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RESTRICTIONS ON THE CURVATURES OF Φ-BOUNDED SURFACES

TILLA KLOTZ MILNOR

1. Introduction

The study of surfaces in the large has often centered on compact surfaces. (See [17], for example.) This paper is meant as a small contribution to the growing body of knowledge about complete open surfaces. Samples of work in this area are [4], [5], [6], [7], [8], [14], [15], [16], [22], [27], [28], [29] and [30].

Given an oriented surface S, a Riemann surface R defined on S (see § 2), and a differential Φ on R, we define in § 3 the notion of Φ-boundedness with respect to R. Since our basic concern is with surfaces smoothly immersed in a Riemannian 3-manifold, we use the concept of Φ-boundedness in cases where Φ or R is connected with the geometry of an immersion. In fact, Φ-boundedness arises naturally in a substantial number of interesting differential geometric situations (see Examples 1 and 2 in § 3), and this paper attempts to present fairly general methods which imitate ad hoc arguments developed in [21], [10], [12] and [35] for handling surfaces of constant mean curvature. Our main results (in § 5) place restrictions on the curvatures of complete Φ-bounded surfaces smoothly immersed in an arbitrary Riemannian 3-manifold. Analogous results can be formulated for an ambient manifold of arbitrary dimension (≥ 3) so long as one works relative to a fixed choice of a unit normal vector field on the immersed surface.

We generate differentials of geometric significance on a Riemann surface R defined on an immersed surface S by associating to any real quadratic form X on S a complex quadratic differential Ω on R, and conformal metrics Γ, $|\Omega|$ and Π on R. (See § 2.) We are especially interested in the case in which X is one of the fundamental forms X_n or skew fundamental forms X'_n defined on S for all integral values of n. While some of the material in this paper was established for surfaces in Euclidean 3-space E^3 in [23], we deal here with a far wider variety of metrics. In addition to the forms X_n and X'_n on S, we introduce the metrics Λ_n (or Λ'_n) which are complete on a complete immersed surface so long as X_n (or X'_n) is positive definite and sufficiently smooth.

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Hopefully, the techniques developed for handling these metrics will prove useful even where our results are not directly applicable.

The methods used in this paper are formal in nature. No proof of a theorem or lemma uses the Codazzi-Mainardi equations [11, p. 78], and only three corollaries (in § 4) use the Gauss equation (22). But in applications it is often these very compatibility conditions which must be used to check whether a given surface satisfies the hypotheses of some result below. (See Example 1 of § 3.)

2. Preliminary definitions and results

Let S be an oriented C^∞ 2-manifold. Our arguments apply to nonorientable surfaces by lifting all considerations to the universal covering surface. We say that a *Riemann surface R is defined on S* if R is a complex 1-manifold on the underlying topological space of S so that each conformal parameter $z = x + iy$ on R yields an oriented C^j coordinate pair x, y on S with $j \geq 1$. The coordinates x, y are said to be R-isothermal on S, and R is said to be C^j-related to S. Generally any C^{k+a} Riemannian metric on S defines a Riemann surface which is C^{k+a+1}-related to S, where $k = 0, 1, \cdots$ and $0 < a < 1$. (See [2, p. 29] or [17, p. 137].) But simpler results often apply. For example, the first fundamental form of a surface S which is C^k-immersed in a locally C^k-conformally Euclidean 3-manifold defines a Riemann surface $R = R_1$ which is C^k-related to S, $k = 2, 3, \cdots$ ([34] and [3]).

Given R-isothermal coordinates x, y on S, any real quadratic form $X = Adx^2 + 2Bdxdy + Cdy^2$ has the expression

$$X = Qdz^2 + Pdzd\bar{z} + \bar{Q}d\bar{z}^2 = 2Re(\Omega) + \Gamma \ , \tag{1}$$

where $z = x + iy$, $4Q = A - C - 2iB$, $2P = A + C$, $\Gamma = \Gamma(X, R) = Pdzd\bar{z} = P(dx^2 + dy^2)$, and $\Omega = \Omega(X, R) = Qdz^2$. Thus $\det X = |P|^2 - 4|Q|^2$, so that the signs of $\det X$ and $|P| - 2|Q|$ must match. On R, Ω is a quadratic differential, while Γ is a conformal metric. One can also associate to X on R the conformal metrics $|\Omega|$ and Π given by

$$\begin{aligned} |\Omega| &= |\Omega(X, R)| = |Q|\,dzd\bar{z} \ , \\ \Pi = \Pi(X, R) &= \sqrt{|\det X|}\,dzd\bar{z} = \sqrt{AC - B^2}(dx^2 + dy^2) \ . \end{aligned} \tag{2}$$

Here $|\Omega|$ is singular only where $Q = 0$, and Π only where $\det X = 0$. As R varies on S, one can think of X as generating families of metrics $\Gamma(X, R)$, $|\Omega(X, R)|$, $\Pi(X, R)$ and $2\text{Re}\,\Omega(X, R)$ on S. The metrics so obtained are of geometric interest on S to the extent that X and R are related to the geometry on S. Denote by R_X the Riemann surface determined on S by any Riemannian metric X which is at least C^a-smooth, $a > 0$. Then

$$\Omega(X, R_X) \equiv 0 , \qquad \Gamma(X, R_X) \equiv X . \tag{3}$$

Lemma 1. *If X is a complete Riemannian metric on a surface S, then $\Gamma = \Gamma(X, R)$ is a complete Riemannian metric on S for any choice of a Riemann surface R on S.*

Proof. Suppose x, y are R-isothermal coordinates and $X = Adx^2 + 2Bdxdy + Cdy^2$. Then $2\Gamma - X = Cdx^2 - 2Bdxdy + Ady^2$ is clearly positive definite. Thus $X \leq 2\Gamma$, and Γ must be complete if X is. (We thank R. Osserman for an improvement of our original argument.)

If X is a pseudo-Riemannian metric on S (i.e., if $\det X \neq 0$), and X is C^2-smooth, then its curvature $K(X)$ is given by a standard formula [31, p. 112]. Even if X is only C^1-smooth, we say that its curvature $K(X)$ *is defined* so long as some continuous function $K(X)$ on S satisfies an integrated version of this formula. (See [9].)

