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CALCULUS ON SUBCARTESIAN SPACES

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Introduction

The notion of a *differentiable subcartesian space* is a generalization of that of a differentiable manifold and includes *arbitrary* subsets of $\boldsymbol{R}^n$ as special examples. In this paper we construct the category of differentiable subcartesian spaces and develop the calculus of differentiable mappings, vector and tensor fields, and exterior differential forms.

More precisely, a differentiable subcartesian space of class C^k is a Hausdorff space equipped with an atlas of local homeomorphisms into various cartesian spaces $\boldsymbol{R}^n$, each pair of which satisfies a condition of C^k-compatibility similar to that satisfied by charts of a C^k-manifold. For the sake of simplicity we shall restrict attention to the C^∞-case. The necessary modifications for other smoothness categories are obvious for the most part, although the C^k-theory, for instance, is not without independent interest (see [4]). The calculus of differential forms gives rise to the de Rham cohomology of a subcartesian space, and we shall introduce that theory in a sequel to this paper.

In brief outline, our results are the following. The category of C^∞-subcartesian spaces possesses a tangent functor T sending each S into its tangent pseudo-bundle TS and each differentiable mapping f into the corresponding induced mapping f_*. As one would guess from the terminology, TS is not a vector bundle but is a fiber space, the dimension of whose fibers may vary on connected components of S. We introduce the notion of *differentiable vector pseudo-bundle* and show how tensor products and other covariant differentiable functors on the category of real vector spaces may be naturally lifted to vector pseudo-bundles. Thus having the "contravariant" tensors and their fields, we introduce the modules of covariant and mixed tensor fields and determine their dual modules. We then introduce the Lie module of Lie derivatives.

In each case the idea is to lift classical objects and constructions (e.g., vector fields and their Lie products) from the ambient cartesian spaces up to S via charts. As in the special case of differentiable manifolds, this method always requires one to check invariance under coordinate changes, but there are now two more things to be checked. The constructions made in $\boldsymbol{R}^n$ with local representatives of objects on S must not depend on the choices of these local

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representatives, and the objects constructed with local representatives must again be local representatives of objects on S. These problematic features are clearly seen, for example, in the construction of Lie derivatives (§ 5).

In the last section we introduce exterior differentiation of alternating forms. Here, the exterior derivative of a form is *not* invariant under change of local representatives, and a multi-valued differential results. We proceed by considering residue classes of alternating forms modulo the indeterminacy ideal $\mathfrak{m}$ of the differential relation. This results in our definition of *differential forms*. We then show that $\mathfrak{m}$ is invariant under exterior, Lie and interior differentiations, and establish several important classical identities relating these. Finally, we establish a singular version of Stokes' identity.

Subcartesian spaces were introduced by N. Aronszajn in the study of Bessel Potentials. (See [3] and *Subcartesian and subriemannian spaces*, Notices, Amer. Math. Soc. **14** (1967) 111.) This paper is in part a response to problems exposed by Aronszajn in a series of lectures on subcartesian spaces in 1966–67 and is an outgrowth of [9]. (Also see *The de Rham cohomology of subcartesian structures*, Notices, Amer. Math. Soc. **18** (1971) 203.)

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1. Structures and maps

A *subcartesian space* S is a Hausdorff space which is locally homeomorphic to (*not necessarily open*) subspaces of cartesian spaces $\boldsymbol{R}^n, n = 0, 1, \cdots$. If φ is one such local homeomorphism, then we denote its domain by U_φ and its range space by $\boldsymbol{R}^{n_\varphi}$. A C^∞-*atlas* on S is a set $\mathfrak{A}$ of local homeomorphisms $\varphi: S \supseteq U_\varphi \to \boldsymbol{R}^{n_\varphi}$ satisfying the following two axioms:

(A1) The domains $\{U_\varphi \mid \varphi \in \mathfrak{A}\}$ form an open cover of S.

(A2) For every $\varphi, \psi \in \mathfrak{A}$ and every $p \in U_\varphi \cap U_\psi$ there exist C^∞-mappings s extending $\psi \circ \varphi^{-1}$ in a neighborhood of φp and t extending $\varphi \circ \psi^{-1}$ in a neighborhood of ψp.

The elements of $\mathfrak{A}$ are called *charts*. Every C^∞-atlas on S is contained in a maximal C^∞-atlas. A maximal C^∞-atlas is called a C^∞-*subcartesian structure*, and S together with a C^∞-subcartesian structure is called a C^∞-*subcartesian space*. When no confusion can result we shall denote the C^∞-subcartesian space $(S, \mathfrak{A})$ by S.

For each $0 \le m \le n$ define $i_{n,m}: \boldsymbol{R}^m \to \boldsymbol{R}^n$ to be the injection $(x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_m, 0, \cdots, 0)$, and define $\pi_{m,n}: \boldsymbol{R}^n \to \boldsymbol{R}^m; (x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_m)$. In the next section we shall show that (A2) implies

(A2′) For every $\varphi, \psi \in \mathfrak{A}$ and every $p \in U_\varphi \cap U_\psi$ there exist neighborhoods U of $i_{N,n_\varphi} \circ \varphi(p)$ and V of $i_{N,n_\psi} \circ \psi(p)$ in $\boldsymbol{R}^N$, and a C^∞-diffeomorphism $f: U \to V$ extending

$$(i_{N,n_\psi} \circ \psi) \circ (i_{N,n_\varphi} \circ \varphi)^{-1}|_U \ ,$$

where $N = \max\{n_\varphi, n_\psi\}$.

Let S and S' be C^∞-subcartesian spaces. A mapping $f: S \to S'$ is of *class* C^∞ if for every $p \in S$, and every $\varphi \in \mathfrak{A}_S$ and $\psi \in \mathfrak{A}_{S'}$ with $p \in U_\varphi$ and $f(U_\varphi) \subseteq U_\psi$, there exists a C^∞-extension of $\psi \circ f \circ \varphi^{-1}$ in a neighborhood of $\varphi(p)$. The set of such mappings is denoted by $C^\infty(S, S')$. The set of C^∞-functions $C^\infty(S, \boldsymbol{R}) =: C^\infty(S)$ is a ring with operations defined pointwise. The C^∞-subcartesian spaces together with the C^∞-mappings form a category which we denote by $\boldsymbol{C}^\infty$. The category of finite dimensional C^∞-manifolds (alternatively, with boundaries or with corners) and C^∞-mppings forms a full subcategory of $\boldsymbol{C}^\infty$.

It is clear that with little or no modification, pseudo-groups Γ, or more conveniently, smoothness categories $\mathscr{C}$ (cf. [12]) other than C^∞ may be used to structure subcartesian spaces, e.g., C^k, $\boldsymbol{R}$-analytic, and Nash. Complex analytic structures can be treated similarly. (See [9] for a general axiomatic treatment.)

If φ is a chart of a C^∞-subcartesian structure $\mathfrak{A}_S$ and U is open in U_φ, then $\varphi|_U \in \mathfrak{A}_S$. If $n \geq n_\varphi$, then $i_{n,n_\varphi} \circ \varphi \in \mathfrak{A}_S$. If S' is a topological subspace of S, then $\{\varphi|_{S'} \mid \varphi \in \mathfrak{A}_S\}$ is a C^∞-atlas on S'. The C^∞-subcartesian structure on S' generated by this atlas is called *the structure induced from S*.

