}\,\Phi$ for all $\theta \in R$. Now choose nonzero $v \in V$; $\{v, Jv\}$ is a basis for V. Let $A = \Phi(v, v)$, $C = \Phi(Jv, Jv)$, $B = \Phi(Jv, v)$, and write $e^{M\theta} = S$,

$$e^{i\theta} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} .$$

Using $A + C = \mathrm{tr}\,\Phi$ we see that the equation $(e^{i\theta}, e^{M\theta})\Phi = \Phi$ is equivalent to

$$(1) \qquad A\cos 2\theta - S^*AS + B''\sin 2\theta + (\mathrm{tr}\,\Phi)\sin^2\theta = 0 ,$$

$$(2) \qquad A\sin 2\theta - B''\cos 2\theta - B' + S^*BS - (\mathrm{tr}\,\Phi)\sin\theta\cos\theta = 0 .$$

The skew-adjoint part of (2) gives $e^{-M\theta}B'e^{M\theta} = B'$. Considering symmetric parts, multiplying (1) by $\sin 2\theta$, (2) by $\cos 2\theta$, and subtracting give

$$S^*AS \sin 2\theta - B'' + S^*B''S \cos 2\theta - (\operatorname{tr} \Phi) \sin\theta \cos\theta = 0 .$$

Conjugating by S^* and subtracting from (2) yield

$$B'' \cos 2\theta = \tfrac{1}{2}(SB''S^* + S^*B''S) . \tag{3}$$

Choosing $\theta = \frac{1}{4}\pi$ gives

$$e^{\pi M/4}B''e^{-\pi M/4} + e^{-\pi M/4}B''e^{\pi M/4} = 0 ,$$

i.e.,

$$B'' = -e^{-M/2}B''e^{M/2} ,$$

and the formulas for A and C now follow from (2) and $\operatorname{tr} \Phi = A + C$. Finally, to show the conditions sufficient, diagonalize B'' and find S explicitly from (3). In fact, one may choose coordinates so that

$$B = \begin{bmatrix} \lambda_1 & & & & & -a_1 \\ & \ddots & & & \iddots & \\ & & \lambda_k & -a_k & & \\ & & a_k & -a_k & & \\ & \iddots & & & \ddots & \\ a_1 & & & & & -\lambda_1 \end{bmatrix}, \quad \lambda_1 \geq \cdots \geq \lambda_k \geq 0 \quad (\text{not all } \lambda_j = 0) ,$$

$$A = \operatorname{tr} \Phi + \begin{bmatrix} & & & & & \lambda_1 \\ & & & & \iddots & \\ & & & \lambda_k & & \\ & & \lambda_k & & & \\ & \iddots & & & & \\ \lambda_1 & & & & & \end{bmatrix}, \quad C = \operatorname{tr} \Phi - \begin{bmatrix} & & & & & \lambda_1 \\ & & & & \iddots & \\ & & & \lambda_k & & \\ & & \lambda_k & & & \\ & \iddots & & & & \\ \lambda_1 & & & & & \end{bmatrix} .$$

Example 5. $\operatorname{Dim} V = 2k$, $\dim W = 2$. This case is almost but not quite dual to the preceding. Again there are three possibilities: (i) $G = G_V \oplus \{I\}$; (ii) $G = G_V \oplus S^1$, $G_W = S^1$; (iii) $G = G_V \oplus S^1$, $G_W = 0$. This time a necessary and sufficient condition for (ii) is that Φ be proper and that there exist a skew-adjoint map $B\colon V \to V$ such that for all $w, w' \in W$, $\Phi^*(w, w') = \langle w, w'\rangle I + \langle Jw, w'\rangle B$, J being an $\langle\ ,\ \rangle$-isometry with $J^2 = -I$. Similarly a necessary and sufficient condition for (iii) is that Φ be proper and that for $w \in W$ there exist an $e^M \in SO_V$ such that:

$$e^{-M\theta}B'e^{M\theta} = B' , \qquad \text{for all } \theta \in R ,$$

$$e^{-M/2}B''e^{M/2} = -B'' ,$$

$$\Phi^*(w, w) = I + \tfrac{1}{2}(e^{-\pi M/4}B''e^{\pi M/4} - e^{\pi M/4}B''e^{\pi M/4}) ,$$
$$\Phi^*(Jw, Jw) = I - \tfrac{1}{2}(e^{\pi M/4}B''e^{\pi M/4} - e^{\pi M/4}B''e^{\pi M/4}) ,$$

where B'' (respectively, B') is the self-adjoint (resp., skew-adjoint) part of $\Phi^*(Jw, w)$.

The necessity for the propriety of Φ is seen as follows. Pick orthonormal bases for V and W with respect to g and $\langle\ ,\ \rangle$; write $A = \Phi^*(w, w)$, $C = \Phi^*(Jw, Jw)$. The orthogonality of the basis for V implies that $A + C = 2I$. The conditions for the propriety of Φ are that $\operatorname{tr} A = \operatorname{tr} C = n$, $\operatorname{tr} B = 0$. But in both cases (ii) and (iii) we must have $A \sim C$, $B \sim -B^*$, so these conditions must be fulfilled. In this case, by taking suitable coordinates we may write

$$B = \begin{bmatrix} \lambda_1 & & & & & -a_1 \\ & \ddots & & & \iddots & \\ & & \lambda_k & -a_k & & \\ & & a_k & -\lambda_k & & \\ & \iddots & & & \ddots & \\ a_1 & & & & & \lambda_1 \end{bmatrix} , \quad \lambda_1 \geq \cdots \geq \lambda_k \geq 0 , \quad \text{not all } \lambda_i = 0 ,$$

$$A = \begin{bmatrix} 1 & & & & & \lambda_1 \\ & \ddots & & & \iddots & \\ & & 1 & \lambda_k & & \\ & & \lambda_k & 1 & & \\ & \iddots & & & \ddots & \\ \lambda_1 & & & & & 1 \end{bmatrix} , \quad C = \begin{bmatrix} 1 & & & & & -\lambda_1 \\ & \ddots & & & \iddots & \\ & & 1 & -\lambda_k & & \\ & & -\lambda_k & 1 & & \\ & \iddots & & & \ddots & \\ -\lambda_1 & & & & & 1 \end{bmatrix} .$$

Example 6. Suppose $j\colon V \to \operatorname{Hom}(W)$ is linear; then $\Phi(v, v') = j(v)j(v')^*$ is a $\operatorname{Hom}(W)$-valued inner product. Suppose $\mathscr{C} \subset \operatorname{Hom}(W)$ is a represented Clifford algebra and image $\Phi \subset \mathscr{C}$, V splitting into subspaces V_i each linearly isomorphic to $\mathscr{C}$ under j and such that $\Phi(v_i, v_k) = 0$ for $v_i \in V_i$, $v_k \in V_k$, $i \neq k$. Then Φ defines a Clifford module structure on V. In this case G_V is a direct sum of spin groups, G_W varies.

The set of $\operatorname{Hom}(W)$-valued inner products on V can be identified, via choices of bases on V and W, with a subset of the symmetric $(mn \times mn)$-matrices. The subset is clearly convex and open, and properly contains the positive $(mn \times mn)$-matrices. It is thus an open $\frac{1}{2}mn(mn + 1)$-cell. The orbit space is the quotient of this cell under the effective action of the $(m^2 + n^2 - 1)$-dimensional group $GL_V \otimes GL_W$, which is a free action off a set of measure zero.

2. Hom ($\mathscr{W}$)-valued metrics

Throughout the sequel unless otherwise noted we shall assume M to be a smooth oriented n-dimensional manifold, and $\mathscr{W}$ an oriented vector bundle on M with m-dimensional fibre, isomorphic with the inner product space W and thus itself inheriting an inner product; that is, we assume *a priori* a (real-valued) metric on $\mathscr{W}$. If Φ is a Hom (W)-valued inner product on the n-dimensional vector space V, a *Hom* ($\mathscr{W}$)*-valued metric of type* Φ on M is a reduction of the principal bundle $B(M) \otimes B(\mathscr{W})$ (with group $GL_V \otimes GL_W$) to a principal bundle $B_{\underline{G}}$ with fibre $\underline{G}$. If the fibre $B(M)_x$ (respectively, $B(\mathscr{W}_x)$ is considered as the collection of isomorphisms $V \to TM_x$ (resp., $W \to \mathscr{W}_x$) we may, given Φ, write $\Phi_x(X, Y) = q \circ \Phi(p^{-1}X, p^{-1}Y) \circ q^{-1}$ for $X, Y \in TM_x$, $(p, q) \in B_{\underline{G}}$. By definition of $\underline{G}$, $\Phi_x(X, Y)$ is defined independently of choice of (p, q). Thus a Hom ($\mathscr{W}$)-valued metric of type Φ may be considered the smooth assignment of Hom ($\mathscr{W}_x$)-valued metric to TM_x, with the metric at each point being in the orbit of Φ.

Example 1 (*cont.*). If $\mathscr{W}$ is the trivial line bundle, then Φ is an affine inner product on V, and a Hom ($\mathscr{W}$)-valued metric on M is just a Riemannian metric on M.