For a conformal metric $\Sigma = \sigma dz d\bar{z}$ on a Riemann surface R, the curvature formula simplifies considerably. In case Σ is C^2-smooth,

$$K(\Sigma) = (-1/2\sigma)\Delta \log \sigma , \tag{4}$$

where $\Delta = \partial^2/dx^2 + \partial^2/dy^2$, and $z = x + iy$ is a conformal parameter on R. The integrated version of (4) is

$$\int_\gamma [(\log \sigma)_y dx - (\log \sigma)_x dy] = 2 \iint_U K(\Sigma)\sigma dxdy , \tag{5}$$

where γ is any C^1 Jordan curve in the domain of z, and U the interior region determined by γ. Thus $K(\Sigma)$ is defined on R if a continuous function $K(\Sigma)$ exists on R which satisfies (5).

Remark 1. By (5), $K(\Sigma) \equiv 0$ if $\log \sigma$ is harmonic on R [1, p. 133]. Thus, if R is defined on S, $\Gamma = \Gamma(X, R)$, and $\log P$ on R is harmonic, then $K(\Gamma) \equiv 0$.

Remark 2. If S is immersed in a Riemannian 3-manifold $\mathcal{M}$, the first fundamental form I on S defines a Riemann surface R_1 on S. If the immersion is C^{k+a}-smooth with $k = 3, 4, \cdots$ and $0 < a < 1$, then I is C^{k+a-1}-smooth, R_1 is C^{k+a}-related to S, and (4) can be used to compute $K(I)$ on R_1. If $\mathcal{M}$ is locally C^3-conformally Euclidean, (4) yields $K(I)$ even if the immersion is just C^3-smooth. When $\mathcal{M} = E^3$, it is known (see [9], [32] and [33]) that $K(I)$ is defined and equal to the extrinsic curvature even if S is only C^2-immersed.

The following elementary result is a modification of Lemma 1 from [21] or [23], and the proof in [23] is easily adapted to establish the present version. For a discussion of subharmonic and superharmonic functions, see [1, p. 135].

Lemma 2. *Let $\Sigma = \sigma dz d\bar{z}$ be a conformal metric on a Riemann surface R. If $K(\Sigma) \leq 0$ (respectively ≥ 0) is defined, then $\log \sigma$ is subharmonic (respectively superharmonic).*

We denote by K the extrinsic curvature on a surface S which is C^2-immersed in a Riemannian 3-manifold. Given a unit normal vector field on S, mean curvature H and the principal curvatures k_1 and k_2 are well defined, with $k_1k_2 = K$, $k_1 + k_2 = 2H$, and $|k_2 - k_1| = 2H'$, where, by definition, $H' = \sqrt{H^2 - K}$. We are interested in the sequence $\{X_n\}$ of fundamental forms I, II, III, $\cdots$ on S given inductively by $X_1 = \mathrm{I}$, $X_2 = \mathrm{II}$, and

$$X_n = 2HX_{n-1} - KX_{n-2} \tag{6}$$

for $n \geq 3$. (See [11, p. 24].) We also work with a companion sequence $\{X'_n\}$ of skew fundamental forms I′, II′, III′, $\cdots$ defined by

$$H'X'_n = X_{n+1} - HX_n \tag{7}$$

for $n \geq 1$ wherever $H' \neq 0$. It is possible (see [25]) to define the skew forms X'_n at some points where $H' = 0$, but we shall not do so here. Wherever $K \neq 0$ on S, use (6) and (7) to define the forms X_n and X'_n for nonpositive integral powers of n. Just as we *automatically* assume that $H' \neq 0$ when working with the X'_n, we *automatically* assume that $K \neq 0$ when dealing with an X_n or X'_n for a nonpositive value of the index n. The relationship between the ordinary and the skew fundamental forms is probably best understood by noting that in terms of lines-of-curvature coordinates x, y on S, $X_n = k_1^{n-1}Edx^2 + k_2^{n-1}Gdy^2$ and $\pm X'_n = -k_1^{n-1}Edx^2 + k_2^{n-1}Gdy^2$, where $\mathrm{I} = Edx^2 + Gdy^2$, and $\pm$ stands for the sign of $k_2 - k_1$. Without the use of special coordinates one can compute that

$$\begin{aligned}(k_2 - k_1)X_n &= (k_2^{n-1} - k_1^{n-1})\mathrm{II} - K(k_2^{n-2} - k_1^{n-2})\mathrm{I}\,, \\ |k_2 - k_1|X'_n &= (k_2^{n-1} + k_1^{n-1})\mathrm{II} - K(k_2^{n-2} + k_1^{n-2})\mathrm{I}\,.\end{aligned} \tag{8}$$

The forms X_n are positive definite for odd n wherever $K \neq 0$, and for even n wherever $K > 0$ (with the unit normal vector field chosen so that $H > 0$). The forms X'_n are positive definite only when n is even and $K < 0$. If S is immersed so that I and II are at least C^a-smooth with $a > 0$, then whenever they are positive definite, the forms X_n and X'_n define Riemann surfaces R_n or R'_n on S. When working on a Riemann surface R_n or R'_n on S, we will *automatically* assume that all conditions necessary for the existence of the Riemann surface are satisfied, *with S so smoothly immersed that the R_n or R'_n in question is actually C^2-related to S.* This allows us to work with all the fundamental and skew fundamental forms on the Riemann surface involved.

For convenience, we write $\Omega_n = \Omega(X_n, R) = Q_n dz^2$, $\Gamma_n = \Gamma(X_n, R) = P_n dz d\bar{z}$, $\Omega'_n = \Omega(X'_n, R) = Q'_n dz^2$, and $\Gamma'_n = \Gamma(X'_n, R) = P'_n dz d\bar{z}$, so that on any given Riemann surface R on S, $X_n = 2\mathrm{Re}\,(\Omega_n) + \Gamma_n$ and $X'_n = 2\mathrm{Re}\,(\Omega'_n) + \Gamma'_n$. We also write $|\Omega_n| = |\Omega(X_n, R)|$, $|\Omega'_n| = |\Omega(X'_n, R)|$, $\Pi_n = \Pi(X_n, R)$, and $\Pi'_n = \Pi(X'_n, R)$. Formulas (6), (7) and (8) remain valid if X is consistently replaced by any one of the symbols Γ, Ω, P or Q.

Remark 3. On an immersed surface S, (8) yields

$$(9)\qquad \det X_n = K^{n-1} \det X_1 = K^{n-j} \det X_j \,, \qquad \det X'_n = -\det X_n$$

for any integers n and j. In terms of R-isothermal coordinates for any R on S, Remark 1 shows that $|P_n| - 2|Q_n| \geq 0$ if n is odd, or if $K \geq 0$ and n is even. If $K \leq 0$ and n is even, $|P_n| - 2|Q_n| \leq 0$. Similarly, if n is odd, or if $K \geq 0$ and n is even, $|P'_n| - 2|Q'_n| \leq 0$. If $K \leq 0$ and n is even, $|P'_n| - 2|Q'_n| \geq 0$. These observations aid in proving Lemmas 11 through 14, and in comparing the hypotheses of various results in this paper.