Example 1.1. Let $S_n = \{(x_1, \cdots, x_n) \in \boldsymbol{R}^n \mid x_1 \in [n-1, n)\}$ and $S = \bigcup_{n \geq 1} S_n$. Define $\varphi_n : \bigcup_{k=1}^n S_k \to \boldsymbol{R}^n$ by $\varphi_n|_{S_k} = i_{n,k}|_{S_k}$. Then $\{\varphi_n \mid n \geq 1\}$ determines a C^∞-subcartesian structure on S. Thus it may be impossible to model a subcartesian space in any single $\boldsymbol{R}^n$.

The category $\boldsymbol{C}^\infty$ admits products. Let S and S' be C^∞-subcartesian spaces. Then

$$\mathfrak{P}_{S \times S'} := \{\varphi \times \psi : U_\varphi \times U_\psi \to \boldsymbol{R}^{n_\varphi} \times \boldsymbol{R}^{n_\psi} \mid \varphi \in \mathfrak{A}_S, \psi \in \mathfrak{A}_{S'}\}$$

is a C^∞-atlas on the topological space $S \times S'$. The product C^∞-subcartesian space is $S \times S'$ equipped with the maximal atlas determined by $\mathfrak{P}_{S \times S'}$. The product functor has the usual universal property in $\boldsymbol{C}^\infty$.

Proposition 1.2. *If S is a paracompact C^∞-subcartesian space and $\mathscr{U}$ is a locally finite open cover of S, then there is a C^∞-partition of unity subordinate to $\mathscr{U}$.*

Proof. Without loss of generality we may assume the elements of $\mathscr{U}$ to be chart domains. Applying the shrinking lemma, let $\mathscr{V} = \{V_U \mid U \in \mathscr{U}\}$ be an open cover of S with $\overline{V}_U \subseteq U$ for each U. For each $U \in \mathscr{U}$ choose $\varphi \in \mathfrak{A}$ with $U_\varphi = U$, and let V_φ be an open subset of $\boldsymbol{R}^{n_\varphi}$ such that $V_U = \varphi^{-1}(V_\varphi)$. We now use the well-known fact (cf. [11]) that every closed subset in $\boldsymbol{R}^n$ is the zero set for some nonnegative real-valued C^∞-function. Choose $g_\varphi \in C^\infty(\boldsymbol{R}^{n_\varphi})$ such that $g_\varphi(x) > 0$ for $x \in V_\varphi$ and $g_\varphi(x) = 0$ for $x \notin V_\varphi$. Define $f_U \in C^\infty(S)$ by $f_U(p) = g_\varphi \circ \varphi(p)$ for $p \in U$ and $f_U(p) = 0$ for $p \notin U$. Define $f = \sum_{U \in \mathscr{U}} f_U$.

Then $f \in C^\infty(S)$ and $f(p) > 0$ for all $p \in S$. Define $z_U = f_U/f$. Then $\{z_U \mid U \in \mathscr{U}\}$ is a C^∞-partition of unity subordinate to $\mathscr{U}$.

2. The tangent functor

Let S be a C^∞-subcartesian space and let $p \in S$. The *structural dimension* of S at p is the number $n_{S,p} = \min\{n_\varphi \mid \varphi \in \mathfrak{A}_S \text{ and } p \in U_\varphi\}$, and $\varphi \in \mathfrak{A}_S$ is *tangential* at p if $n_\varphi = n_{S,p}$. The structural dimension function $p \mapsto n_{S,p}$ is upper semi-continuous.

Lemma 2.1. *Let $p \in S$ and let $\varphi \in \mathfrak{A}_S$ be tangential at p. Let V be a neighborhood of $\varphi(p)$ in $\boldsymbol{R}^n$, $n = n_\varphi$, and let $f_1, f_2 \in C^\infty(V, \boldsymbol{R}^m)$. If $f_1 \circ \varphi|_{\varphi^{-1}(V)} = f_2 \circ \varphi|_{\varphi^{-1}(V)}$, then $Df_1(\varphi p) = Df_2(\varphi p)$.*

Proof. Suppose first that $m = 1$. If $Df_1(p) \neq Df_2(\varphi p)$, then $f_1 - f_2$ is of maximal rank at p. By the implicit function theorem there is a C^∞-coordinate system θ around $\varphi(p)$ such that θ sends $U_\theta \cap (f_1 - f_2)^{-1}(0)$ into the hyperplane $i_{n,n-1}(\boldsymbol{R}^{n-1})$. Then $\pi_{n-1,n} \circ \theta \circ \varphi$ is a chart in $\mathfrak{A}_S$ in a neighborhood of p and its range space is $\boldsymbol{R}^{n-1}$. This contradicts the assumption that φ is tangential at p. Thus $Df_1(\varphi p) = Df_2(\varphi p)$. For $m > 1$ the lemma follows by considering coordinate functions.

Corollary 2.2. *Axiom* A2 *implies* A2$'$.

Proof. Suppose $n_\psi \leq n_\varphi$. We first assume that ψ is tangential at p. Then $s \circ t|_{\psi(U_\varphi \cap U_\psi)} = \mathrm{Id}_{\boldsymbol{R}^{n_\psi}}|_{\psi(U_\varphi \cap U_\psi)}$, and it follows from (2.1) that $D(s \circ t)(\psi p) = \mathrm{Id}_{\boldsymbol{R}^{n_\varphi}}$. Thus t is of maximal rank at $\psi(p)$. Let E be a complementary subspace to Image $Dt(\psi p)$ in $\boldsymbol{R}^{n_\varphi}$.

Define

$$u\colon \boldsymbol{R}^{n_\psi} \times E \to \boldsymbol{R}^{n_\varphi};\ (x, y) \mapsto t(x) + y\ .$$

Then $Du(\psi p) = Dt(\psi p) + \mathrm{Id}_E$ is an isomorphism, and it follows that u is a C^∞-diffeomorphism in some neighborhood of $(\psi p, 0) = i_{n_\varphi, n_\psi} \circ \psi(p)$. Now let ψ be arbitrary and let $\theta \in \mathfrak{A}_S$ be tangential at p. There are then local diffeomorphisms u extending $i_{n_\varphi, n_\theta} \circ \theta \circ \varphi^{-1}$ in a neighborhood of φp and v extending $\psi \circ (i_{n_\psi, n_\theta} \circ \theta)^{-1}$ in a neighborhood of $i_{n_\psi, n_\theta} \circ \theta(p)$. Define w in a neighborhood of $i_{n_\varphi, n_\theta} \circ \theta p$ by

$$w = i_{n_\varphi, n_\psi} \circ v \circ \pi_{n_\psi, n_\varphi} + \pi_F\ ,$$

where F is a complement of $i_{n_\varphi, n_\psi} \boldsymbol{R}^{n_\psi}$ in $\boldsymbol{R}^{n_\varphi}$. Then $w \circ u$ is a local C^∞-diffeomorphism extending $(i_{n_\varphi, n_\psi} \circ \psi) \circ \varphi^{-1}$ in a neighborhood of φp. q.e.d.