Example 2 (*cont.*). If $\mathscr{W}$ is a trivial R^2-bundle, and Φ is as in Example 2, then a Hom (R^2)-valued metric of type Φ defines an almost-complex structure with hermitian metric on M.

Example 3 (*cont.*). If Φ is as in example 3, then a Hom ($\mathscr{W}$)-valued metric on M defines an almost-Hamiltonian structure (cf. [5, p. 36]) on M, with a "Clifford-valued" metric.

We consider the problem of the existence of a Hom ($\mathscr{W}$)-valued metric of type Φ.

2.1. Proposition. *The existence of a* Hom ($\mathscr{W}$)*-valued metric of type* Φ *on M is equivalent to the reduction of* $B_{SO_V}(M) \otimes B_{SO_W}(\mathscr{W})$ *to a principal bundle* B_G *with group G. If* $G' = 0$, *then this reduction is possible if and only if* $B_{SO_V}(M)$ *and* $B_{SO_W}(\mathscr{W})$ *reduce to* $B_{G_V}(M)$ *and* $B_{G_W}(\mathscr{W})$, *respectively, i.e., if and only if the associated bundles* $B_{SO_V}(M)/G_V$ *and* $B_{SO_W}(\mathscr{W})/G_W$ *admit sections.*

Proof. The first statement is clear, since $B(M) \otimes B(\mathscr{W})$ can always be reduced to $B_{SO_V}(M) \otimes B_{SO_W}(\mathscr{W})$, a principal bundle with group the maximal compact subgroup of $GL_V \otimes GL_W$. The obstructions to reducing to B_G lie in $H^*(M, \pi_*(SO_V \oplus SO_W/G))$, and when $G' = 0$ this is just $H^*(M, \pi_*(SO_V/G_V \oplus SO_W/G_W)) = H^*(M, \pi_*(SO_V/G_V)) + H^*(M, \pi_*(SO_W/G_W))$, proving the other assertion. q.e.d.

For trivial $\mathscr{W}$ the metric thus exists if and only if $B_{SO_V}(M)/G_V$ is sectionable. If $\mathscr{W}$ is nontrivial, then both bundles must be tested.

Example 2 (*cont.*). $G_W = SO_W \approx U(1)$, so $B_{SO_W}(\mathscr{W})/G_W$ is trivial and admits a section whether $\mathscr{W}$ is trivial or not, and M admits a Hom ($\mathscr{W}$)-valued metric of type Φ if and only if M admits an almost-complex structure.

Example. The preceding discussion has assumed M to be oriented. Suppose for this example that M is the (nonorientable) Mobius strip. M does not admit a Hom (R^2)-valued metric of type Φ in Example 2 because it is not almost-complex. However, M does admit a Hom (TM)-valued metric of type Φ as follows: pick a Riemannian metric on M, and define $\Phi(u, v) =$ rotation of the tangent plane so as to carry the unit vector in the direction of u to the unit vector in the direction of v, and scalar multiplication by $|u|\cdot|v|$. The existence of this Φ may be thought to say that M is "twisted almost-complex" (its double cover is almost-complex).

Example. If $V = R^4 = W$,

$$\Phi : \begin{cases} (e_i, e_i) \to I\,, \\ \left.\begin{matrix}(e_1, e_2)\\(e_3, e_4)\end{matrix}\right\} \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}, \\ \left.\begin{matrix}(e_2, e_1)\\(e_4, e_3)\end{matrix}\right\} \to \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \\ (e_i, e_j) \to 0\,, \qquad \text{otherwise}, \end{cases}$$

then $G_V = U(2)$ and $G_W = S^1 \times S^1$, $SO_W/G_W \sim S^2 \times S^2$, and $B_{SO_W}(\mathscr{W})/G_W$ is sectionable when the obstruction $c \in H^3(M, \pi_2(SO_W/G_W))$ vanishes [6, p. 178]. Thus a Hom $(\mathscr{W})$-valued metric of type Φ exists if and only if M is almost-complex (so B_{SO_V} reduces to $B_{G_V} = B_{U(2)}$) and $\mathscr{W}$ admits an almost-product structure (so $B_{SO_W}(\mathscr{W})$ reduces to $B_{G_W} = B_{S^1 \times S^1}$; *cf.* [5, p. 36]).

Example 3 (*cont.*). The obstruction to reducing $B_{SO_V}(M)$ to $B_{Sp_V}(M)$ is the second Stiefel-Whitney class of M. When this vanishes, i.e., when M is a spin manifold, a bundle with fibre W can be defined such that $C(TM_x) =$ Hom $(\mathscr{W}_x)$. Then $SO_W/G_W = 0$, so a metric of type Φ exists.

Now suppose $G' \neq 0$. If $B_{SO_V}(M)$ and $B_{SO_W}(\mathscr{W})$ reduce to $B_{G_V}(M)$ and $B_{G_W}(\mathscr{W})$, then a Hom $(\mathscr{W})$-valued metric of type Φ exists. However, the condition is not necessary. Let $j: G' \to (SO_V \oplus SO_W)/(G_V \oplus G_W)$, $q: (SO_V \oplus SO_W)/(G_V \oplus G_W) \to (SO_V \oplus SO_W)/G$. Then we have an exact sequence

$$\cdots \longrightarrow \pi_{i+1}((SO_V \oplus SO_W)/(G_V \oplus G_W)) \xrightarrow{q_*} \pi_{i+1}((SO_V \oplus SO_W/G) \longrightarrow$$
$$\longrightarrow \pi_i(G') \xrightarrow{j_*} \pi_i((SO_V \oplus SO_W)/(G_V \oplus G_W) \longrightarrow \cdots$$

whence a short exact sequence

$$0 \longrightarrow \frac{\pi_i((SO_V \oplus SO_W)/(G_V \oplus G_W))}{j_*\pi_i(G')} \longrightarrow \pi_i((SO_V \oplus SO_W)/G) \longrightarrow$$

$$\xrightarrow{\Delta_i} \ker\{j_* : \pi_{i-1}(G') \longrightarrow \pi_{i-1}((SO_V \oplus SO_W)/(G_V \oplus G_W))\} \longrightarrow 0$$

and an exact cohomology sequence

$$\cdots \longrightarrow H^{k-1}(M, \ker j_*) \xrightarrow{\beta^*} H^k\left(M, \frac{\pi_i((SO_V \oplus SO_W)/(G_V \oplus G_W))}{j_*\pi_i(G')}\right) \longrightarrow$$

$$\xrightarrow{q^*} H^k(M, \pi_i((SO_V \oplus SO_W)/G)) \xrightarrow{\Delta^*} H^k(M, \ker j_*) \longrightarrow \cdots .$$

Suppose $c^{k+1} \in H^{k+1}(M, \pi_k((SO_V \oplus SO_W)/(G_V \oplus G_W)))$ is the primary obstruction to the sectioning of $B_{SO_V}(M)/G_V$ and $B_{SO_W}(\mathscr{W})/G_W$. Then c^{k+1} induces an element $\tilde{c}^{k+1} \in H^{k+1}(M, \pi_k((SO_V \oplus SO_W)/G))$, which is the primary obstruction to sectioning $(B_{SO_V}(M) \oplus B_{SO_W}(\mathscr{W}))/G$. We also have an exact sequence

$$0 \longrightarrow j_*\pi_k(G') \xrightarrow{\alpha_*} \pi_k((SO_V \oplus SO_W)/(G_V \oplus G_W)) \longrightarrow$$

$$\xrightarrow{\rho_*} \pi_k((SO_V \oplus SO_W)/(G_V \oplus G_W))/j_*\pi_k(G') \longrightarrow 0 ,$$

where ρ_* is defined by the sequence, whence a cohomology sequence

$$\cdots \longrightarrow H^{k+1}(M, j_*\pi_k(G')) \xrightarrow{\alpha^*} H^{k+1}(M, \pi_k((SO_V \oplus SO_W)/(G_V \oplus G_W)))$$

$$\xrightarrow{\rho^*} H^{k+1}\left(M, \frac{\pi_k((SO_V \oplus SO_W)/(G_V \oplus G_W))}{j_*\pi_k(G')}\right) \longrightarrow \cdots .$$

Hence

2.2. Proposition. *The primary obstruction to reducing to B_G, $\tilde{c}^{k+1} = 0$, if and only if*

$$c^{k+1} = \alpha^* h , \qquad h \in H^{k+1}(M, j_*\pi_k G')) ,$$

or

$$\rho^* c^{k+1} = \beta^* \hat{h} , \qquad \hat{h} \in H^k(M, \ker j_*) ,$$

(*notations as in preceding discussion*).