The next lemma establishes some basic arithmetic relationships. We use $\pm$ to denote the sign of $k_2 - k_1$.

Lemma 3. *Let S be a surface at least C^2-immersed in a Riemannian 3-manifold. If R_n is C^2-related to S, then on R_n*

$$(10)\qquad 2\Gamma_j = (k_2^{j-n} + k_1^{j-n})\Gamma_n \,, \qquad \pm 2\Gamma'_j = (k_2^{j-n} - k_1^{j-n})\Gamma_n \,,$$

$$(11)\qquad \pm 2\Omega_j = (k_2^{j-n} - k_1^{j-n})\Omega'_n \,, \qquad 2\Omega'_j = (k_2^{j-n} + k_1^{j-n})\Omega'_n$$

for all integers j. In particular, $\Gamma'_n \equiv \Omega_n \equiv 0$ on R_n, while

$$(12)\qquad H'\Gamma_n = 2|\Omega_{n+1}| \,, \qquad \Gamma_n = 2|\Omega'_n| \,.$$

If R'_n is C^2-related to S, then on R'_n

$$(13)\qquad \pm 2\Gamma_j = (k_2^{j-n} - k_1^{j-n})\Gamma'_n \,, \qquad 2\Gamma'_j = (k_2^{j-n} + k_1^{j-n})\Gamma'_n \,,$$

$$(14)\qquad 2\Omega_j = (k_2^{j-n} + k_1^{j-n})\Omega_n \,, \qquad \pm 2\Omega'_j = (k_2^{j-n} - k_1^{j-n})\Omega_n$$

for all integers j. In particular, $\Gamma_n \equiv \Omega'_n \equiv 0$ on R'_n, while

$$(15)\qquad H\Gamma'_n = 2|\Omega'_{n+1}| \,, \qquad \Gamma'_n = 2|\Omega_n| \,.$$

Proof. The derivation of (10), (11) and (12) will be outlined. Similar computation yields (13), (14) and (15). Use R_n-isothermal coordinates x, y on S, and let $\mathrm{I} = Edx^2 + 2Fdxdy + Gdy^2$, $\mathrm{II} = Ldx^2 + 2Mdxdy + Ndy^2$. Since $X_n = P_n(dx^2 + dy^2)$ on R_n, (8) yields

$$(k_2^{n-1} - k_1^{n-1})L - K(k_2^{n-2} - k_1^{n-2})E = (k_2 - k_1)P_n \,,$$
$$(k_2^{n-1} - k_1^{n-1})N - K(k_2^{n-2} - k_1^{n-2})G = (k_2 - k_1)P_n \,,$$
$$(k_2^{n-1} - k_1^{n-1})M - K(k_2^{n-2} - k_1^{n-2})F = 0 \,.$$

Multiply these equations by G, E and $-2F$ (or by N, L and $-2M$) respectively, add and simplify to get

$$2P_1P_n = (EG - F^2)(k_2^{n-1} + k_1^{n-1}) \,, \qquad 2P_2P_n = (LN - M^2)(k_2^{n-1} + k_1^{n-1}) \,,$$

using $2H(EG - F^2) = EN + GL - 2FM$ and $K(EG - F^2) = LN - M^2$. Division yields an expression for P_2 which, when substituted into the equation for P_2' provided by (8), gives $\Gamma_n' \equiv 0$. This and (7) yield $H\Gamma_n = \Gamma_{n+1}$. Thus, by (8),

$$(k_2 - k_1)\Gamma_n = (k_2^{n-1} - k_1^{n-1})\Gamma_2 - K(k_2^{n-2} - k_1^{n-2})\Gamma_1 ,$$
$$(k_2 - k_1)H\Gamma_n = (k_2^{n} - k_1^{n})\Gamma_2 - K(k_2^{n-1} - k_1^{n-1})\Gamma_1 .$$

Cramer's rule now gives Γ_1 and Γ_2 in terms of Γ_n. Place these values into the equations for Γ_j and Γ_j' provided by (8) to get (10). Similarly, because $\Omega_n \equiv 0$ on R_n, (7) yields $H'\Omega_n' = \Omega_{n+1}$. By (8),

$$|k_2 - k_1|\,\Omega_n' = (k_2^{n-1} + k_1^{n-1})\Omega_2 - K(k_2^{n-2} + k_1^{n-2})\Omega_1 ,$$
$$(k_2 - k_1)H'\Omega_n' = (k_2^{n} - k_1^{n})\Omega_2 - K(k_2^{n-1} - k_1^{n-1})\Omega_1 .$$

Here Cramer's rule gives Ω_1 and Ω_2 in terms of Ω_n'. Place these values into the expressions for Ω_j and Ω_j' provided by (8) to get (11). Finally, (3), (9), Remark 3 and $\det X_1 = |P_1|^2 - 4|Q_1|^2$ yield $P_n^2 = K^{n-1}(|P_1|^2 - 4|Q_1|^2)$. Substitute the values obtained from (10) and (11) for P_1 and Q_1. This gives $|P_n| = 2|Q_n'|$. Now use $H'\Omega_n = \Omega_{n+1}$ and $H' \geq 0$ to get (12).

Remark 4. Lemma 3 yields various statements of the following sort. On R_n, Γ_n is complete if Γ_{n+1} is complete and $H \neq 0$ bounded, while Γ_{n+1} is complete if Γ_n is complete and $H \geq$ constant > 0. On R_n', Γ_n' is complete if Γ_{n+1}' is complete and $H' \neq 0$ bounded, while Γ_{n+1}' is complete if Γ_n' is complete and $H' \geq$ constant > 0.

Because there are methods available for studying the global properties of a surface provided with a complete Riemannian metric, it seems worthwhile to identify as many distinct metrics as possible on an open immersed surface which are generated by the geometry of the immersion, and complete whenever I is complete. For this reason we will be concerned with the metrics Λ_n and Λ_n' defined wherever $K \neq 0$ on a C^2-immersed surface S by

(16) $$2\Lambda_n = (k_1^{1-n} + k_2^{1-n})X_n , \qquad \pm 2\Lambda_n' = (k_2^{1-n} - k_1^{1-n})X_n' ,$$

where $\pm$ stands for the sign of $k_2 - k_1$. On R_n, (3) and (10) yield $\Lambda_n = \Gamma_1$, so that Λ_n is complete on S if I is, and if X_n is positive definite on S. Similarly, on R_n', (3) and (13) yield $\Lambda_n' = \Gamma_1$, so that Λ_n' is complete on S if I is, and if X_n' is positive definite on S. In particular, Λ_n and Λ_n' are as smooth as I is on R_n and R_n' respectively. (The metrics Λ_2 and Λ_2' were described in [24]. For an interesting use of Λ_2' on R_2', see Appendix 1 of [22].)