If φ and $\psi \in \mathfrak{A}_S$ are tangential at p, it follows that the linear map $f_{*\varphi p}\colon T_{\varphi p}\boldsymbol{R}^n \to T_{\psi p}\boldsymbol{R}^n$, $n := n_\varphi = n_\psi$, is independent of the choice of f among all local diffeomorphisms extending $\psi \circ \varphi^{-1}$ near $\varphi(p)$. In the set of triples (φ, p, v), where $p \in S$, $\varphi \in \mathfrak{A}_S$ is tangential at p, and $v \in T_{\varphi p}\boldsymbol{R}^n$, we define the relation

$(\varphi, p, v) \sim (\psi, q, w)$ if and only if $p = q$ and $w = f_* v$ for some local diffeomorphism f extending $\psi \circ \varphi^{-1}$ near $\varphi(p)$.

This is an equivalence relation. We denote the equivalence class of (φ, p, v) by $[\varphi, p, v]$, and call p the *footpoint* of $[\varphi, p, v]$. Now let S_p be the set of all equivalence classes of $\sim$ with footpoint p. For each $\varphi \in \mathfrak{A}_S$ tangential at p and each $X_p \in S_p$, there is a unique $v \in T_{\varphi p}\boldsymbol{R}^n$ with $[\varphi, p, v] = X_p$. Thus for each $p \in S$ and $\varphi \in \mathfrak{A}_S$ tangential at p, we have the bijection

$$\varphi_{*p} : S_p \to T_{\varphi p}\boldsymbol{R}^n ; \qquad \varphi_{*p}([\varphi, p, v]) = v \ .$$

If $\psi \in \mathfrak{A}_S$ is also tangential at p, and $f_* : T_{\varphi p}\boldsymbol{R}^n \to T_{\psi p}\boldsymbol{R}^n$ is the unique linear map induced by local diffeomorphisms f extending $\psi \circ \varphi^{-1}$ near φp, then $f_* \circ \varphi_{*p} = \psi_{*p}$. Thus the bijections φ_{*p} and ψ_{*p} induce the same vector space structure on S_p.

Definition 2.3. The tangent space of S at p is the vector space S_p. We also denote S_p by T_pS.

Proposition 2.4. *If $f : S \to S'$ is a C^∞-mapping, $p \in S, \varphi \in \mathfrak{A}_S$ is tangential at $p, \varphi' \in \mathfrak{A}_{S'}$ is tangential at $f(p)$, and F is a C^∞-extension of $\varphi' \circ f \circ \varphi^{-1}$ in a neighborhood of $\varphi(p)$, then $F_{*\varphi p} : T_{\varphi p}\boldsymbol{R}^{n_\varphi} \to T_{\varphi' \circ f p}\boldsymbol{R}^{n_{\varphi'}}$ induces a linear map*

$$f_{*p} := (\varphi'_{*fp})^{-1} \circ F_{*\varphi p} \circ \varphi_{*p} : T_pS \to T_{fp}S' \ ,$$

and this map is independent of the choices of φ, θ, and F.

Proof. Let θ, θ' and G be alternate choices of φ, φ' and F, and let s and t be local C^∞-diffeomorphisms extending $\theta \circ \varphi^{-1}$ and $\theta' \circ \varphi'^{-1}$ near $\varphi(p)$ and $\varphi'(fp)$, respectively. We have already seen that $s_{*\varphi p} \circ \varphi_{*p} = \theta_{*p}$ and $t_{*\varphi'(fp)} \circ \varphi'_{*fp} = \theta'_{*fp}$. Since $G \circ s \circ \varphi = t \circ F \circ \varphi$ in a neighborhood of p, Lemma 2.1 implies $(G \circ s)_{*\varphi p} = (t \circ F)_{*\varphi p}$. An easy computation yields

$$(\theta'_{*fp})^{-1} \circ G_{*\theta p} \circ \theta_{*p} = (\varphi'_{*fp})^{-1} \circ F_{*\varphi p} \circ \varphi_{*p} \ .$$

Thus f_{*p} is independent of the choices of φ, φ' and F. Linearity of f_{*p} is obvious. q.e.d.

For each $f \in C^\infty(S, S')$ define $f_* : \bigcup_{p \in S} T_pS \to \bigcup_{q \in S'} T_qS'$ by $f_*|_{T_pS} = f_{*p}$. If $\varphi, \theta \in \mathfrak{A}_S$ have the same range space $\boldsymbol{R}^n$, $p \in U_\varphi \cap U_\theta$, and s is a local C^∞-diffeomorphism extending $\theta \circ \varphi^{-1}$ in a neighborhood V of $\varphi(p)$, then s_* is a C^∞-diffeomorphism extending $\theta_* \circ (\varphi_*)^{-1}$ in the bundle neighborhood $\tau^{-1}(V)$, where $\tau : T\boldsymbol{R}^n \to \boldsymbol{R}^n$ is the tangent projection. Now define TS to be $\bigcup_{p \in S} T_pS$ equipped with the topology induced by the bijections $\varphi_*, \varphi \in \mathfrak{A}_S$. TS is a Hausdorff space, and each bijection φ_* is a homeomorphism of TU_φ into $T\boldsymbol{R}^{n_\varphi}$. Then $\mathfrak{B}_{TS} := \{\varphi_* | \varphi \in \mathfrak{A}_S\}$ is a C^∞-atlas on TS.

Definition 2.5. Define $(TS, \mathfrak{A}_{TS})$ to be the C^∞-subcartesian space determined by $\mathfrak{B}_{TS}$. Define $\tau : TS \to S$ to be the footpoint projection.

Restricted to the subcategory of C^∞-manifolds, T agrees with the classical tangent functor. Let $\mathbf{0}: S \to TS$ be the zero section. Then $\mathbf{0}$ is of class C^∞. If $\varphi \in \mathfrak{A}_S$ is tangential at p, then $\varphi_* \in \mathfrak{A}_{TS}$ is tangential at $\mathbf{0}p$. If $X_p \in T_pS$ is not zero, however, then φ_* need not be tangential at X_p (see Example 2.7).

Theorem 2.6. *The correspondence* $T: S \mapsto TS$, $f \mapsto f_*$ *in* C^∞ *is a covariant functor, and the footpoint* map $\tau: TS \to S$ *is of class* C^∞.

Proof. The proof of functorality is straightforward. If $\varphi \in \mathfrak{A}_S$, then the tangent projection $T\boldsymbol{R}^{n_\varphi} \to \boldsymbol{R}^{n_\varphi}$ is a C^∞-extension of $\varphi \circ \tau \circ \varphi_*^{-1}$. Thus $\tau \in C^\infty(TS, S)$.