2.3. Corollary. *Suppose $j: G' \to (SO_V \oplus SO_W)/(G_V \oplus G_W)$ induces homotopy injections at all levels. Then the primary obstruction is an element of $H^{k+1}(M, \pi_k((SO_V \oplus SO_W)/(G_V \oplus G_W)))/H^{k+1}(M, \pi_k G')$.*

Proof. For $H^k(M, \ker j_*) = 0$ for all k, so $\tilde{c}^{k+1} = 0$ if and only if $c^{k+1} = \alpha^* h$ for $h \in H^{k+1}(M, j_*\pi_k G')$, or alternatively, $c^{k+1} = \alpha'^* h'$, where $h' \in H^{k+1}(M, \pi_k G')$ and

$$\alpha'^* : H^*(M, \pi_k G') \xrightarrow{\approx} H^*(M, j_*\pi_k G')$$
$$\xhookrightarrow{\alpha^*} H^*(M, \pi_k((SO_V \oplus SO_W)/(G_V \oplus G_W))) \ .$$

Example 5 (*cont.*). If $\dim W = 2$, then G' vanishes unless $G_W = 0$, and then $G' = 0$ or S^1. In the latter case we have topologically $G = G_V \times S^1$, so $j: S^0 \sim (SO_V/G_V) \times S^1$ induces an injection in homotopy. Hence the obstructions to a Hom $(\mathscr{W})$-valued metric of type Φ lie in $H^{k+1}(M, \pi_k((SO_V/G_V) \times S^1))/H^{k+1}(M, \pi_k(S^1))$. Thus we have a corollary to the previous proposition:

2.4. Corollary. *In this example a metric of type Φ is possible only if the only obstruction to sectioning $(SO_V/G_V) \times S^1$ lies in $H^2(M, \pi_1((SO_V/G_V) \times S^1))$.*

For instance, if $\dim M = 2$ and Φ is given by

$$(e_1, e_1) \to \begin{bmatrix} 1 & 1/4 \\ 1/4 & 1 \end{bmatrix} , \qquad (e_2, e_2) \to \begin{bmatrix} 1 & -1/4 \\ -1/4 & 1 \end{bmatrix} ,$$
$$\left.\begin{matrix} (e_1, e_2) \\ (e_2, e_1) \end{matrix}\right\} \to \begin{bmatrix} 1/4 & 0 \\ 0 & -1/4 \end{bmatrix}$$

for some $e_1 \in V$ and $e_2 = Je_1$ ($J^2 = -I$), then $SO_V/G_V = S^1$, and $j_* : \pi_1(G') = \pi_1(S^1) \to \pi_1((SO_V/G_V) \times S^1) = Z \oplus Z : n \to (n, n)$. Since $H^2(M, Z \oplus Z) = Z \oplus Z$, M will not admit a Hom $(\mathscr{W})$-valued metric of type Φ with $\mathscr{W}$ trivial unless M is flat. But M will always admit a Hom (TM)-valued metric of type Φ. (Choose an almost-complex structure J on TM and for any unit vector $v \in TM$ define

$$\Phi(v, v) : v \to v + \tfrac{1}{4}Jv \ , \qquad Jv \to \tfrac{1}{4}v + Jv \ ,$$
$$\Phi(Jv, v) : v \to \tfrac{1}{4}v \ , \qquad Jv \to \tfrac{1}{4}Jv.)$$

In fact, the discussion shows that M admits a Hom $(\mathscr{W})$-valued metric of type Φ if and only if the characteristic class (in the sense of [6, p. 178]) $c \in H^2(M, \pi_1(SO_W))$ of $\mathscr{W}$ generates $H^2(M, Z)$.

Now the fibre $(B_{\underline{G}})_x$ of $B_{\underline{G}}$ ($x \in M$) may be considered as a set of isomorphisms $V \otimes W \to TM_x \otimes W_x$. The right action of $\underline{G}$ on $B_{\underline{G}}$ is then given by

$$(R_{(a,b)}(p, q))(v, w) = (p, q)(av, bw) \ .$$

As previously noted, $\underline{G}$ may be considered naturally as a subgroup of $SL_V \oplus SL_W$, where W is given its *a priori* inner product and V has the inner product g induced by Φ.

If $\pi : B_{\underline{G}} \to M$ is the projection, define a V-valued 1-form ω on $B_{\underline{G}}$ by $\omega(X_{(p,q)}) = p^{-1}\pi_* X$. Similarly the *fundamental linear form* $\overset{w}{\omega}$ on $B_{\underline{G}}$ is the Hom $(V^*, \mathrm{Hom}\,(W, W)) \approx V \otimes W^* \otimes W$ -valued 1-form on $B_{\underline{G}}$ defined thus: for $X \in T_{(p,q)}B_{\underline{G}}$, $v^* \in V^*$, $w \in W$, $x = \pi(p, q)$,

$$(\overset{w}{\omega}(X))(v^*)w = q^{-1}(\Phi_x(\pi_* X, \tilde{g}_x p^{*-1} v^*))(qw)) \ .$$

Observe that $\overset{w}{\omega} = \tilde{\Phi} \circ \omega$ and also that

$$\begin{aligned} ((R^*_{(a,b)}\overset{w}{\omega})(X_{(p,q)}))(v^*)w &= (\overset{w}{\omega}(R_{(a,b)*}X_{(p,q)}))(v^*)w \\ &= b^{-1}q^{-1}(\Phi_x(\pi_* R_{(a,b)*}X_{(p,q)}\tilde{g}_x(p^{*-1}a^{*-1}v^*))(qbw)) \\ &= b^{-1}q^{-1}(\Phi_x(\pi_* X_{(p,q)}, \tilde{g}_x(p^{*-1}a^{*-1}v^*))(qbw)) \\ &= b^{-1}((\overset{w}{\omega}(X))(a^{*-1}v^*)(bw)) \ . \end{aligned} \tag{4}$$

Alternatively, think of $\overset{w}{\omega}$ as a $(V \otimes W^* \otimes W)$-valued form, and (4) becomes

$$R^*_{(a,b)}\overset{w}{\omega} = (a^{-1}, b, b^{-1})\overset{w}{\omega} \ , \tag{5}$$

where a and b^{-1} act on the left and b acts on the right.

Infinitessimally, let $(A, B) \in \underline{\mathfrak{G}} =$ Lie algebra of $\underline{G}$. Then $(\exp tA, \exp tB) \in \underline{G}$ and $R_{(\exp tA,\, \exp tB)}$ is a one-parameter group of diffeomorphisms of $B_{\underline{G}}$. Let its infinitessimal generator be $\widetilde{(A, B)}$. Clearly $\sim$ commutes with Lie bracket and $\pi_*\widetilde{(A, B)} = 0$. Thus we have described a canonical isomorphism of $\underline{\mathfrak{G}}$ with the tangent space to the fibre of $B_{\underline{G}}$ at any point. The infinitessimal version of (5) becomes

$$\begin{aligned} \widetilde{(A, B)} \lrcorner\, d\overset{w}{\omega} &= (-A, I, I)\overset{w}{\omega} + (I, B, I)\overset{w}{\omega} + (I, I, -B)\overset{w}{\omega} \\ &= (-A \otimes I \otimes I + I \otimes B \otimes I - I \otimes I \otimes B)\overset{w}{\omega} \ , \end{aligned} \tag{6}$$

where $\lrcorner$ denotes interior product. Write

$$\psi(A, B) = (-A \otimes I \otimes I + I \otimes B \otimes I - I \otimes I \otimes B) \ .$$

Let $(p, q) \in B_{\underline{G}}$, and let $H_{(p,q)}$ be an n-dimensional subspace of $T_{(p,q)}B_{\underline{G}}$. $H_{(p,q)}$ is *horizontal* if $\pi_* H_{(p,q)} = T_{\pi(p,q)}M$, i.e., if $H_{(p,q)}$ is complementary to the tangent space of the fibre. Then $\omega : H_{(p,q)} \to V$ is an isomorphism, and the restriction $d\overset{w}{\omega} : H_{(p,q)} \wedge H_{(p,q)} \to V \otimes W^* \otimes W$ is now given by the linearity of Φ as $d\overset{w}{\omega} = \tilde{\Phi} \circ d\omega$.

3. Hom ($\mathscr{W}$)-valued connexions

In this section we shall assume M to be an oriented n-dimensional manifold, $\mathscr{W}$ a vector bundle of inner product spaces on M with fibre dimension n, and Φ a Hom ($\mathscr{W}$)-valued metric on M. The real-valued bilinear form $g(X, Y) = \dfrac{1}{m} \operatorname{tr} \Phi(X, Y)$ on TM is a Riemannian metric on M, called the metric *induced* by Φ. g induces maps $\hat{g} : TM \to T^*M : X \to g(X, \cdot)$, and $\tilde{g} = \hat{g}^{-1} : T^*M \to TM$. Define $\tilde{\Phi} : TM \to L(T^*M, \operatorname{Hom}(\mathscr{W})) : Y \to \Phi(Y, \tilde{g}(\cdot))$.