Remark 5. Remark 1 shows that just as $K(\mathrm{I}) \equiv 0$ if $\log P_1$ on R_1 is harmonic, $K(\Lambda_n) \equiv 0$ if $\log P_1$ on R_n is harmonic, and $K(\Lambda_n') \equiv 0$ if $\log P_1$ on R_n' is harmonic.

By $\tilde{S}$ we denote the universal covering surface of S. Any immersion of S

and all associated structures on S lift easily to $\tilde{S}$. Moreover, any Riemann surface R on S lifts to a Riemann surface $\tilde{R}$ on $\tilde{S}$. Clearly, $\tilde{R}$ is just the universal covering surface of R. Thus $\tilde{R}$ is conformally equivalent to the sphere, or the disc, or the plane. (See [1, p. 181].) If R is conformally equivalent to the plane, R is *parabolic*, and S is said to be *R-parabolic*. To avoid the introduction of new notation $\tilde{X}_n, \tilde{X}'_n, \tilde{P}_n, \tilde{Q}_n$, etc., we speak simply of X_n, X'_n, P_n, Q_n, etc., *on* $\tilde{S}$ or $\tilde{R}$ (meaning, of course, lifted to $\tilde{S}$ or $\tilde{R}$). The following elementary observation will prove useful.

Remark 6. A subharmonic function bounded from above on a parabolic Riemann surface must be constant. (See [1, p. 209].) Moreover, if a subharmonic function v is bounded from above by a harmonic function u, then $v - u$ is still subharmonic. Thus on a parabolic Riemann surface a subharmonic function bounded from above by a harmonic function must itself be harmonic. Similarly, a superharmonic function bounded from below by a harmonic function on a parabolic Riemann surface must itself be harmonic.

3. Φ-boundedness

Let $\Phi = F dz^n d\bar{z}^m$ be a differential of type (n, m) on a Riemann surface R. (See [2, p. 222].) The integers n and m need not be positive. Thus, if Φ never vanishes on R, the differential $1/\Phi = (1/F) dz^{-n} d\bar{z}^{-m}$ is well defined.

We call z a *global* conformal parameter on a Riemann surface R if z provides a conformal immersion of R into the plane. (Most authors require that the immersion be one-one, but this is too strict a requirement for our purposes.) Since a global conformal parameter is locally a conformal parameter in the usual sense of that word, one can express any differential Φ on R in terms of a global conformal parameter z as $\Phi = F dz^n d\bar{z}^m$, where F is a function on R.

Definition 1. Φ is said to be *harmonically bounded* on R if there exists a global conformal parameter on the universal covering surface $\tilde{R}$ of R in terms of which $|F| \leq e^u$ for some harmonic function u on $\tilde{R}$. Lemma 4 shows that this concept does not depend upon the particular choice of the global conformal parameter z.

Remark 7. To illustrate Definition 1, note that any continuous differential on a compact Riemann surface of genus one is harmonically bounded. As a further example, let $\Phi = F dz^2$ be a holomorphic quadratic differential which never vanishes on a Riemann surface R. Since all such differentials vanish identically on the sphere [1, p. 325], $\tilde{R}$ is not conformally equivalent to the sphere. We can thus choose a global conformal parameter z on $\tilde{R}$. In terms of z, $\log|F|$ is harmonic, so that both Φ and $1/\Phi$ are harmonically bounded on R. Similar reasoning shows that a quadratic differential $\Phi = F dz^2$ which never vanishes on R is harmonically bounded if there exists a complex valued function f on $\tilde{R}$ which is bounded away from zero so that $f\Phi$ is holomorphic on $\tilde{R}$.

Definition 2. Let S be an oriented C^∞ surface. Suppose R is a Riemann surface defined on S, and $\Phi = Fdz^n dz^m$ a differential of type (n, m) on R. Then S is *Φ-bounded/R* (read as Φ-bounded with respect to R) if Φ is harmonically bounded on R.

By Remark 7, a surface S is simultaneously Φ-bounded/R and $(1/\Phi)$-bounded/R if $\Phi = Fdz^2 \neq 0$ is holomorphic on R. In geometrically interesting examples, Ω or R is in some way connected with the geometry of an immersion of the surface.

Example 1. Suppose S is immersed in a 3-manifold $\mathscr{M}$ of constant sectional curvature. For simplicity we assume that the immersion is C^∞. The proofs of the following three facts, though originally established for $\mathscr{M} = E^3$, extend with little or no modification to cover the present situation. First, Ω_2 on R_1 is holomorphic if $H \equiv$ constant on S, with $\Omega_2 = 0$ only where $H' = 0$ [13]. Next, Ω_1 on R_2 is holomorphic if $K \equiv$ constant > 0 on S, with $\Omega_1 = 0$ only where $H' = 0$ [18]. Finally, $H'\Omega_2$ is holomorphic on R_2' and never vanishes if $K \equiv$ constant < 0 on S. (See [19] or [22, Appendix 1].) There is a unified explanation of these statements. If $\alpha + \beta H + \gamma K \equiv 0$ on S for some choice of constants α, β and γ, then setting $\varepsilon_1 = \alpha + \beta k_1 + \gamma k_1^2$, and $\varepsilon_2 = \alpha + \beta k_2 + \gamma k_2^2$, one can find a Riemann surface R^* defined on S wherever $H'\varepsilon_1\varepsilon_2 \neq 0$, so that the quadratic differential $\phi\Omega_1 + \psi\Omega_2 \neq 0$ is holomorphic on R^* for any functions ϕ and ψ on S which satisfy $\pm\, \varepsilon_1\varepsilon_2 = \phi(|\varepsilon_2| - |\varepsilon_1|) + \psi(k_1|\varepsilon_2| - k_2|\varepsilon_1|)$.

Example 2. If S is immersed in a 3-manifold so that $H \equiv 0$, then S is called a minimal surface. On R_1, $\Gamma_1 H = \Gamma_2$, so that S is *minimal* if and only if $\Gamma_2 \equiv 0$ on R_1. S is said to be *R-minimal* in case there is a Riemann surface R defined on S on which $\Gamma_2 \equiv 0$, and Ω_1 is holomorphic. Of course, S is minimal if and only if it is R_1-minimal by (5). The properties of R-minimal surfaces are remarkably much like those of ordinary minimal surfaces. (See [20] or [26].)