Proposition 2.7. *Let* S, S' *be* C^∞*-subcartesian spaces. Then there is a unique* C^∞*-diffeomorphism* $i: T(S \times S') \to TS \times TS'$ *such that for every* C^∞-map $f: S'' \to S \times S'$,

$$i \circ f_* = (\mathrm{proj}_S \circ f)_* \times (\mathrm{proj}_{S'} \circ f)_* \,.$$

Proof. The product $S \leftarrow S \times S' \to S'$ gives rise to the product mapping $\mathrm{proj}_{S*} \times \mathrm{proj}_{S'*}: T(S \times S') \to TS \times TS'$. Then

$$\begin{aligned}(\mathrm{proj}_{S*} \times \mathrm{proj}_{S'*}) \circ f_* &= \mathrm{proj}_{S*} \circ f_* \times \mathrm{proj}_{S'*} \circ f_* \\ &= (\mathrm{proj}_S \circ f)_* \times (\mathrm{proj}_{S'} \circ f)_* \,.\end{aligned}$$

Substituting the appropriate injections $S \to S \times S'$ and $S' \to S \times S'$ for f shows $\mathrm{proj}_{S*} \times \mathrm{proj}_{S'*}$ to be surjective. Let $\varphi \in \mathfrak{A}_S$ and $\theta \in \mathfrak{A}_{S'}$ be tangential at p and q, respectively. Since $\varphi \times \theta \in \mathfrak{A}_{S \times S'}$, $\dim T_{(p,q)}S \times S' \le n_\varphi + n_\theta = \dim T_pS \times T_qS'$, which together with surjectivity implies that $\mathrm{proj}_{S*} \times \mathrm{proj}_{S'*}$ is bijective. Thus $\dim T_{(p,q)}S \times S' = \dim T_pS \times T_qS'$, and $\varphi \times \theta$ is tangential at (p, q). Let $\mathrm{proj}_\varphi: \boldsymbol{R}^{n_\varphi + n_\theta} \to \boldsymbol{R}^{n_\varphi}$ and $\mathrm{proj}_\theta: \boldsymbol{R}^{n_\varphi + n_\theta} \to \boldsymbol{R}^{n_\theta}$ be the obvious C^∞-maps. Then a straightforward calculation shows

$$(\mathrm{proj}_{\varphi*} \times \mathrm{proj}_{\theta*}) \circ (\varphi \times \theta)_* = (\varphi_* \times \theta_*) \circ (\mathrm{proj}_{S*} \times \mathrm{proj}_{S'*}) \,.$$

Thus $\mathrm{proj}_{S*} \times \mathrm{proj}_{S'*}$ is of class C^∞. Similarly, $(\mathrm{proj}_{S*} \times \mathrm{proj}_{S'*})^{-1}$ is of class C^∞. Thus we take $i := \mathrm{proj}_{S*} \times \mathrm{proj}_{S'*}$. The uniqueness of i is obvious.

q.e.d.

Let $X_p \in T_pS$, and $\varphi \in \mathfrak{A}_S$ be tangential at p. Let $f \in C^\infty(S)$ and let f_φ be a C^∞-map extending $f \circ \varphi^{-1}$ in some neighborhood of φp. Then $(\varphi_* X_p) \cdot f_\varphi = Df_\varphi(\varphi p, \varphi_* X_p) \in \boldsymbol{R}$ is independent of the choice of f_φ by virtue of Lemma 2.1. If $\theta \in \mathfrak{A}_S$ is also tangential at p, then $(\varphi_* X_p) \cdot f_\varphi = (\theta_* X_p) \cdot f_\theta$. We denote this real number by $X_p \cdot f$. Each X_p thus determines a real-valued derivation on the ring of germs of C^∞-functions at p. Conversely, if D is such a derivation at p, then there is a unique $X_p \in T_pS$ such that $X_p \cdot [f] = D([f])$ for every germ $[f]$ at p. (The proof carries over unchanged from the case $S = \boldsymbol{R}^n$.) If D is a derivation on $C^\infty(S)$, then for every $p \in S$, $f \mapsto Df(p)$ is an $\boldsymbol{R}$-valued derivation. It follows as usual that this derivation factors through the natural map $f \mapsto [f]_p$ sending f into its germ at p. Then to each $p \in S$ there is a unique $X_p \in T_pS$

such that $(Df)p = X_p \cdot f$ for every $f \in C^\infty(S)$. Now if $\varphi \in \mathfrak{A}_S$ and $p \in U_\varphi$, then define $a^i = D(x^i \circ \varphi) \in C^\infty(U_\varphi)$, where $x^1, \cdots, x^{n_\varphi}$ is the standard coordinate system on $\boldsymbol{R}^{n_\varphi}$. Let U be a neighborhood of φp on which there are local representatives b^i of $a^i, i = 1, \cdots, n_\varphi$. Then $q \mapsto X_q$ is smooth because

$$\varphi_* X_q = \sum_{i=1}^{n_\varphi} b^i(\varphi q) \frac{\partial}{\partial x^i}(\varphi q) \qquad \text{for all } q \in \varphi^{-1}U \ .$$

Thus, as in the case of C^∞-manifolds, each derivation on $C^\infty(S)$ can be realized by a unique C^∞-vector field on S.

If $g \in C^\infty(S, R), f \in C^\infty(R)$ and $X_p \in T_pS$, then $(g_* X_p) \cdot f = X_p \cdot (g^* f)$, where as usual $g^* f := f \circ g$.

Example 2.8. Let S be the union of the x, y-plane and the positive z-axis in $\boldsymbol{R}^3$. The inclusion is a chart tangential at $(0, 0, 0)$ and determines a C^∞-structure on S. $n_{S,(0,0,0)} = 3$, and $n_{S,(x,y,0)} = 2$ and $n_{S,(0,0,z)} = 1$ for all x, y, z different from 0. The tangential dimensions of TS at points $X_p, p \neq (0, 0, 0)$, are $2n_{S,p}$. Let $X = (\partial/\partial x)(0, 0, 0)$ and $Z = (\partial/\partial z)(0, 0, 0)$. Then $n_{TS,X+Z} = 3$, $n_{TS,Z} = 4$, $n_{TS,X} = 5$ and $n_{TS,\mathbf{0}(0,0,0)} = 6$.

The structural dimension $n_{S,p}$ clearly dominates the topological dimension dim (U) of a sufficiently small neighborhood U of p. The difference, $n_{S,p}$-dim (U) however, can be arbitrarily large.

Example 2.9. Let $g_0 : [0, 1] \rightarrow \boldsymbol{R}$ be continuous but nowhere differentiable, and define $g_{k+1}(t) = \int_0^t g_k(x)dx$. Define $S \subseteq \boldsymbol{R} \times \boldsymbol{R}^{m+1}$ to be the graph of $t \mapsto (g_0(t), \cdots, g_m(t))$. Then for every $p \in S, n_{S,p} = m + 2$ while the topological dimension of S is 1.

A differentiable subcartesian space S together with its sheaf of smooth functions is a reduced differentiable space, [13]. When S is an analytic space, the tangent space T_pS coincides with what Whitney calls the full or Zariski tangent space, [16]. When a differentiable space S is a differentiable subcartesian space, then its C_3-tangent space coincides with T_pS.

The construction of the C^∞-tangent functor can be similarly carried through for other smoothness categories. If S is structured with $\mathscr{C}^+$, then TS is structured with $\mathscr{C}$, e.g., if S is of class C^{k+1}, then TS is of class C^k.

3. Contravariant tensors and tensor fields

The purpose of this section is to introduce contravariant tensors and tensor fields on subcartesian spaces. Rather than repeat certain arguments for each kind of tensor to be considered, we shall begin with a convenient common generalization.