A Hom $(\mathscr{W})$-valued connexion on (M, Φ) is a linear map $\overset{w}{\nabla} : \Gamma(TM) \otimes \Gamma(TM) \to \Gamma(L(T^*M, \operatorname{Hom}(\mathscr{W})))$ such that for all $X, Y \in \Gamma(TM)$ and smooth $f : M \to R$,

$$\overset{w}{\nabla}_{fX} Y = f \overset{w}{\nabla}_X Y\,, \qquad \overset{w}{\nabla}_X fY = (Xf)\tilde{\Phi}(Y) + f\overset{w}{\nabla}_X Y\,.$$

Locally, a Hom $(\mathscr{W})$-valued connexion may be constructed thus. Let x^i be local coordinates on M, inducing coordinates ∂_i on TM and dual coordinates dx^i on T^*M. Choosing $\overset{w}{\Gamma}{}^k_{ij} \in \operatorname{Hom}(\mathscr{W})$ and setting $\overset{w}{\nabla}_{\partial_i}\partial_j = \sum_k \overset{w}{\Gamma}{}^k_{ij}\partial_k$ completely determine $\overset{w}{\nabla}$. For, given vector fields $X = \sum x^i\partial_i$, $Y = \sum y^j\partial_j$, we have

$$\begin{aligned}
\overset{w}{\nabla}_X Y &= \sum_i x^i \overset{w}{\nabla}_{\partial_i} Y = \sum_{i,j} x^i \overset{w}{\nabla}_{\partial_i}(y^j\partial_j) \\
&= \sum_{i,j} x^i (y^j \overset{w}{\nabla}_{\partial_i}\partial_j + (\partial_i y^j)\tilde{\Phi}(\partial_j)) \\
&= \sum_k \Big(\sum_i x^i \Big(\sum_j y^j \overset{w}{\Gamma}{}^k_{ij} + (\partial_i y^j) \sum_l \Phi_{jl} g^{lk}\Big)\Big)\partial_k\,,
\end{aligned}$$

where $\Phi_{jl} = \Phi(\partial_j, \partial_l)$, $g_{lk} = g(\partial_l, \partial_k)$, $(g^{lk}) = (g_{lk})^{-1}$. One may now construct global Hom $(\mathscr{W})$-valued connexions from local ones, using a partition of unity, for one verifies at once from the local equation that convex linear combinations of How $(\mathscr{W})$-valued connexions are themselves such connexions. Given $\overset{w}{\nabla}$, define the *affine connexion ∇ induced by $\overset{w}{\nabla}$* by $\nabla_X Y = \dfrac{1}{m} \operatorname{tr} \overset{w}{\nabla}_X Y \in \Gamma(L(T^*M, R)) = \Gamma(TM)$.

Any vector field V along a curve $c : R \to M$ determines a map $T^*M \to \operatorname{Hom}(\mathscr{W})$ along c, denoted $\dfrac{\overset{w}{D}V}{dt}$ and called the *Hom $(\mathscr{W})$-valued covariant derivative of V (related to Φ)*. The operation $V \to \dfrac{\overset{w}{D}V}{dt}$ is characterized by

$$\frac{\overset{w}{D}(U + V)}{dt} = \frac{\overset{w}{D}U}{dt} + \frac{\overset{w}{D}V}{dt}\,.$$

If $f : R \to R$ is smooth, then

$$\frac{\overset{w}{D}(fV)}{dt} = \frac{df}{dt}\tilde{\Phi}(V) + f\frac{\overset{w}{D}V}{dt}\,.$$

If V is induced by a vector field Y on M, then

$$\frac{\overset{w}{D}V}{dt} = \overset{w}{\nabla}_{\frac{dc}{dt}} Y \,.$$

3.1. Lemma. *There is a unique operator* $\dfrac{\overset{w}{D}}{dt}$ *satisfying these conditions.*

Proof. Choose local coordinates x^i, and write $c(t) = (u^i(t))$, $V = \sum v^j \partial_j$. Then

$$\begin{aligned}\frac{\overset{w}{D}V}{dt} = \sum_j \frac{\overset{w}{D}(v^j\partial_j)}{dt} &= \sum_j \frac{dv^j}{dt}\tilde{\Phi}(\partial_j) + v^j \overset{w}{\nabla}_{(\sum_i \frac{du^i}{dt}\partial_i)}\partial_j \\ &= \sum_j \frac{dv^j}{dt}\tilde{\Phi}(\partial_j) + \sum_i v^j \frac{du^i}{dt} \sum_k \overset{w}{\Gamma}{}^k_{ij}\partial_k \\ &= \sum_{i,j,k}\left(\frac{dv^j}{dt}\Phi_{ji}g^{ik} + v^j\frac{du^i}{dt}\overset{w}{\Gamma}{}^k_{ij}\right)\partial_k \,,\end{aligned}$$

which proves uniqueness. Conversely, if $\dfrac{\overset{w}{D}}{dt}$ is defined locally by this last expression, it has the desired properties. q.e.d.

Let $\underline{\mathfrak{G}}$ be the Lie algebra of the group $\underline{G}$ of the bundle $B_{\underline{G}}$ associated to Φ. Then a Hom $(\mathscr{W})$-valued connexion on M induces a $\underline{\mathfrak{G}}$-valued *connexion 1-form* $\overset{w}{\phi}$ on $B_{\underline{G}}$ defined by

$$(\overset{w}{\phi}_{(p,q)}X)(v,w) = (p,q)^{-1}((\overset{w}{\nabla}_{\pi_* X}pv)(qw)) - X(v,w) \,,$$

where $X \in T_{(p,q)}B_{\underline{G}}$, and $(v,w) : M \to V \otimes W$. Note that if X is vertical, and $X = \widetilde{(A,B)}$ for $(A,B) \in \underline{\mathfrak{G}}$, then the first term of $\overset{w}{\phi}(X)$ vanishes and the second term just becomes $(A,B)\cdot(v,w)$. The kernel of $\overset{w}{\phi}$ is called the *horizontal space* of the connexion.

We would like to define a compatibility condition on a connexion suggesting invariance of Φ under parallel translation. The obvious candidate for such a condition would be

$$\frac{d}{dt}\Phi(U,V) = \Phi\left(\frac{DU}{dt}, V\right) + \Phi\left(U, \frac{DV}{dt}\right)$$

$\left(\text{using some connexion in } \mathscr{W} \text{ to define } \dfrac{d}{dt}\Phi(U,V)\right)$, where $\dfrac{DU}{dt} = \dfrac{1}{m}\operatorname{tr}\dfrac{\overset{w}{D}U}{dt}$.

Unfortunately this condition cannot always be realized. A similar but somewhat weaker condition would "reverse the order of mapping into Hom $(\mathscr{W})$" on the right hand side of the equation and say in essence:

$$\text{“}\ \frac{d}{dt}\Phi(U, V) = g\left(\frac{\overset{w}{D}U}{dt}, V\right) + g\left(U, \frac{\overset{w}{D}V}{dt}\right).\text{”}$$

This equation does not make sense, however, and needs rephrasing. To do so we fix (throughout the sequel) some (real-valued) connexion on $\mathscr{W}$, also to be denoted ∇. (In case $\mathscr{W}$ is trivial we use the trivial connexion.) Then define a Hom $(\mathscr{W})$-valued connexion $\overset{w}{\nabla}$ on (M, Φ) to be *compatible* (with Φ) if for any parametrized curve $c: R \to M$ and vector fields U and V along c:

$$\frac{D}{dt}(\Phi(U, V)w) = \left(\frac{\overset{w}{D}U}{dt}(\hat{g}V) + \frac{\overset{w}{D}V}{dt}(\hat{g}U)^*\right)w + \Phi(U, V)\frac{dw}{dt}.$$

It is clear that if $\overset{w}{\nabla}$ is compatible with Φ, then ∇ is compatible with g.

3.1. Lemma. *If $\overset{w}{\nabla}$ is compatible with Φ, then for any vector fields Y, Y' on M, and $X \in TM_x$,*

$$\nabla_X(\Phi(Y, Y')) = (\overset{w}{\nabla}_X Y)(\hat{g}(Y')) + (\overset{w}{\nabla}_X Y')(\hat{g}Y)^*.$$

A connexion $\overset{w}{\nabla}$ is *symmetric* if $\overset{w}{\nabla}_X Y - \overset{w}{\nabla}_Y X = \tilde{\Phi}([X, Y])$ for all $X, Y \in \Gamma(TM)$.

3.2. Lemma. *A convex linear combination of compatible (respectively, symmetric) connexions is compatible (resp., symmetric).*

3.3. Proposition. *A necessary and sufficient condition for the existence on M of symmetric connexions compatible with Φ is that $(\nabla_i\Phi_{jk} + \nabla_j\Phi_{ki} + \nabla_i\Phi_{jk})' \equiv 0$. The self-adjoint part of such a connexion is determined by these characteristics.*

Proof. Let u^i be local coordinates. We have

$$\nabla_i\Phi_{jk} = (\overset{w}{\nabla}_{\partial_i}\partial_j)(\hat{g}\partial_k) + (\overset{w}{\nabla}_{\partial_i}\partial_k)(\hat{g}\partial_{j_s})^*.$$

Permuting indices and using symmetry yield

$$\nabla_i\Phi_{jk} = (\overset{w}{\nabla}_{\partial_i}\partial_j)(\hat{g}\partial_k) + (\overset{w}{\nabla}_{\partial_k}\partial_i)(\hat{g}\partial_j)^*, \tag{7}$$

$$\nabla_j\Phi_{ki} = (\overset{w}{\nabla}_{\partial_j}\partial_k)(\hat{g}\partial_i) + (\overset{w}{\nabla}_{\partial_i}\partial_j)(\hat{g}\partial_k)^*, \tag{8}$$

$$\nabla_k\Phi_{ij} = (\overset{w}{\nabla}_{\partial_k}\partial_i)(\hat{g}\partial_j) + (\overset{w}{\nabla}_{\partial_j}\partial_k)(\hat{g}\partial_i)^*. \tag{9}$$

Call the adjoint equations (7)*, (8)*, and (9)*. Then (7) + (8) − (9)* gives

$$(\overset{w}{\nabla}_i \partial_j)(\hat{g}\partial_k)'' = \tfrac{1}{2}(\nabla_i \Phi_{jk} + \nabla_j \Phi_{ki} - \nabla_k \Phi_{ji}) \,. \tag{10}$$

This proves the second assertion. To prove the first assertion it suffices by the previous lemma to construct symmetric compatible connexions locally and use a partition of unity. To construct locally a suitable skew-adjoint part, note first that for a symmetric compatible connexion, $\partial_i \Phi_{jj}$ is self-adjoint, and (7) − (7)* + (8) − (8)* + (9) − (9)* is identically zero. It follows that if for each triplet $\{i, j, k\}$ (at least two distinct) an arbitrary value is assigned to (say) $(\overset{w}{\nabla}_{\partial_i} \partial_j)(\hat{g}\partial_k)'$, the other permutations can be determined by (7), (8), (9). These then determine the connexion. q.e.d.