Remark 8. Since no global conformal parameter exists on the sphere, $\tilde{R}$ must be conformally equivalent to the disc or the plane in case any Φ is harmonically bounded on R. Thus S is not compact with genus zero if S is Φ-bounded/R for some choice of R and Φ.

The following lemma shows that Definition 1 is independent of the special choice of a global conformal parameter on $\tilde{R}$. It also shows that when $n + m \neq 0$, Φ-boundedness and F-boundedness, as defined in § 2 of [23], are equivalent. Our proof of Lemma 4 was influenced by a lemma due to Osserman. (See [28, p. 78].)

Lemma 4. *Suppose R is a Riemann surface whose universal covering surface is not conformally equivalent to the sphere. Let $\Phi = Fdz^n d\bar{z}^m$ be a differential of type (n, m) on R. Then statements* (i) *and* (ii) *are equivalent, and if $n + m \neq 0$, all three statements* (i), (ii) *and* (iii) *are equivalent.*

(i) *There is a global conformal parameter z on $\tilde{R}$ in terms of which $|F| \leq e^u$ for some harmonic function u on $\tilde{R}$.*

(ii) *For each global conformal parameter z on $\tilde{R}$ there is a harmonic function u on $\tilde{R}$ such that* $|F| \leq e^u$.

(iii) *There is a global conformal parameter z on $\tilde{R}$ in terms of which* $|F| \leq 1$.

Proof. Let z and w be global conformal parameters on $\tilde{R}$. Then locally, $\Phi = F dz^n d\bar{z}^m = \hat{F} dw^n d\bar{w}^n$ where $\hat{F} = F(dz/dw)^n(d\bar{z}/d\bar{w})^m$. Here $z = z(w)$ is locally biholomorphic, so that $\log|dz/dw| = \log|d\bar{z}/d\bar{w}|$ is harmonic on $\tilde{R}$. Statement (ii) follows from (i) because

$$|\hat{F}| \leq e^{u+(n+m)\log|dz/dw|}\,. \tag{17}$$

Since (i) follows trivially from (ii), the two statements are equivalent.

Assume now that $n + m \neq 0$ and that (i) is valid. Then for some global conformal parameter z on $\tilde{R}$ there is a harmonic function u on $\tilde{R}$ such that $|F| \leq e^u$. We need a complex valued function w on $\tilde{R}$ so that the locally induced function $w = w(z)$ is biholomorphic with $|e^u(dz/dw)^n(d\bar{z}/d\bar{w})^m| = 1$, or equivalently, with $u + (n + m)\operatorname{Re}\log(dw/dz) = 0$. On the simply connected Riemann surface $\tilde{R}$, we can find a harmonic conjugate function v for u. Set

$$w = \int e^{(u+iv)/(n+m)} dz \tag{18}$$

on $\tilde{R}$. Although w need not be one-one from $\tilde{R}$ to the plane, the integral is well defined on $\tilde{R}$ because $\tilde{R}$ is simply connected. Since w is a global conformal parameter on $\tilde{R}$ in terms of which $|\hat{F}| \leq 1$, statement (iii) follows from (i) if $n + m \neq 0$. But (i) follows trivially from (iii) by taking $u \equiv 0$. Thus the proof of Lemma 4 is complete.

Remark 9. If $\Phi = F\, dz^n d\bar{z}^{-n}$ is harmonically bounded on R, then by (17) there is a *single* harmonic function u on $\tilde{R}$ such that $|F| \leq e^u$ for *any* choice of a global conformal parameter z on $\tilde{R}$. In particular, if F is bounded for any one global conformal parameter on R, it is bounded for any other global conformal parameter on $\tilde{R}$.

The next lemma is most often applied with $\Sigma = \Gamma(X, R)$ where R is a Riemann surface C^2-related to an immersed surface R, and X is some quadratic form on S.

Lemma 5. *Suppose the curvature $K(\Sigma)$ of the conformal metric $\Sigma = \sigma dz d\bar{z}$ on a parabolic Riemann surface R is defined. If Σ is harmonically bounded on R and $K(\Sigma) \leq 0$, or if $1/\Sigma$ is harmonically bounded on R and $K(\Sigma) \geq 0$, then* $K(\Sigma) \equiv 0$.

Proof. Suppose Σ is harmonically bounded on R, and $K(\Sigma) \leq 0$. Use a global conformal parameter on $\tilde{R}$ so that $\log \sigma \leq u$ for some harmonic function u on R. By Lemma 2 and Remark 6, $\log \sigma$ must itself be harmonic. Thus Remark 1 yields $K(\Sigma) \equiv 0$. A similar argument gives $K(\Sigma) \equiv 0$ if $1/\Sigma$ if harmonically bounded and $K(\Sigma) \geq 0$.

Corollary to Lemma 5. *Let R be a Riemann surface C^2-related to a surface S, and let X be a complete Riemannian metric on S with $\Gamma = \Gamma(X, R)$. If S is $(1/\Gamma)$-bounded/R and $K(\Gamma) \geq 0$ is defined, then S is R-parabolic and $K(\Gamma) \equiv 0$.*

Proof. By Remark 8, S is not compact with genus zero since it is $(1/\Gamma)$-bounded/R. If $K(\Gamma) \geq 0$, S is not compact with genus greater than one, by the Gauss-Bonnet theorem. But Γ is complete since X is by Lemma 1. The theorem of Blank-Fiala-Huber implies that S must be R-parabolic [15]. Thus by Lemma 5, $K(\Gamma) \equiv 0$.

Lemma 6. *Suppose $\Sigma = \sigma dz d\bar{z}$ is a complete, continuous, positive definite, conformal metric on a Riemann surface R. Any global conformal parameter z on $\tilde{R}$ in terms of which σ is bounded yields a conformal homeomorphism of $\tilde{R}$ onto the plane.*

Proof. Because $|\sigma| < c$ in terms of z for some constant c, the Σ-length of an arc on $\tilde{R}$ is less than c times the length induced by its immersion in the plane by z. If D is the image of $\tilde{R}$ under z, any arc in D lifts to an arc in $\tilde{R}$. Otherwise an arc $\gamma\colon [0, 1] \to D$ of finite length would exist whose restriction to $[0, 1]$ lifts to a divergent arc on $\tilde{R}$ of finite Σ-length, contradicting the assumption that Σ is complete. Similarly, D can have no finite boundary point p. Otherwise, an arc $\gamma\colon [0, 1] \to \bar{D}$ of finite length would exist with $\gamma(1) = p$ and $\gamma([0, 1)) \subset D$ whose restriction to $[0, 1)$ lifts to a divergent arc on $\tilde{R}$ of finite length, giving the same contradiction. Thus z yields a covering of the whole plane by $\tilde{R}$, which is possible only if z is one-one.