A *C^∞-family of $\boldsymbol{R}$-vector spaces* is a pair of C^∞-subcartesian spaces B and S, and a mapping $\pi \in C^\infty(B, S)$ onto S such that for each $p \in S$, $\pi^{-1}(p)$ has an

$\boldsymbol{R}$-vector space structure and such that the vector operations

$$+ : B \times_S B \to B \quad \text{and} \quad \cdot : \boldsymbol{R} \times B \to B$$

are of class C^∞. A *morphism* $(B, \pi, S) \to (E, \tau, R)$ is a pair $f \in C^\infty(B, E)$ and $g \in C^\infty(S, R)$ such that $\tau \circ f = g \circ \pi$ and such that f is linear along fibers. Since f determines g, we shall often denote the pair simply by f. We shall also sometimes denote g by f_b. As usual we denote the set of C^∞-sections in $\xi = (B, \pi, S)$ by $\Gamma\xi$. If $\Gamma\xi \neq \emptyset$, then $\Gamma\xi$ is a $C^\infty(S)$-module. If $\xi = (B, \pi, S)$ and $f \in C^\infty(R, S)$ then the fiber space pull-back $f^\dagger\xi := (R \times_S B, \pi_R, R)$ (cf. [5]) is a C^∞-family, and $R \leftarrow R \times_S B \to B$ is the pull-back of $R \to S \leftarrow B$ in $\boldsymbol{C}^\infty$. As usual, we denote $\pi^{-1}(p)$ also by ξ_p or ξ^p and $f|_{\xi_p}$ by f_p or f^p.

Definition 3.1. A C^∞-vector pseudo-bundle is a C^∞-family of $\boldsymbol{R}$-vector spaces $\xi = (B, \pi, S)$ such that $\mathfrak{A}_B$ has a subatlas $\mathfrak{B}$ whose charts satisfy the following two conditions:

(VPB1) For each $\beta \in \mathfrak{B}$, $U_\beta = \pi^{-1}\pi U_\beta$.

(VPB2) Each $\beta \in \mathfrak{B}$ is a morphism $U_\beta \to (\boldsymbol{R}^{n_\beta}, \pi_{n,n_\beta}, \boldsymbol{R}^n)$ of C^∞-families, where $n = n_{\beta_b}$.

Such an atlas $\mathfrak{B}$ is called a *pseudo-bundle atlas* on B. If S is a C^∞-subcartesian space, then (TS, τ, S) is a C^∞-vector pseudo-bundle, and $\{\varphi_* \mid \varphi \in \mathfrak{A}_S\}$ is the maximal pseudo-bundle atlas on TS. Similarly, if $\xi = (B, \pi, S)$ is an arbitrary C^∞-vector pseudo-bundle, then B has a unique maximal pseudo-bundle atlas $\mathfrak{B}_\xi \subseteq \mathfrak{A}_B$. If $\beta \in \mathfrak{B}_\xi$, then $\beta_b \in \mathfrak{A}_S$, and for every $\varphi \in \mathfrak{A}_S$ and $p \in U_\varphi$ there exists a $\beta \in \mathfrak{B}_\xi$ with $p \in \pi U_\beta \subseteq U_\varphi$. The zero section $\mathbf{0}: S \to B$ is of class C^∞; thus for C^∞-vector pseudo-bundles ξ, $\Gamma\xi \neq \emptyset$. Any family of subspaces $\{\zeta_p \subseteq \xi_p \mid p \in S\}$ with the induced structure on $\bigcup_{p \in S} \zeta_p$ forms a C^∞-vector pseudo-bundle.

A *morphism* $f: \xi \to \zeta$ *of* C^∞*-vector pseudo-bundles* is simply a morphism of C^∞-families. The resulting category VPB contains as full subcategories the C^∞-vector bundles over C^∞-subcartesian spaces and, more specially, the C^∞-vector bundles over C^∞-manifolds. We denote by $VPB(S)$ the subcategory of pseudo-bundles over a fixed space S and morphisms f satisfying $f_b = \mathrm{Id}_S$. Note that $VPB(S)$ is abelian without any additional conditions being imposed on the morphisms. If $f \in C^\infty(R, S)$, and $\xi = (B, \pi, S)$ is a C^∞-vector pseudo-bundle, then $f^\dagger\xi$ is also a C^∞-vector psesudo-bundle.

Proposition 3.2. *Let* $f: (B, \pi, S) \to (E, \tau, R)$ *be a morphism of* C^∞*-vector pseudo-bundles. If* $\alpha \in \mathfrak{B}_{(B,\pi,S)}$ *and* $\beta \in \mathfrak{B}_{(E,\tau,R)}$ *with* $f(U_\alpha) \subseteq U_\beta$, *and if* $p \in U_\varphi$, *then there exist a neighborhood* V *of* φp (*where* $\varphi := \alpha_b$) *and a* C^∞*-vector bundle map*

$$F: (\boldsymbol{R}^{n_\alpha}, \pi_{n_\varphi, n_\alpha}, V) \to (\boldsymbol{R}^{n_\beta}, \pi_{n_\psi, n_\beta}, \boldsymbol{R}^{n_\psi}), \qquad (\psi := \beta_b)$$

extending $\beta \circ f \circ \alpha^{-1}|_{(\pi^{-1}_{n_\varphi, n_\alpha} V) \cap U_\alpha}$.

Proof. There is a C^∞-extension g of $\beta \circ f \circ \alpha^{-1}$ in some neighborhood W of $\alpha \circ \mathbf{0}(p)$. Let $V := \pi(\mathbf{0}(S) \cap W)$, and define $F\colon V \times \boldsymbol{R}^{n_\alpha - n_\varphi} \to \boldsymbol{R}^{n_\psi} \times \boldsymbol{R}^{n_\beta - n_\psi}$;

$$(q, v) \mapsto (\tau g(q, 0), D_2 P \circ g_{(q,0)}(v)) \ ,$$

where $P\colon \boldsymbol{R}^{n_\psi} \times \boldsymbol{R}^{n_\beta - n_\psi} \to \boldsymbol{R}^{n_\beta - n_\psi}$, and D_2 is the partial differential along $\boldsymbol{R}^{n_\alpha - n_\varphi}$. This F satisfies the requirements of the proposition.

Proposition 3.3. *A C^∞-mapping $\gamma\colon S \to B$ belongs to $\Gamma(B, \pi, S)$ if and only if for every $\beta \in \mathfrak{B}_{(B,\pi,S)}$ and every $p \in \pi U_\beta$, there exists a local section σ in $(\boldsymbol{R}^{n_\beta}, \pi_{n,n_\beta}, \boldsymbol{R}^n)$, where $n := n_{\beta_b}$, extending $\beta \circ \gamma \circ \beta_b^{-1}$ near $\beta_b p$. σ is called a local representative of γ relative to β.*

Proof. Sufficiency is clear. To show necessity, let f be any C^∞-mapping extending $\beta \circ \gamma \circ \beta_b^{-1}$ in a neighborhood of $\beta_b p$. Then $\sigma\colon q \mapsto (q, P \circ fq)$ is a smooth local section extending $\beta \circ \gamma \circ \beta_b^{-1}$ near $\beta_b p$. q.e.d.

Combining the arguments of Corollary 2.2 and Proposition 3.2 we see that if $\alpha, \beta \in \mathfrak{B}_\xi$ take their values in the same trivial bundle $(\boldsymbol{R}^n, \pi_{m,n}, \boldsymbol{R}^m)$, then the connecting map $\alpha \circ \beta^{-1}$ admits local extensions which are C^∞-vector bundle isomorphisms. With this observation the proof of the following theorem becomes routine, following as in the case of differentiable vector bundles over differentiable manifolds (cf. [8]).