It should not be surprising that compatibility and symmetry do not determine the connexion. In the usual case of a complex manifold with hermitian metric, the connexion is not determined by these conditions—one may also specify that the anti-holomorphic part of the covariant derivative vanish on holomorphic vector fields (a skew-adjoint condition).

There are two natural ways to define a Hom $(\mathscr{W})$-valued second covariant derivative, namely, as $\overset{w}{\nabla}_X(\nabla_Y Z)$ or $\nabla_X(\overset{w}{\nabla}_Y Z)$. (The latter definition depends on the fixed connexion in $\mathscr{W}$: specifically

$$(\nabla_X(\overset{w}{\nabla}_Y Z))(\hat{g}U)w = \nabla_X((\overset{w}{\nabla}_Y Z)(\hat{g}U)w) - (\overset{w}{\nabla}_Y Z)(\nabla_X \hat{g}U)w - (\overset{w}{\nabla}_Y Z)(\hat{g}U)\nabla_X w \,,$$

where on the right the first and last ∇ refer to the connexion in $\mathscr{W}$.) In general these are unequal. When they are equal the connexion ∇ is said to be *invariant.*

3.4. Proposition. *That $\overset{w}{\nabla}$ is invariant implies $\overset{w}{\nabla} = \tilde{\Phi} \circ \nabla$. If $\overset{w}{\nabla}$ is symmetric, compatible with Φ, and satisfies $\overset{w}{\nabla} = \tilde{\Phi} \circ \nabla$, then $\overset{w}{\nabla}$ is invariant.*

Remark. This proposition implies that for a given connexion on $\mathscr{W}$ there is at most one compatible invariant symmetric connexion.

Proof. The computation is local. We adopt the summation convetion. Let $X = x^i \partial_i$, $Y = y^j \partial_j$, $Z = z^k \partial_k$. Then

$$\begin{aligned}
\overset{w}{\nabla}_X(\nabla_Y Z) &= x^\alpha(\partial_\alpha(y^i(\partial_i z^\beta + z^j \Gamma^\beta_{ij})\Phi_{\beta\delta} g^{\delta\gamma} \\
&\qquad + y^i(\partial_i z^\beta + z^j \Gamma^\beta_{ij})\overset{w}{\Gamma}{}^\gamma_{\alpha\beta}))\partial_\gamma \,, \\
\nabla_X(\overset{w}{\nabla}_Y Z) &= x^\alpha(\nabla_\alpha(y^p(\partial_p z^q \Phi_{qs} g^{s\gamma} + z^q \overset{w}{\Gamma}{}^\gamma_{pq}) \\
&\qquad + y^p(\partial_p z^q \Phi_{qs} g^{s\mu} + z^q \overset{w}{\Gamma}{}^\mu_{pq})\Gamma^\gamma_{\alpha\mu}))\partial_\gamma \,.
\end{aligned}$$

Expanding and cancelling like terms, this implies that $\overset{w}{\nabla}_X(\nabla_Y Z) = \nabla_X(\overset{w}{\nabla}_Y Z)$ if and only if for all α, γ,

$$
\begin{aligned}
&(\partial_\alpha y^i)z^j(\Gamma^\beta_{ij}\Phi_{\beta\delta}g^{\delta\gamma}) + y^i(\partial_\alpha z^j)(\Gamma^\beta_{ij}\Phi_{\beta\delta}g^{\delta\gamma}) \\
&\quad + y^i z^j(\partial_\alpha \Gamma^\beta_{ij})\Phi_{\beta\delta}g^{\delta\gamma} + y^i(\partial_i z^\beta)\overset{w}{\Gamma}{}^\gamma_{\alpha\beta} + y^i z^j \Gamma^\beta_{ij}\overset{w}{\Gamma}{}^\gamma_{\alpha\beta} \\
(11)\quad &= (\partial_\alpha y^p)z^q\overset{w}{\Gamma}{}^\gamma_{pq} + y^p(\partial_\alpha z^q)\overset{w}{\Gamma}{}^\gamma_{pq} + y^p z^q(\partial_\alpha \overset{w}{\Gamma}{}^\gamma_{pq}) \\
&\quad + y^p(\partial_p z^q)\Phi_{qs}g^{s\mu}\Gamma^\gamma_{\alpha\mu} + y^p z^q \overset{w}{\Gamma}{}^\mu_{pq}\Gamma^\gamma_{\alpha\mu} + y^p(\partial_p z^q)\partial_\alpha(\Phi_{qs}g^{s\gamma}) \ .
\end{aligned}
$$

If Y and Z are chosen locally constant, (11) becomes

$$
y^i z^j(\partial_\alpha \Gamma^\beta_{ij}\Phi_{\beta\delta}g^{\delta\gamma} + \Gamma^\beta_{ij}\overset{w}{\Gamma}{}^\gamma_{\alpha\beta}) = y^i z^j(\partial_\alpha \overset{w}{\Gamma}{}^\gamma_{ij} + \overset{w}{\Gamma}{}^\beta_{ij}\Gamma^\gamma_{\alpha\beta}) \ .
$$

Whence for all α, i, j, γ,

$$
(12)\qquad (\partial_\alpha \Gamma^\beta_{ij})\Phi_{\beta\delta}g^{\delta\gamma} + \Gamma^\beta_{ij}\overset{w}{\Gamma}{}^\gamma_{\alpha\beta} = \partial_\alpha \overset{w}{\Gamma}{}^\gamma_{ij} + \overset{w}{\Gamma}{}^\beta_{ij}\Gamma^\gamma_{\alpha\beta} \ .
$$

Cancelling these terms in (11) and then choosing only Z locally constant give

$$
(13)\qquad \Gamma^\beta_{ij}\Phi_{\beta\delta}g^{\delta\gamma} = \overset{w}{\Gamma}{}^\gamma_{ij} \qquad \text{for all } i, j, \gamma \ .
$$

But this is just the assertion that for all i, j,

$$
\tilde{\Phi}(\nabla_{\partial_i}\partial_j)dx^\gamma = (\overset{w}{\nabla}_{\partial_i}\partial_j)dx^\gamma \ .
$$

Since both $\tilde{\Phi}(\nabla_{\partial_i}\partial_j)$ and $\overset{w}{\nabla}_{\partial_i}\partial_j$ are function-linear maps $T^*M \to \mathrm{Hom}\,(\mathscr{W})$, this implies that $\tilde{\Phi}(\nabla_{\partial_i}\partial_j) = \overset{w}{\nabla}_{\partial_i}\partial_j$ for all i, j, whence clearly $\tilde{\Phi}(\nabla_X\partial_j) = \overset{w}{\nabla}_X\partial_j$ for all X, j. Now if $Y = y^j\partial_j$, then

$$
\begin{aligned}
\overset{w}{\nabla}_X Y = \overset{w}{\nabla}_X(y^j\partial_j) &= (Xy^j)\Phi(\partial_j) + y^j\overset{w}{\nabla}_X\partial_j \\
&= \tilde{\Phi}((Xy^j)\partial_j) + y^j\Phi(\nabla_X\partial_j) = \tilde{\Phi}(\nabla_X Y) \ .
\end{aligned}
$$

Conversely, suppose $\overset{w}{\nabla}$ is compatible with Φ and $\overset{w}{\nabla} = \tilde{\Phi}\circ\nabla$. Then

$$
\begin{aligned}
\nabla_X\Phi(U, V) &= (\overset{w}{\nabla}_X U)(\hat{g}V) + (\overset{w}{\nabla}_X V)(\hat{g}U)^* && \text{by compatibility} \\
(14)\qquad &= \Phi(\nabla_X U, V) + \Phi(\nabla_X V)(\hat{g}U)^* && \text{by (13)} \\
&= \Phi(\nabla_X U, V) + \Phi(U, \nabla_X V) \ .
\end{aligned}
$$

But then

$$
\begin{aligned}
(\nabla_X\overset{w}{\nabla}_Y Z)(\hat{g}U) &= \nabla_X((\overset{w}{\nabla}_Y Z)(\hat{g}U)) - (\overset{w}{\nabla}_Y Z)(\nabla_X(\hat{g}U)) \\
&= \nabla_X(\Phi(\nabla_Y Z, U)) - \Phi(\nabla_Y Z, \nabla_X U) && \text{by (13) and compatibility}
\end{aligned}
$$

$$= \Phi(\nabla_X \nabla_Y Z, U) + \Phi(\nabla_Y Z, \nabla_X U) - \Phi(\nabla_Y Z, \nabla_X U) \qquad \text{by (14)}$$
$$= \Phi(\nabla_X \nabla_Y Z, U) = ((\tilde{\Phi} \circ \nabla_X)(\nabla_Y Z))\hat{g}U$$
$$= (\overset{w}{\nabla}_X \nabla_Y Z)(\hat{g}U) \qquad \text{by (13).} \quad \text{q.e.d.}$$

It should be noted that invariant compatible connexions do not necessarily exist.