The next result shows, for example, that a complete surface S with global Tchebychev coordinates x, y is R-parabolic with respect to the Riemann surface R defined on S by the conformal parameter $z = x + iy$. For another application of Lemma 6, see Corollary 2 to Lemma 3 in [23]. (The hypotheses of that corollary are meant to include the assumption that R_1 is defined on S, a fact which is not automatic on a surface C^1-immersed in E^3, as shown in [3].)

Corollary to Lemma 6. *Let R be a Riemann surface C^2-related to a surface S, and let X be a quadratic form on S such that $\Gamma = \Gamma(X, R)$ is a complete Riemannian metric. If S is Γ-bounded/R, then S is R-parabolic.*

Proof. By Lemma 4, there is a global conformal parameter z on $\tilde{R}$ in terms of which P is bounded, where $\Gamma = P\, dz d\bar{z}$. Lemma 6 shows that S is R-parabolic.

4. Auxiliary results

Throughout this section we assume that S is at least C^2-immersed in a Riemannian 3-manifold $\mathcal{M}$. (Only in the corollaries to Lemmas 11, 12 and 13 is any restriction placed on $\mathcal{M}$.) We also assume that any Riemann surface mentioned is defined, and C^2-related to S. Finally, in any result involving the curvature of a metric, we assume that the curvature is defined. Note that

Lemma 7 places no condition on the sign of K unless n is even.

Lemma 7. *Let $j - n > 0$ be odd, and X_n complete. Suppose S is Ω_j-bounded$/R_n$ or Γ'_j-bounded$/R_n$ with H' bounded away from zero, or Γ_j-bounded$/R_n$ or Ω'_j-bounded$/R_n$ with H bounded away from zero. Then S is Γ_n-bounded$/R_n$ and R_n-parabolic.*

Proof. On R_n, $\Gamma_n = X_n$, and by Lemma 3,

$$(19)\qquad \begin{aligned} 4|Q_j| &= 2|P'_j| = |k_2^{j-n} - k_1^{j-n}|P_n\,, \\ 2|P_j| &= 4|Q'_j| = |k_2^{j-n} + k_1^{j-n}|P_n \end{aligned}$$

in terms of any global conformal parameter on $\tilde{R}_n$. If $j - n > 0$ is odd, $H' = |k_2 - k_1|/2 \geq \text{constant} > 0$ yields $|k_2^{j-n} - k_1^{j-n}| \geq \text{constant} > 0$, while $H = |k_1 + k_2|/2 \geq \text{constant} > 0$ yields $|k_2^{j-n} + k_1^{j-n}| \geq \text{constant} > 0$. Given the stated hypotheses, (19) shows that S must be Γ_n-bounded$/R_n$. By the corollary to Lemma 6, S is R_n-parabolic.

Lemma 8. *Let $j - n > 0$ be odd, and X'_n complete. Suppose S is Ω_j-bounded$/R'_n$ or Γ'_n-bounded$/R'_n$ with H bounded away from zero, or Γ_n-bounded$/R'_n$ or Ω'_j-bounded$/R'_n$ with H' bounded away from zero. Then S is Γ'_n-bounded$/R_n$ and R_n-parabolic.*

Proof. On R'_n, $\Gamma'_n = X'_n$, and by Lemma 3,

$$(20)\qquad \begin{aligned} 4|Q_j| &= 2|P'_j| = |k_2^{j-n} + k_1^{j-n}|P'_n\,, \\ 2|P_j| &= 4|Q'_j| = |k_2^{j-n} - k_1^{j-n}|P'_n \end{aligned}$$

in terms of any global conformal parameter z on R'_n. Now follow the reasoning which established Lemma 7.

Lemma 9. *Suppose S is R_n-parabolic with $K(X_n) \geq 0$. If S is $(1/\Omega_{n+1})$-bounded$/R_n$ or $(1/\Gamma'_{n+1})$-bounded$/R_n$ with H' bounded or $(1/\Omega'_{n+1})$-bounded$/R_n$ or $(1/\Gamma_{n+1})$-bounded$/R_n$ with H bounded, then $K(X_n) \equiv 0$. If $j - n > 0$ is odd with both H and H' bounded, and S is $(1/\Omega_j)$-bounded$/R_n$ or $(1/\Gamma'_j)$-bounded$/R_n$ or $(1/\Omega'_j)$-bounded$/R_n$ or $(1/\Gamma_j)$-bounded$/R_n$, then $K(X_n) \equiv 0$.*

Proof. Use a global conformal parameter z on $\tilde{R}_n$ which maps $\tilde{R}_n$ one-one onto the plane. By (19), if S is $(1/\Omega_j)$-bounded$/R_n$ or $(1/\Gamma'_j)$-bounded$/R_n$ or $(1/\Omega'_j)$-bounded$/R_n$ or $(1/\Gamma_j)$-bounded$/R_n$ for $j - n > 0$ odd, bounds on H and H' force $\log P_n$ to be bounded from below by a harmonic function. (When $j = n + 1$, H' bounded is enough in the first two cases, and H bounded is enough in the last two cases.) Since $K(X_n) = K(\Gamma_n) \geq 0$ on R_n, Lemma 2 and Remarks 1 and 6 yield $K(X_n) \equiv 0$.

Similar reasoning based upon (20) yields the following.

Lemma 10. *Suppose S is R'_n-parabolic with $K(X'_n) \geq 0$. If S is $(1/\Omega_{n+1})$-bounded$/R'_n$ or $(1/\Gamma'_{n+1})$-bounded$/R'_n$ with H bounded or $(1/\Omega'_{n+1})$-bounded$/R'_n$ or $(1/\Gamma_{n+1})$-bounded$/R'_n$ with H' bounded, then $K(X'_n) \equiv 0$. If $j - n > 0$ is odd with both H and H' bounded, and S is $(1/\Omega_j)$-bounded$/R'_n$ or $(1/\Gamma'_j)$-*

bounded$/R'_n$ *or* $(1/\Omega'_j)$*-bounded*$/R'_n$ *or* $(1/\Gamma_j)$*-bounded*$/R'_n$*, then* $K(X'_n) \equiv 0$.