Theorem 3.4. *Let λ be a C^∞-covariant functor of k arguments on the category of finite dimisional $\boldsymbol{R}$-vector spaces. There is then a unique functor Λ on VPB satisfying the following conditions:*

(i) *For all $S \in \boldsymbol{C}^\infty$, $\Lambda(\mathrm{VPB}(S)) \subseteq \mathrm{VPB}(S)$.*

(ii) *For all $S \in \boldsymbol{C}^\infty$ and $\xi_i \in \mathrm{VPB}(S), i = 1, \cdots, k$, and every $p \in S$, $\Lambda(\xi_1, \cdots, \xi_k)_p = \lambda(\xi_1^p, \cdots, \xi_k^p)$.*

(iii) *For any $f^i\colon \xi_i \to \zeta_i$ in $\mathrm{VPB}(S), i = 1, \cdots, k$, and every $p \in S$, $\Lambda(f^1, \cdots, f^k)_p = \lambda(f_p^1, \cdots, f_p^k)$.*

(iv) *If $\xi_i, i = 1, \cdots, k$, are the trivial bundles $S \times \boldsymbol{R}^{n_i}$, then $\Lambda(\xi_1, \cdots, \xi_k)$ is the trivial bundle $S \times \lambda(\boldsymbol{R}^{n_1}, \cdots, \boldsymbol{R}^{n_k})$.*

(v) *If $h \in C^\infty(R, S)$, then $\Lambda(h^\dagger\xi_1, \cdots, h^\dagger\xi_k) = h^\dagger\Lambda(\xi_1, \cdots, \xi_k)$.*

Further, if λ' is another such covariant C^∞-functor, and $t\colon \lambda \to \lambda'$ is a natural transformation, then the mapping $T\colon \Lambda \to \Lambda'$ defined on each fiber by

$$T(\xi_1, \cdots, \xi_k)_p = t(\xi_1^p, \cdots, \xi_k^p)$$

is a natural transformation of functors on VPB(S).

Let $S \in \boldsymbol{C}^\infty$. In view of the previous theorem we have the C^∞-vector pseudo-bundles $\otimes^k TS$, $\wedge^k TS = \mathrm{Alt} \otimes^k TS$ (the *pseudo-bundle of k-vectors*), $\circledcirc^k TS = \mathrm{Sym} \otimes^k TS$ (the *pseudo-bundle of symmetric tensors of rank k*), and their modules of sections $\mathscr{T}_k(S), \mathscr{X}^k(S)$, and $\mathscr{S}^k(S)$. Alt and Sym are the obvious natural transformations. If $f \in C^\infty(S, R)$, we shall often write simply f_* for any of the induced maps $\otimes^k f_*$, etc. The externally graded modules $\mathscr{T}(S) = \{\mathscr{T}_k(S) \mid k \geq 1\}$, $\mathscr{X}^\bullet(S) = \{\mathscr{X}^k(S) \mid k \geq 1\}$, and $\mathscr{S}(S) = \{\mathscr{S}^k(S) \mid k \geq 1\}$ with the point-wise defined operations (i.e., as provided by the theorem) form graded $C^\infty(S)$-algebra with the obvious symmetry properties. In contrast with the case of differentiable manifolds, these algebras need not be generated locally by $\mathscr{X}(S) := \mathscr{X}^1(S)$, nor need the homogeneous submodules be locally free, nor locally of finite type. These facts are born out by the following two examples.

Example 3.5. Let S be the space of Example 2.8. Then every $X \in \mathscr{X}(S)$ vanishes at $(0,0,0)$, every $Y \in \mathscr{X}^2(S)$ vanishes along the z-axis, and every $Z \in \mathscr{X}^3(S)$ vanishes identically. Every $X \in \mathscr{X}(S) \wedge \mathscr{X}(S) \subseteq \mathscr{X}^2(S)$ has a zero of order 2 at $(0,0,0)$, but $x\partial/\partial x \wedge \partial/\partial y \in \mathscr{X}^2(S)$ has a zero only of order 1 at $(0,0,0)$.

Example 3.6. Let $S \subseteq \boldsymbol{R}$ be the closed left half-line together with the points $\{1/n \mid n \in \boldsymbol{N}\}$. Let $\mathfrak{i} = \{[f]_0 \in C^\infty(S)_0 \mid f(1/n) = 0, \forall n \in \boldsymbol{N}\}$. Then $\mathscr{X}(S)_0 = \mathfrak{i}[\partial/\partial x]_0$ is neither free nor finitely generated over $C^\infty(S)_0$. Since $C^\infty(S)_0$ has a unique maximal ideal, it follows from a theorem of Kaplansky [6] that $\mathscr{X}(S)_0$ cannot be projective, contrasting further the difference between vector bundles and vector pseudo-bundles.

4. Covariant tensor fields and alternating forms

Definition 4.1. Let $\xi = (B, \pi, S)$ be a C^∞-vector pseudo-bundle. Define $\mathscr{F}(\xi) = \mathrm{VPB}\,(S)(\xi, S \times \boldsymbol{R})$, that is, the $C^\infty(S)$-module of footpoint-preserving morphisms of ξ into the trivial line bundle over S. Let $P: S \times \boldsymbol{R} \to \boldsymbol{R}$ be the principal part projection. We write $\mathscr{F}^k(S)$ for $\mathscr{F}(\otimes^k TS)$, $\mathscr{F}^k_{\mathrm{alt}}(S)$ for $\mathscr{F}(\wedge^k TS)$, $\mathscr{F}^k_{\mathrm{sym}}(S)$ for $\mathscr{F}(\circledcirc^k TS)$, and $\mathscr{F}^0(S)$ for $\Gamma(S \times \boldsymbol{R})$. The elements of $\mathscr{F}^k_{\mathrm{alt}}(S)$ are called k-forms.

We could equivalently define $\mathscr{F}(\xi)$ to be the smooth functions on B which are linear along fibers.

From Proposition 3.2 it follows that $\phi \in \mathscr{F}(\xi)$ if and only if for each $\beta \in \mathfrak{B}_\xi$ with $\varphi := \beta_b$, and each $p \in U_\varphi$, there exist a neighborhood V of φp and a C^∞-section

$$\omega \in \Gamma(V \times (\boldsymbol{R}^{n_\beta - n_\varphi})^*)$$

such that $\omega \circ \beta|_{\pi^{-1}(U_\varphi \cap \varphi^{-1}V)} = \phi|_{\pi^{-1}(U_\varphi \cap \varphi^{-1}V)}$. Such an ω is called a *local representative of ϕ relative to β* (on $U_\varphi \cap \varphi^{-1}V$). From Theorem 3.4, $\phi \in \mathscr{F}(\Lambda\xi)$ if and only if ω can be chosen from $\Gamma(V \times (\lambda \boldsymbol{R}^{n_\beta - n_\varphi})^*)$. Thus, for example, $\phi \in \mathscr{F}^k_{\mathrm{alt}}(S)$ if and only if it has local representatives which are alternating forms of rank k.

Condition (iii) of Theorem 3.4 defines

$$\Lambda(\phi^1, \cdots, \phi^k) \in \mathrm{VPB}\,(S)(\Lambda(\xi_1, \cdots, \xi_k), S \times \lambda\boldsymbol{R}) .$$

Equipped with the obvious corresponding operations, $\mathscr{F}(S) := \{\mathscr{F}^k(S) \mid k \geq 0\}$, $\mathscr{F}_{\mathrm{alt}}(S) := \{\mathscr{F}^k_{\mathrm{alt}}(S) \mid k \geq 0\}$, and $\mathscr{F}_{\mathrm{sym}}(S) := \{\mathscr{F}^k_{\mathrm{sym}}(S) \mid k \geq 0\}$ are graded $C^\infty(S)$-algebras. As with the contravariant tensor fields, the modules of germs

$\mathscr{F}^k(S)_p$ need not be free (see Example 4.6). In distinction with the contravariant case, however, the modules $\mathscr{F}(S)$, $\mathscr{F}_{\mathrm{alt}}(S)$ and $\mathscr{F}_{\mathrm{sym}}(S)$ are locally generated (in positive degrees) by $\mathscr{F}^1(S)$.