Examples 1 (*cont.*). A Hom (R)-valued connexion is just a usual real-valued connexion; it equals its induced affine connexion. Compatibility is the usual Riemannian compatibility $\frac{d}{dt} g(U, V) = g\left(\frac{DU}{dt}, V\right) + g\left(U, \frac{DV}{dt}\right)$. Symmetry is the usual symmetry of an affine connexion, and since Φ has no skew-adjoint part, the compatible symmetric connexion is unique. Also, since $\overset{w}{\nabla} = \nabla$, it is trivially invariant.

Example 2 (*cont.*). In the Hermitian case a Hom (R^2)-valued connexion is a real-linear map $TM \otimes TM \to L(T^*M, \text{Hom}(R^2))$. If $\overset{w}{\nabla}$ is compatible and symmetric, then (10) gives $(\overset{w}{\nabla}_{\partial_i} \partial_j)(\hat{g}\partial_k)'' = \frac{1}{2}(\partial_i \Phi_{jk} + \partial_j \Phi_{ki} - \partial_k \Phi_{ji})$. But for this Φ the self-adjoint parts of the Φ_{ij}, and hence of the $\partial_k \Phi_{ij}$, are always multiples of I. Hence the self-adjoint part of $\overset{w}{\nabla}$ must be a multiple of I, so we actually have $\overset{w}{\nabla}: TM \otimes TM \to TM \otimes C$. Also, Proposition 3.3 then specifies that compatibility implies the closure of the Kähler form Φ', i.e., that (M, Φ) is Kähler. Equivalently, if Y is a vector field, choose $X_1, \cdots, X_{n-2}$ locally independent and perpendicular to Y and JY; set $Y = X_{n-1}$, $JY = X_n$. Then by compatibility,

$$(\overset{w}{\nabla}_{X_i} JY)(\hat{g}X_j) = (\overset{w}{\nabla}_{X_i} X_j)(\hat{g}JY)^* = 0 \,,$$
$$J(\overset{w}{\nabla}_{X_i} Y)(\hat{g}X_j) + J(\overset{w}{\nabla}_{X_i} X_j)(\hat{g}Y)^* = 0 \,,$$

so

$$(\overset{w}{\nabla}_{X_i} JY - J\overset{w}{\nabla}_{X_i} Y)(\hat{g}X_j) = -(J(\overset{w}{\nabla}_{X_i} X_j)(\hat{g}Y) + (\overset{w}{\nabla}_{X_i} X_j)(\hat{g}JY))^* = 0 \,,$$

and J is parallel with respect to $\overset{w}{\nabla}$. In this case

$$\begin{aligned} \Phi(\nabla_X U, V) + \Phi(U, \nabla_X V) &= \langle \nabla_X U, V\rangle I - \langle \nabla_X U, JV\rangle J \\ &\quad + \langle U, \nabla_X V\rangle I - \langle U, J\nabla_X V\rangle J \\ &= \nabla_X(\langle U, V\rangle I - \langle U, JV\rangle J) = \nabla_X \Phi(U, V) \,, \end{aligned}$$

so the connexion $\overset{w}{\nabla} = \tilde{\Phi} \circ \nabla$ is invariant.

Example 6 (*cont.*). In the Clifford situation as in the previous example compatibility and symmetry imply that the self-adjoint part of $\overset{w}{\nabla}$ is a multiple of I. In this case, though, there is no guarantee that $\overset{w}{\nabla}_X Y$ is automatically in $TM \otimes \mathscr{C}$. However, it is clear that if the free choices of skew-adjoint parts are chosen in $\mathscr{C}$, then the connexion will take values in $TM \otimes \mathscr{C}$. Now suppose $w \in \mathscr{C}$. Since TM_x has structure of Clifford module, we may form the vector field wY for any vector field Y on M. Using precisely the same argument as in the previous example, we find that compatibility and symmetry of connexion imply that $\overset{w}{\nabla}_X wY = w\overset{w}{\nabla}_X Y$, while invariance implies that $\nabla_X wY = w\nabla_X Y$.

4. Torsion and curvature

Throughout, $\mathscr{W}$ is considered to have fixed connexion ∇ with covariant derivative $\dfrac{D}{dt}$.

The notion of Hom $(\mathscr{W})$-valued torsion is developed following the G-structure method of Singer-Sternberg [5]. If $H_{(p,q)}$ is a horizontal subspace of $T_{(p,q)}B_G$, then via the identification of $H_{(p,q)}$ with V under ω, we get a map

$$c_{H_{(p,q)}}: V \wedge V \to V \otimes W^* \otimes W: (u, v) \to d\overset{w}{\omega}(X \wedge Y) = \tilde{\Phi}(d\omega(X \wedge Y)) ,$$

where $X, Y \in H_{(p,q)}$, $\omega(X) = u$, $\omega(Y) = v$. If $H^1_{(p,q)}$ and $H^2_{(p,q)}$ are two horizontal subspaces, there exists a map $S: V \to \mathfrak{J}$ defined by $\widetilde{S(V)}_{(p,q)} = Y_1 - Y_2$ where $\omega(Y_1) - \omega(Y_2) = v$. Let $X_i, Y_i \in H^i_{(p,q)}$ $(i = 1, 2)$ such that $\omega(X_i) = u$, $\omega(Y_i) = v$. Then

$$\begin{aligned} c_{H^1_{(p,q)}}(u \wedge v) - c_{H^2_{(p,q)}}(u \wedge v) &= d\overset{w}{\omega}((X_1 - X_2) \wedge Y_1) + d\overset{w}{\omega}(X_2 \wedge (Y_1 - Y_2)) \\ &= d\overset{w}{\omega}(\tilde{S}(u)_{(p,q)} \wedge Y_1) + d\overset{w}{\omega}(X_2 \wedge \tilde{S}(v)_{(p,q)}) \\ &= (\tilde{S}(u)_{(p,q)} \lrcorner\, d\overset{w}{\omega}(Y_1)) - (\tilde{S}(v)_{(p,q)} \lrcorner\, d\overset{w}{\omega}(X_2)) \\ &= (\psi(S(u)\overset{w}{\omega})Y_1 - (\psi(S(v))\overset{w}{\omega})X_2 \\ &= (\psi(S(u))\Phi)v - (\psi(S(v))\Phi)u . \end{aligned} \tag{15}$$

Now if $\Psi: V \to gl(V \otimes W^* \otimes W)$ is any linear map, we may define the 2-form $\partial\Psi: V \wedge V \to V \otimes W^* \otimes W: u \wedge v \to \Psi(u)(\tilde{\Phi}(v)) - \Psi(v)(\tilde{\Phi}(u))$. The span of 2-forms $\partial\Psi$, where $\Psi = \Psi \circ S$ for $S: V \to \mathfrak{J}$, is denoted $\partial(\Psi \circ \mathrm{Hom}\,(V, \mathfrak{J}))$. Then (15) implies that

$$c_{H^1_{(p,q)}} - c_{H^2_{(p,q)}} \in \partial(\psi \circ \mathrm{Hom}\,(V, \mathfrak{J})) .$$

Hence there is a well-defined function $c: B_G \to \dfrac{\tilde{\Phi} \circ \mathrm{Hom}\,(V \wedge V, V)}{(\partial \circ \mathrm{Hom}\,(V, \mathfrak{J}))}$, (note incidentally that $\tilde{\Phi}$ and ψ are injective), called the (*first-order*) *structure function* of $B_{\underset{\sim}{G}}$.

Suppose that $\overset{w}{\nabla}$ is a connexion on M and $H_{(p,q)}$ is its horizontal space. Then define a Hom$(T^*M, \mathrm{Hom}(\mathscr{W}))$-valued 2-form T on M as follows. For $X, Y \in T_xM$, let $\bar{X}, \bar{Y} \in H_{(p,q)}$ such that $\pi_*\bar{X} = X$, $\pi_*\bar{Y} = Y$, and define

$$T(X, Y) = (p, q)\overset{w}{\omega}(\bar{X}, \bar{Y}) = (p, q)c_{H_{(p,q)}}(\omega\bar{X}, \omega\bar{Y}) .$$

T is the *torsion* of $(M, \Phi, \overset{w}{\nabla})$. It is clear that if $T = 0$, then $c = 0$. On the other hand, Fujimoto has shown [2, Corollary 3.4.1] that if c vanishes a connexion 1-form with vanishing torsion does exist.