Remark 10. One cannot conclude that X_n in Lemma 9 or X'_n in Lemma 10 is complete. For a positive definite metric $\Sigma = \sigma\, dz d\bar{z}$ on the x, y-plane need not be complete even if $\log|\sigma| \geq v$, where v is harmonic. As an example, let $\Sigma = e^x dz d\bar{z}$. This metric assigns finite length to the nonpositive x-axis. The metric X_n or X'_n in Lemmas 10 and 11 is complete if and only if the global conformal parameter w associated with the uniformizing parameter z by (18) is one-one. *Thus Lemmas* 6, 7, *and* 8 *in* [23] *must be suitably corrected.*

Lemma 11. *Suppose S is either* $(1/\Pi_j)$*-bounded*$/R_n$ *or* $(1/\Pi'_j)$*-bounded*$/R_n$ *with* $j \geq n$*, K bounded, and* X_n *complete. If* $K(X_n) \geq 0$ *then* $K(X_n) \equiv 0$*. The same result holds if* R_n *is replaced by* R'_n*, and* X_n *by* X'_n*.*

Proof. On R_n, (9) yields

$$|\det X_j| = |\det X'_j| = |K^{j-n}|\, P_n^2 \,, \tag{21}$$

since $\Gamma_n = X_n$. Thus, if K is bounded and S is either $(1/\Pi_j)$-bounded$/R_n$ or $(1/\Pi'_k)$-bounded$/R_n$, S is $(1/\Gamma_n)$-bounded$/R_n$. By the corollary to Lemma 5, $K(X_n) \equiv 0$ if $K(X_n) \geq 0$. Similar reasoning applies if we replace R_n by R'_n and $X_n = \Gamma_n$ by $X'_n = \Gamma'_n$.

Corollary to Lemma 11. *Suppose S is a complete surface* C^2*-immersed in* E^3 *with* $K = K(I)$ *bounded. If S is either* $(1/\Pi_j)$*-bounded*$/R_1$ *or* $(1/\Pi'_j)$*-bounded*$/R_1$ *for some* $j > 1$*, then* $K = K(I) < 0$ *on S.*

Proof. On R_1, we use (21) with $n = 1$. Thus, if S is either $(1/\Pi_j)$-bounded$/R_1$ or $(1/\Pi'_j)$-bounded$/R_1$, $K = K(I)$ never vanishes on S. If $K(I) > 0$ anywhere on S, it follows that $K(I) > 0$ over all of S, and Lemma 11 yields $K(I) \equiv 0$, a contradiction. It follows that $K(I) < 0$ on S.

Lemma 12. *Suppose S is either* $(1/\Pi_j)$*-bounded*$/R_n$ *or* $(1/\Pi'_j)$*-bounded*$/R_n$ *with* $j < n$*, K bounded away from zero, and* X_n *complete. If* $K(X_n) \geq 0$*, then* $K(X_n) \equiv 0$*. The same result holds if* R_n *is replaced by* R'_n *and* X_n *by* X'_n*.*

Proof. Since $j < n$ and K is bounded away from zero, (21) shows that S must be $(1/\Gamma_n)$-bounded$/R_n$ if it is either $(1/\Pi_j)$-bounded$/R_n$ or $(1/\Pi'_j)$-bounded$/R_n$. Again, the corollary to Lemma 5 yields $K(X_n) \equiv 0$ if $K(X_n) \geq 0$, since $X_n = \Gamma_n$ on R_n. Similar reasoning applies if we replace R_n by R'_n and X_n by X'_n.

Corollary to Lemma 12. *Suppose S is a complete surface* C^2*-immersed in a* 3*-manifold with sectional curvature* $\mathscr{K} <$ *constant* < 0*. If S is* $(1/\Pi_j)$*-bounded*$/R_1$ *or* $(1/\Pi'_j)$*-bounded*$/R_1$ *for some* $j < 1$*, then* $K(I) \geq 0$ *implies* $K(I) \equiv 0$*.*

Proof. The Gauss equation [30, p. 527] states that

$$K(I) - \mathscr{K} = K \,. \tag{22}$$

Thus, if $K(I) \geq 0$, K is bounded away from zero since $\mathscr{K} \leq$ constant < 0, and Lemma 12 with $n = 1$ yields $K(I) \equiv 0$.

Lemma 13. *Suppose S is either Π_j-bounded$/R_n$ or Π'_j-bounded$/R_n$ with $j \geq n$, K bounded away from zero and X_n complete. If $K(X_n) \leq 0$ then $K(X_n) \equiv 0$. The same result holds if R_n is replaced by R'_n and X_n by X'_n.*

Proof. Since $j \geq n$ and K is bounded away from zero, (21) shows that S must be Γ_n-bounded$/R_n$ if it is either Π_j-bounded$/R_n$ or Π'_j-bounded$/R_n$. By the corollary to Lemma 6, S is R_n-parabolic since $\Gamma_n = X_n$ is complete on R_n. Thus Lemma 5 yields $K(X_n) \equiv 0$ if $K(X_n) \leq 0$. Similar reasoning applies if we replace R_n by R'_n and X_n by X'_n.

Corollary to Lemma 13. *Suppose S is a complete surface C^2-immersed in a 3-manifold with sectional curvature $\mathscr{K} \geq$ constant > 0. If S is Π_j-bounded$/R_1$ or Π'_j-bounded$/R_1$ for some $j \geq 1$, then $K(I) \leq 0$ implies $K(I) \equiv 0$.*

Proof. Here (22) shows that K is bounded away from zero if $K(I) \leq 0$ since $\mathscr{K} \geq$ constant > 0, and Lemma 13 with $n = 1$ yields $K(I) \equiv 0$.

Lemma 14. *Suppose S is either Π_j-bounded$/R_n$ or Π'_j-bounded$/R_n$ with $j < n$, K bounded and X_n complete. If $K(X_n) \leq 0$ then $K(X_n) \equiv 0$. The same result holds if R_n is replaced by R'_n and X_n by X'_n.*

Proof. Since $j < n$ and K is bounded, (21) shows that S must be Γ_n-bounded$/R_n$ if it is either Π_j-bounded$/R_n$ or Π'_j-bounded$/R_n$. Now follow the reasoning which established Lemma 13.