If $f \in \mathrm{VPB}\,(\xi, \zeta)$, then

$$f^* : \mathscr{F}(\zeta) \to \mathscr{F}(\xi) ; \qquad f^*\phi = \phi \circ f$$

is a functorial homomorphism of $\boldsymbol{R}$-algebras. From Theorem 3.4,

$$(4.2) \qquad f^* \Lambda(\phi^1, \cdots, \phi^k) = \Lambda(f^*\phi^1, \cdots, f^*\phi^k) \,.$$

If $g \in C^\infty(S, R)$, then

$$(4.3) \qquad ((\Lambda g_*)^*\phi)X = \phi(\Lambda g_* X) \,, \quad \phi \in \mathscr{F}(\Lambda TS) \,, \quad X \in \Lambda TS \,.$$

We shall usually write simply g^* for $(\Lambda g_*)^*$.

Lemma 4.4. *Let $\xi = (B_\xi, \pi_\xi, S)$ and $\zeta = (B_\zeta, \pi_\zeta, S)$ be C^∞-vector pseudobundles. Let $\mu \colon B_\xi \to B_\zeta$ be linear along fibers and satisfy $\pi_\zeta \circ \mu = \pi_\xi$. Then a necessary and sufficient condition for μ to be a* VPB (S)*-morphism is that $\omega \circ \mu$ be smooth for every $\omega \in \mathscr{F}(\zeta)$.*

Proof. Necessity is obvious. To prove the converse, it is sufficient to consider only the case $S \subseteq \boldsymbol{R}^n, B_\xi \subseteq \boldsymbol{R}^n \times \boldsymbol{R}^k, B_\zeta \subseteq \boldsymbol{R}^n \times \boldsymbol{R}^m$. The proof then follows by a straightforward use of local coordinates.

Proposition 4.5. *Let $\xi, \zeta \in$* VPB (S). *Suppose that for every $p \in S$ there is a $\beta \in \mathfrak{B}_\zeta$ with $p \in U_\varphi$ and* $\dim \zeta_p = n_\beta - n_\varphi$, *where $\varphi := \beta_b$. Then the natural map*

$$\kappa \colon \mathrm{VPB}\,(S)(\xi, \zeta) \to \mathrm{Hom}_{C^\infty(S)}\,(\mathscr{F}(\zeta), \mathscr{F}(\xi)) ; \qquad \kappa\mu = \mu^*$$

is an isomorphism of $C^\infty(S)$-modules.

Proof. Evidentally, κ is a monomorphism. Let $h \in \mathrm{Hom}_{C^\infty(S)}\,(\mathscr{F}(\zeta), \mathscr{F}(\xi))$. If $\phi \in \mathscr{F}(\zeta)$ satisfies $\phi|_U = 0$ for some open $U \subseteq S$, then $(h\phi)|_U = 0$, i.e, h is local. We show that h is punctual. Let $p \in S$, $\phi \in \mathscr{F}(\zeta)$ with $\phi p = 0$, and $\beta \in \mathfrak{B}_\zeta$ with $\varphi := \beta_b$ and $n_\beta = n_\varphi + \dim \zeta_p =: n_\varphi + m$. Then there exist a neighborhood U of φp and a local representative $\omega \in \Gamma(U \times \boldsymbol{R}^m)^*$ of ϕ relative to β. Writing $\omega = \sum_{i=1}^m a_i e^i$, where $a_i \in C^\infty(U)$ and e^i is the canonical i^{th}-coordinate section in $(U \times \boldsymbol{R}^m)^*$, we have for each $q \in \varphi^{-1}U$

$$(h\varphi^*\omega)_q = \sum_{i=1}^m a_i(\varphi q)(h\varphi^* e^i)_q \,.$$

Since $\dim \zeta_p = m$, it follows that $a_i(\varphi_p) = 0, i = 1, \cdots, m$. Thus $(h\phi)_p = 0$. Then for each $p \in S$ there is a unique linear map $h_p : \zeta_p{}^* \to \xi_p{}^*$ such that for all $\phi \in \mathscr{F}(\zeta), (h\phi)_p = h_p(\phi_p)$. For each $p \in S$, define $\mu_p = h_p{}^* : \xi_p \to \zeta_p$. To show that $p \mapsto \mu_p$ is smooth, it suffices to show that $p \mapsto \phi \circ \mu_p$ is smooth for each $\phi \in \mathscr{F}(\zeta)$ (Lemma 4.4). But $\phi \circ \mu_p = h_p(\phi_p) = (h\phi)_p$. Thus $\mu: p \mapsto \mu_p$ belongs to VPB $(S)(\xi, \zeta)$, and $\kappa\mu = h$.

Corollary 4.6. *Let $\xi \in$ VPB (S). Then the natural* map

$$\kappa: \Gamma\xi \to \mathrm{Hom}_{C^\infty(S)}(\mathscr{F}(\xi), C^\infty(S))$$

is an isomorphism of $C(S)^\infty$-modules.

The dual module of $\Gamma\xi$, however, need not be $\mathscr{F}(\xi)$ (cf. Example 4.9). Trivial examples show that in general the dimensional hypothesis in Proposition 4.5 cannot be avoided.

Example 4.7. Let $S = \{0\} \cup \{1/n \mid n \in \boldsymbol{N}\} \subseteq \boldsymbol{R}^1$ be equipped with the induced C^∞-structure. Then $n_{S,0} = 1$ and $n_{S,1/n} = 0$. If i is the inclusion $S \hookrightarrow \boldsymbol{R}$, then i^*dx generates $\mathscr{F}^1(S)$, and i^*dx is nonzero only at 0. If $x \in C^\infty(\boldsymbol{R})$ is the identity map, then the germs $[i^*x]_0$ and $[i^*dx]_0$ are nonzero, but $[i^*x]_0[i^*dx]_0 = 0$. Thus $\mathscr{F}^1(S)$ is not free.

Example 4.8. Let S be as in Example 3.5, and $i: S \hookrightarrow \boldsymbol{R}^3$. Then $i^*(dx + dz)$ is nowhere 0 on S, $i^*(zdx \wedge dy)$ is identically 0 on S, and $i^*(dx \wedge dy \wedge dz)$ is nonzero only at $(0, 0, 0)$. $\mathscr{F}^3_{\mathrm{alt}}(S)$ is generated by $i^*(dx \wedge dy \wedge dz)$, but

$$i^*(x^2 + y^2 + z^2)dx \wedge dy \wedge dz = 0.$$

Example 4.9. Every $\phi \in \mathscr{F}(\xi)$ gives an element of $\mathrm{Hom}_{C^\infty(S)}(\Gamma\xi, C^\infty(S))$, but the converse is not true. Let S be the space of Example 3.6. Define $\omega(x) = (1/x)dx$ for $x < 0$ and 0 otherwise. Then $i^*\omega$ maps $\mathscr{X}(S)$ into $C^\infty(S)$ linearly but is not an element of $\mathscr{F}(S)$.