We can also define a $V \otimes W^* \otimes W$-valued *torsion 2-form* $\mathscr{T}$ on $B_{\underset{\sim}{G}}$ by

$$\mathscr{T}(X, Y) = d\overset{w}{\omega}(HX, HY) = d[(p, q)^{-1}(\tilde{\Phi} \circ \pi_*)(p, q)](HX, HY) ,$$

where H is the projection onto the horizontal space of the connexion $\overset{w}{\nabla}$.

4.1. Proposition. $T(X, Y) = \nabla_X\tilde{\Phi}(Y) - \nabla_Y\tilde{\Phi}(X) - \tilde{\Phi}([X, Y])$.

Proof.

$$\begin{aligned} T(X, Y) &= (p, q)(\bar{X}\overset{w}{\omega}(\bar{Y}) - \bar{Y}\overset{w}{\omega}(\bar{X}) - \overset{w}{\omega}([\bar{X}, \bar{Y}])) \\ &= (p, q)(\bar{X}q^{-1}(\Phi_x(\pi_*\bar{Y}, \tilde{g}_x p^{*-1}\cdot)q\cdot) \\ &\quad - \bar{Y}q^{-1}(\Phi_x(\pi_*\bar{X}, \tilde{g}_x p^{*-1}\cdot)q\cdot) \\ &\quad - q^{-1}(\Phi(\pi_*[\bar{X}, \bar{Y}], \tilde{g}_x p^{*-1}\cdot)q\cdot)) . \end{aligned} \tag{16}$$

Now let C be the integral curve of X through x, and $\bar{C}$ the horizontal lift over C through (p, q). Then

$$\begin{aligned} &(p, q)(\bar{X}q^{-1}(\Phi_x(\pi_*\bar{Y}, \tilde{g}_x p^{*-1}\cdot)) \\ &= \bar{C}(0) \lim_{t\to 0} \frac{1}{t}(q_{\bar{C}(t)}\Phi_{C(t)}(\pi_*\bar{Y}_t, \tilde{g}_{C(t)} p^{*-1}_{\bar{C}(t)}\cdot q_{\bar{C}(t)}) \\ &\quad - q^{-1}\Phi_x(\pi_*\bar{Y}, \tilde{g}_x p^{*-1}\cdot)q\cdot) \\ &= \bar{C}(0) \lim_{t\to 0} \frac{1}{t}(\bar{C}(t)^{-1}\tilde{\Phi}_{C(t)}(Y_t) - \bar{C}(0)^{-1}\tilde{\Phi}_x(Y)) \\ &= \lim_{t\to 0} \frac{1}{t}(\bar{C}(0)\bar{C}(t)^{-1}\tilde{\Phi}_{C(t)}(Y_t) - \tilde{\Phi}_x(Y)) = \nabla_X\tilde{\Phi}_x(Y) . \end{aligned}$$

Using a similar argument for Y, from (16) we have

$$T(X, Y) = \nabla_X\tilde{\Phi}_x(Y) - \nabla_Y\tilde{\Phi}_x(X) - \tilde{\Phi}_x([X, Y]) .$$

4.2. Proposition. *An invariant symmetric compatible connexion has zero torsion.*

Proof. For all $X, Y, U \in TM_x$,

$$(\nabla_X\tilde{\Phi}(Y))(\hat{g}U) = \nabla_X(\Phi(Y, U)) - \tilde{\Phi}(Y)(\hat{g}\nabla_X U)$$
$$= (\overset{w}{\nabla}_X Y)(\hat{g}U) + (\overset{w}{\nabla}_X U)(\hat{g}Y)^* - \Phi(Y, \nabla_X U) \quad \text{by compatibility.}$$

Therefore

$$\begin{aligned} T(X, Y)(\hat{g}U) &= (\nabla_X\tilde{\Phi}(Y) - \nabla_Y\tilde{\Phi}(X) - \Phi([X, Y]))(\hat{g}U) \\ &= (\overset{w}{\nabla}_X Y)(\hat{g}U) + (\overset{w}{\nabla}_X U)(\hat{g}Y)^* - \Phi(Y, \nabla_X U) - (\overset{w}{\nabla}_Y X)(\hat{g}U) \\ &\quad - (\overset{w}{\nabla}_Y U)(\hat{g}X)^* + \Phi(X, \nabla_Y U) - \Phi([X, Y], U) \\ &= (\overset{w}{\nabla}_X U)(\hat{g}Y)^* - \Phi(Y, \nabla_X U) - (\overset{w}{\nabla}_Y U)(\hat{g}X)^* + \Phi(X, \nabla_Y U) \\ &= 0 \qquad \text{by symmetry and invariance.} \quad \text{q.e.d.} \end{aligned}$$

Note that a torsion zero metric may not have invariant connexion. (For instance, in Example 2 a complex manifold need not be Kählerian.) Also note that with any compatible symmetric connexion, $\operatorname{tr} T = 0$.

Example 1 (*cont.*). Here the torsion equals its trace and is automatically zero.

Examples 2 and 3 (*cont.*). The torsion zero condition is that the almost-complex structure be complex, or the almost-Hamiltonian structure be Hamiltonian.

Example 5. In case (ii) recall that choosing suitable coordinates we obtain $\Phi: A = I = C$,

$$B = \begin{bmatrix} & & & & & -a_1 \\ & & & & \cdot^{\cdot^{\cdot}} & \\ & & & -a_k & & \\ & & a_k & & & \\ & \cdot^{\cdot^{\cdot}} & & & & \\ a_1 & & & & & \end{bmatrix}.$$

Suppose $a_1 = \cdots = a_{e_1}, a_{e_1+1} = \cdots = a_{e_1+e_2}, \cdots = a_k$. If M is simply connected, then torsion zero implies that M is a product of complex manifolds of dimensions $e_1, e_2, \cdots$. In case (iii) for trivial $\mathscr{W}$ and M simply connected, torsion zero imples that M is a product of manifolds of dimensions according to the numbers of equal pairs (λ, a_i). These manifolds will have certain additional structure induced by Φ.

Define the *Hom* $(\mathscr{W})$*-valued curvature* form

$$\overset{w}{R}(X, Y)Z = \nabla_X(\overset{w}{\nabla}_Y Z) - \nabla_Y(\overset{w}{\nabla}_X Z) - \overset{w}{\nabla}_{[X,Y]}Z \,,$$

where ∇ is the affine connexion in $\mathscr{W}$.

4.3. Proposition. *If $s\colon U \subset R^2 \to M$ is a smooth surface, and V a vector field on M along s, then*

$$\overset{w}{R}\left(\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}\right)V + \frac{D}{dx}\frac{\overset{w}{D}V}{dy} - \frac{D}{dy}\frac{\overset{w}{D}V}{dx} .$$

Proof. This is just a local computation. By linearity it will suffice to show that for some local coordinate system,

$$\overset{w}{R}(\partial_i, \partial_j)\partial_k = \frac{D}{dx^i}\frac{\overset{w}{D}}{dx^j}\partial_k - \frac{D}{dx^j}\frac{\overset{w}{D}}{dx^i}\partial_k ,$$

But

$$\begin{aligned}\frac{D}{dx^i}\frac{\overset{w}{D}}{dx^j}\partial_k - \frac{D}{dx^j}\frac{\overset{w}{D}}{dx^i}\partial_k &= \frac{D}{dx^i}(\overset{w}{\nabla}_{\partial_j}\partial_k) - \frac{D}{dx^j}(\overset{w}{\nabla}_{\partial_i}\partial_k) \\ &= \nabla_{\partial_i}\overset{w}{\nabla}_{\partial_j}\partial_k - \nabla_{\partial_j}\overset{w}{\nabla}_{\partial_i}\partial_k = \overset{w}{R}(\partial_i, \partial_j)\partial_k .\end{aligned}$$

4.4. Proposition. $\overset{w}{R}(X, Y) = -\overset{w}{R}(Y, X)$. *If $\overset{w}{\nabla}$ is compatible, then*

$$\overset{w}{R}(X, Y)Z + \overset{w}{R}(Y, Z)X + \overset{w}{R}(Z, X)Y = 0 .$$

Proof. The first relation follows at once from the definitions. By multilinearity it suffices to prove the second only in the case where $[X, Y] = [Y, Z] = [X, Z] = 0$. In this case,

$$\begin{aligned}&\overset{w}{R}(X, Y)Z + \overset{w}{R}(Y, Z)X + \overset{w}{R}(Z, X)Y \\ &\quad = \nabla_X(\overset{w}{\nabla}_Y Z - \overset{w}{\nabla}_Z Y) + \nabla_Y(\overset{w}{\nabla}_Z X - \overset{w}{\nabla}_X Z) + \nabla_Z(\overset{w}{\nabla}_X Y - \overset{w}{\nabla}_Y X) \\ &\quad = \nabla_X\tilde{\Phi}([Y, Z]) + \nabla_Y\tilde{\Phi}([Z, X]) + \nabla_Z\tilde{\Phi}([X, Y]) \qquad \text{by compatibility} \\ &\quad = 0 .\end{aligned}$$

4.5. Propoition. *Suppose that $\overset{w}{\nabla}$ is invariant, and $x \in M$. Then $\overset{w}{R}(x) = 0$ if and only if $R(x) = \operatorname{tr}\overset{w}{R}(x) = 0$.*

Proof. $\Rightarrow$ is obvious. Conversely suppose $R(x) = 0$. Then $\overset{w}{R}(X, Y)Z = \overset{w}{\nabla}_X\nabla_Y Z - \overset{w}{\nabla}_Y\nabla_X Z - \overset{w}{\nabla}_{[X,Y]}Z = \tilde{\Phi}(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z) - \tilde{\Phi}(R(X, Y)Z) = 0$. q.e.d.