5. Main results

Theorems 1 through 4 generalize results due to Klotz and Osserman [21], Hawley [10], Hoffman [12], and Yau [35]. These theorems can be used, therefore, to characterize all complete surfaces of constant mean curvature in a 3-manifold of constant sectional curvature. (See Example 1 of § 3.) When applied to surfaces in E^3, Theorems 1 through 4 check *special cases* of a conjecture due to John Milnor. (See [21] or [22].) Thus, for example, Theorem 1 verifies the conjecture for all Ω_2-bounded$/R_1$ surfaces in E^3 with $K < 0$. The remaining Theorems 5 through 8 are meant to facilitate use of the metrics Λ_n and Λ'_n defined by (16) in the study of global properties of open complete immersed surfaces.

Throughout this section, we assume that S is a *complete* surface C^2-immersed in a Riemannian 3-manifold. As in § 4, we assume that any Riemann surface mentioned is defined, and C^2-related to S. Similarly, if the curvature of a metric is referred to, we assume it is defined.

Theorem 1. *Suppose that H' is bounded away from zero and that S is either Ω_j-bounded$/R_1$ or Γ'_j-bounded$/R_1$ where $j > 1$ is even. Then $K(I) \equiv 0$ if $K(I) \leq 0$ on S.*

Proof. By Lemma 7, S is Γ_1-bounded$/R_1$ and R_1-paɹabolic. By Lemma 5, $K(I) \equiv 0$ if $K(I) \leq 0$ on S, since $\Gamma_1 = I$ on R_1.

Corollary to Theorem 1. *Suppose that H' is bounded away from zero and that $f\Omega_j$ is holomorphic on $\tilde{R}_1$ where f is a complex valued function bounded* *away from zero, and $j > 1$ is even. Then $K(I) \equiv 0$ if $K(I) \leq 0$ on S.*

Proof. By (11) and (12), we have $4|\Omega_j| = |k_2^{j-1} - k_1^{j-1}|\Gamma_1$. Since Γ_1 is positive definite on R_1 and j is even, Ω_j vanishes only if $k_1 = k_2$, an impossibility here since $H' \geq$ constant > 0. By Remark 7, we conclude that S is Ω_j-bounded$/R_1$ since $|f| \geq$ constant > 0. Theorem 1 thus yields $K(I) \equiv 0$ if $K(I) \leq 0$ on S.

Theorem 2. *Suppose H is bounded away from zero and that S is either Γ_j-bounded$/R_1$ or Ω'_j-bounded$/R_1$ where $j > 1$ is even. Then $K(I) \equiv 0$ if $K(I) \leq 0$ on S.*

Proof. By Lemma 8, S is Γ_1-bounded$/R_1$ and R_1-parabolic. By Lemma 5, $K(I) \equiv 0$ if $K(I) \leq 0$ on S.

Theorem 3. *Suppose $K(I) \geq 0$ while H and H' are bounded. If S is $(1/\Omega_j)$-bounded$/R_1$ or $(1/\Gamma'_j)$-bounded$/R_1$ or $(1/\Omega'_j)$-bounded$/R_1$ or $(1/\Gamma_j)$-bounded$/R_1$ for an even $j > 1$, then $K(I) \equiv 0$. (If $j = 2$, H bounded is not needed in the first two cases, and H' bounded is not needed in the last two cases.)*

Proof. Use $n = 1$ in (19) and apply the corollary to Lemma 5.

Theorem 4. *Suppose H and H' are bounded while $f\Omega_j \not\equiv 0$ is holomorphic on $\tilde{R}_1$ for a bounded complex valued function f and an even integer j. Then $K(I) \equiv 0$ if $K(I) \geq 0$ on S. (If $j = 2$, the bound on H is not needed.)*

Proof. Since holomorphic quadratic differentials on the sphere vanish identically [1, p. 325], S is not compact with zero. If $K(I) \geq 0$, S cannot be compact with genus greater than 1 by the Gauss-Bonnet theorem. Thus a theorem of Blank-Fiala-Huber [15] states that S is R_1-parabolic. Using a global conformal parameter on $\tilde{R}_1$, (19) with $n = 1$ yields $|f|H'|k_2^{j-2} + \cdots + k_1^{j-2}|P_1 = |2fQ_j|$, so that $\log P_1 = \log|2fQ_j| - \log|f|H'|k_2^{j-2} + \cdots + k_1^{j-2}|$, where $\log|2fQ_j|$ is harmonic except at the isolated zeros of fQ_j where $\log|2fQ_j| = -\infty$. Because f, H and H' are bounded, there is a constant c such that $\log P_1 \geq u$ where $u = \log|2fQ_j| - c$ is harmonic except at the isolated points where $u = -\infty$. (If $j = 2$, we get this with no bound on H.) It follows [1, p. 135] that $\log P_1$ is superharmonic, so that the difference $u - \log P_1 \leq 0$ is subharmonic. By Remarks 1 and 6, $u - \log P_1$ is constant, so that u is always finite, and $\log P_1$ is itself harmonic. By Remark 5, $K(I) \equiv 0$ on S.

The remaining results involve the metrics Λ_n and Λ'_n discussed in § 2, and defined by (16).

Theorem 5. *If S is Γ_1-bounded$/R'_n$, then $K(\Lambda'_n) \equiv 0$ if $K(\Lambda'_n) \leq 0$.*

Proof. On R'_n, $\Lambda'_n = \Gamma_1$ is complete. By the corollary to Lemma 6, S is R'_n-parabolic, and by Lemma 5, $K(\Lambda') \equiv 0$ if $K(\Lambda'_n) \leq 0$.

Theorem 6. *If S is $(1/\Gamma_1)$-bounded$/R'_n$, then $K(\Lambda'_n) \equiv 0$ if $K(\Lambda'_n) \geq 0$.*

Proof. Again, $\Lambda'_n = \Gamma_1$ on R'_n is complete. Here the corollary to Lemma 5 yields $K(\Lambda') \equiv 0$ if $K(\Lambda') \leq 0$.

Theoren 7. *If S is $(1/\Gamma_1)$-bounded$/R_n$, and $K(\Lambda_n) \geq 0$, then $K(\Lambda_n) \equiv 0$.*

Proof. On R_n, $\Lambda_n = \Gamma_1$ is complete. By the corollary to Lemma 5, $K(\Lambda_n) \equiv 0$ if $K(\Lambda_n) \geq 0$.

Theorem 8. *If S is Γ_1-bounded$/R_n$, and $K(\Lambda_n) \leq 0$, then $K(\Lambda_n) \equiv 0$.*

Proof. Again, $\Lambda_n = \Gamma_1$ on R_n is complete. The corollary to Lemma 6 shows that S is R_n-parabolic. By Lemma 5, $K(\Lambda_n) \equiv 0$ if $K(\Lambda_n) \leq 0$.

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