We wish to investigate the implications of constant curvature in the Hom $(\mathscr{W})$-valued setting. We shall say that M has *constant Hom $(\mathscr{W})$-valued curvature* at x if $(\overset{w}{R}_x(v_1, v_2)v_1)(\hat{g}v_2)$ depends only on $|v_1|, |v_2|, g(v_1, v_2)$ and the orientation $(v_1 : v_2)$. (We continue to suppose $\overset{w}{\nabla}$ to be invariant through this discussion.) In that case $\operatorname{tr}(R_x(v_1, v_2)v_1)(\hat{g}v_2) = g(R(v_1, v_2)v_1, v_2)$ will depend only on these quantities also, so that the sectional curvature $K(p)$, which is independent of them, will be constant. Hence we suppose in this discussion that M has constant sectional curvature at x. Let v_1, v_2 be an orthonormal basis for a plane $p \subset TM_x$, and $w_1 = av_1 + bv_2$, $w_2 = -bv_1 + av_2$, $a^2 + b^2 = 1$, another basia with the same orientation. Then

$$\begin{aligned}
(\overset{w}{R}(w_1, w_2)w_1)(\hat{g}w_2) &= \tilde{\Phi}(R(w_1, w_2)w_1)(\hat{g}w_2) = \Phi(R(w_1, w_2)w_1, w_2) \\
&= \Phi(R(av_1 + bv_2, -bv_1 + av_2)(av_1 + bv_2), \\
&\qquad - bv_1 + av_2) \\
&= \Phi(R(v_1, v_2)(av_1 + bv_2), -bv_1 + av_2) \\
&= -ab\Phi(R(v_1, v_2)v_1, v_2) + ab\Phi(R(v_1, v_2)v_2, v_2) \\
&\qquad + a^2\Phi(R(v_1, v_2)v_1, v_2) - b^2\Phi(R(v_1, v_2)v_2, v_1) \\
&= \Phi(R(v_1, v_2)v_1, v_2) + \text{skew-adjoint terms.}
\end{aligned}$$

Using the basis $w_1, -w_2$ with opposite orientation gives the same result. Hence the symmetric part of $(R(v_1, v_2)v_1)(\hat{g}v_2)$ is invariant under change of orthonormal basis.

If v_1, v_2 are an arbitrary basis, by considering the orthonormal basis

$$\frac{v_1}{|v_1|}, \qquad \frac{|v_1|^2 v_2 - g(v_1, v_2)v_1}{|v_1|(|v_1|^2|v_2|^2 - g(v_1, v_2)^2)^{1/2}}$$

we find that symmetric part of

$$\frac{(\overset{w}{R}(v_1, v_2)v_1)(\hat{g}v_2)}{|v_1|^2|v_2|^2 - g(v_1, v_2)^2}$$

is invariant for all $v_1, v_2 \in p$. We define this to be $K(p)''$.

Now as M has constant sectional curvature at x, if $n \geq 3$ we have

$$R_x(v_1, v_2)v_3 = c(\hat{g}(v_1, v_3)v_2 - \hat{g}(v_2, v_3)v_1) ,$$

and therefore for any $v_1, v_2 \in TM_x$

$$\begin{aligned}
(\overset{w}{R}_x(v_1, v_2)v_1)(\hat{g}v_2) &= c\Phi(g(v_1, v_1)v_2 - g(v_2, v_1)v_1, v_2) \\
&= c(|v_1|^2\Phi(v_2, v_2) - g(v_2, v_1)\Phi(v_1, v_2)) .
\end{aligned}$$

Hence

$$\frac{c(|v_1|^2\Phi(v_2, v_2) - \frac{1}{2}g(v_1, v_2)(\Phi(v_1, v_2) + \Phi(v_2, v_1)))}{|v_1|^2|v_2|^2 - g(v_1, v_2)^2}$$
$$= \frac{c(|v_2|^2\Phi(v_1, v_1) - \frac{1}{2}g(v_1, v_2)(\Phi(v_1, v_2) + \Phi(v_2, v_1)))}{|v_1|^2|v_2|^2 - g(v_1, v_2)^2},$$

or

$$\frac{\Phi(v_2, v_2)}{|v_2|^2} = \frac{\Phi(v_1, v_1)}{|v_1|^2}$$

for all $v_1, v_2 \in TM_x$, i.e., there exists $\phi_x'' \in \mathrm{Hom}\,(\mathscr{W}_x)$ such that $\Phi(v, v) = |v|^2\phi_x''$ for all $v \in TM_x$, and whence $\Phi(u, v)'' = g(u, v)\phi_x''$ for all $u, v \in TM_x$.

The skew-adjoint part of $(\overset{w}{R}_x(v_1, v_2)v_1)(\hat{g}v_2)$ is

$$\frac{c}{2}g(v_1, v_2)(\Phi(v_2, v_1) - \Phi(v_1, v_2)) = -\frac{c}{2}g(v_1, v_2)\Phi(v_1, v_2)'.$$

For any oriented basis v_1, v_2 for $p, \Phi(v_1, v_2)'/(|v_1|^2|v_2|^2 - g(v_1, v_2)^2)^{1/2}$ is invariant. This element of Hom $(\mathscr{W}_x)$ is denoted $K(p)'$. Hence, if M has constant curvature $(n \geq 3)$ at x, we may write for all $v_1, v_2 \in TM_x$,

$$\begin{aligned}(\overset{w}{R}_x(v_1, v_2)v_1)(\hat{g}v_2) = c((&|v_1|^2|v_2|^2 - g(v_1, v_2)^2)\phi_x'' \\ &- g(v_1, v_2)(|v_1|^2|v_2|^2 - g(v_1, v_2)^2)^{1/2}K(p)'),\end{aligned}$$

where v_1, v_2 is a basis for the oriented plane $p \subset TM_x$. But for $n \geq 3$ the bundle of oriented planes in TM forms a connected double cover of $V_2(TM)$. Thus, if $n \geq 3$, then $K(p)' \equiv 0$. Hence

4.6. Proposition. *Unless $\Phi(u, v)$ is self-adjoint for all $u, v \in V$, $(n \geq 3)$, there are no manifolds with* Hom $(\mathscr{W})$*-valued metric of type Φ and constant* Hom $(\mathscr{W})$*-valued curvature. The self-adjoint part of the curvature is constant if an only if the metric Φ has the form $\Phi(u, v) = g(u, v)\phi_x''$ for some fixed $\phi_x'' \in$* Hom $(\mathscr{W}_x)$.

We now return to the general (not necessarily invariant) case. One may derive the curvature form from the connexion 1-form $\overset{w}{\phi}$, giving a structural equation. Indeed, define a $\mathfrak{J}$-valued *curvature form* $\mathscr{R}$ on $B_{\underline{G}}$ by

$$\mathscr{R}(X, Y)(v, w) = (p, q)^{-1}[(\overset{w}{R}(\pi_* X, \pi_* Y)pv)qw],$$

where $X, Y \in T_{(p,q)}\, B_{\underline{G}}$. Note that $\mathscr{R}(X, Y) = d\overset{w}{\phi}(HX, HY)$ where H is projection onto the horizontal subspace of $\overset{w}{\phi}$.

4.7. Proposition (*structural equation*).

$$d\overset{w}{\phi}(X, Y) = -\overset{w}{\phi} \wedge \overset{w}{\phi}(X, Y) + \mathscr{R}(X, Y) .$$

Proof. If X and Y are vertical, and $X = \tilde{A}$, $Y = \tilde{B}$, then

$$\begin{aligned} d\overset{w}{\phi}(X, Y) &= X\overset{w}{\phi}(Y) - Y\overset{w}{\phi}(X) - \overset{w}{\phi}([X, Y]) = XB - YA - [A, B] \\ &= -(\overset{w}{\phi} \wedge \overset{w}{\phi})(X, Y) = -(\overset{w}{\phi} \wedge \overset{w}{\phi})(X, Y) + \mathscr{R}(X, Y) , \end{aligned}$$

since $\mathscr{R}$ is horizontal. If $X = \tilde{A}$ is vertical, and Y horizontal, then

$$d\overset{w}{\phi}(X, Y) = 0 = -(\overset{w}{\phi} \wedge \overset{w}{\phi})(X, Y) + \mathscr{R}(X, Y) .$$

If X and Y are both horizontal, then due to $\overset{w}{\phi}(X) = \overset{w}{\phi}(Y) = 0$ we have

$$d\overset{w}{\phi}(X, Y) = d\overset{w}{\phi}(HX, HY) = \mathscr{R}(X, Y) = -(\overset{w}{\phi} \wedge \overset{w}{\phi})(X, Y) + \mathscr{R}(X, Y) .$$

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