Let $f: \xi \to \zeta$ be a morphism in VPB (S), $\alpha \in \mathfrak{B}_\xi$, $\beta \in \mathfrak{B}_\zeta$ with $\varphi := \alpha_b = \beta_b$, $m := n_\alpha - n_\varphi$, and $n := n_\beta - n_\varphi$. Let $p \in U_\varphi$, let U be a neighborhood of φp, and suppose $F: U \to \mathrm{Hom}(\boldsymbol{R}^m, \boldsymbol{R}^n)$ extends $q \mapsto \beta \circ f \circ \alpha_q^{-1}$, $q \in U$, i.e., F is a *local representative of f relative to α and β*. Then with the usual interpretation we write $F = \sum y_i \otimes x^j$, where $i = 1, \cdots, n$, $j = 1, \cdots, m$, $y_i \in C^\infty(U, \boldsymbol{R}^n)$, and $x^j \in C^\infty(U, \boldsymbol{R}^{m*})$. When $\xi = \otimes^k \eta$, $\zeta = \otimes^l \eta$, we have

$$F = \sum_{\mu,\nu} y_{\mu_1} \otimes \cdots \otimes y_{\mu_l} \otimes x^{\nu_1} \otimes \cdots \otimes x^{\nu_k},$$

where $y_{\mu_i} \in \Gamma(U \times \boldsymbol{R}^n)$ and $x^{\nu_i} \in \Gamma(U \times \boldsymbol{R}^m)^*$. Applying the contraction C^i_j, we obtain

$$C^i_j(f_p) = \otimes^{l-1} \beta_p{}^{-1} \circ C^i_j F \circ \otimes^{k-1} \alpha_p .$$

Thus $C^i_j f: \otimes^{k-1} \eta \to \otimes^{l-1} \eta$ is a VPB (S)-morphism.

Definition 4.10. Let $\mathscr{F}^k_m(S)$ denote the $C^\infty(S)$-module VPB $(S)(\otimes^k TS, \otimes^m TS)$. We call $\mathscr{F}^k_m(S)$ the module of tensor fields of type (k, m). We shall also denote $\mathscr{F}^k(S)$ by $\mathscr{F}^k_0(S)$.

Clearly, $C^i_j: \mathscr{F}^k_m(S) \to \mathscr{F}^{k-1}_{m-1}(S)$ is a natural homomorphism of $C^\infty(S)$-modules. If $\omega \in \mathscr{F}^k_{\mathrm{alt}}(S)$ and $X \in \mathscr{X}(S)$, then as usual we define

$$i_X\omega = \begin{cases} \frac{1}{k} C_1^1 X \otimes \omega \ , & k \geq 1 \ , \\ \quad 0 \qquad , & k = 0 \ . \end{cases}$$

Thus for $X_2, \cdots, X_k \in T_pS$,

$$i_X\omega(p)(X_2 \wedge \cdots \wedge X_k) = \omega(p)(X(p) \wedge X_2 \cdots \wedge X_k) \ .$$

Definition 4.11. Let ξ be a vector pseudo-bundle. Define $\check{\xi}$ to be the vector pseudo-bundle whose fibers are

$$\check{\xi}_p := \{\gamma(p) \in \xi_p \,|\, \gamma \in [\Gamma\xi]_p\}$$

and whose total space structure is that induced from the total space of ξ.

For any C^∞-functor λ on the category of $\boldsymbol{R}$-vector spaces,

$$\Lambda(\check{\xi}_1, \cdots, \check{\xi}_k) \subseteq (\Lambda(\xi_1, \cdots, \xi_k))^\vee \ .$$

Proposition 4.12. *Let $\xi_i \in$ VPB (S), $i = 1, \cdots, k$. Then*

$$\bigotimes_{i=1}^k \check{\xi}_i = \left(\bigotimes_{i=1}^k \xi_i\right)^\vee \ .$$

If all $\xi_i = \xi$, then

$$\wedge^k \check{\xi} = (\wedge^k \xi)^\vee \ .$$

Proof. Let $X \in \Gamma \bigotimes_{i=1}^k \xi_i$ and $p \in S$. Assume without loss of generality that $Xp = X_1^p \otimes \cdots \otimes X_k^p$, where $X_i^p \in \xi_i^p$. For each i, let $\phi^i \in \mathscr{F}(\xi_i)$ be such that $\phi_p^i X_i^p = 1$, and define

$$\psi^j : \bigotimes_{i=1}^k \xi_i \to \xi_j ; \qquad \psi_q^j(Y_1^q \otimes \cdots \otimes Y_k^q) = \Big(\prod_{i \neq j} \phi_q^i Y_i^q\Big) Y_j^q \ ,$$
$$Y_i^q \in \xi_i^q , \ q \in S \ .$$

It is routine to show that each ψ^j is a VPB (S)-morphism. Thus $\psi^j X \in \Gamma\xi_j$ for each j. Since $\psi^1 X \otimes \cdots \otimes \psi^k X(p) = Xp$, the first part is proved.

To see the second, note that for each $p \in S$, $(\wedge^k \xi)_p^\vee \subseteq \otimes^k \check{\xi}_p$, and that each element X_p of the former is alternating. Then $X_p \in \mathrm{Alt} \otimes^k \check{\xi}_p = \wedge^k \check{\xi}_p$.

Example 4.13. It is not generally true that a morphism $f : \xi \to \zeta$ will map $\check{\xi}$ into $\check{\zeta}$. Let S be the space of Example 4.8 and let $j : \boldsymbol{R}^2 \hookrightarrow S$ be the obvious inclusion. Then

$$j_* \check{T}_{(0,0)} \boldsymbol{R}^2 = \operatorname{span}_{\boldsymbol{R}} \left\{ \frac{\partial}{\partial x}(0,0,0), \frac{\partial}{\partial y}(0,0,0) \right\} \subseteq T_{(0,0,0)} S ,$$

while $\check{T}_{(0,0,0)} S = 0$.

There seems to be no theorem analogous to Theorem 3.4 for contravariant functors F. The trouble is that for each $\beta \in \mathfrak{B}_\xi$, $F(\beta)$ maps in the wrong direction in order to be a chart on $\bigcup_{p \in S} F\xi_p$, and it may not be fiber-wise invertible (e.g., when F is the dual-space functor). When S is paracompact, there exist Riemannian metrics on ξ, and one of these may be used to equip $\bigcup_{p \in S} \xi_p^* = \xi^*$ with a VPB-structure. Then $\xi^* \cong \xi$ and $\Gamma\xi \cong \Gamma\xi^* \subseteq \mathscr{F}(\xi)$, where as in Examples 3.5–4.8 the inclusion may be proper. The module $\Gamma\xi^*$ is, however, unsuitably small for the applications which follow.

5. Lie derivatives of tensor fields

We first consider Lie derivatives of contravariant tensor fields. Let $X \in \mathscr{X}(S)$ and $Y \in \mathscr{T}_k(S)$. Let $\varphi \in \mathfrak{A}_S$, $p \in U_\varphi$, and suppose X_1 and Y_1 are local representatives relative to φ of X and Y in some neighborhood V of φp. We shall show that $\mathscr{L}_{X_1} Y_1$ is a local representative of some $W \in \mathscr{T}_k(S)$ on V relative to φ, and that W_p is independent of the choices of φ, X_1 and Y_1.

We first show that $\mathscr{L}_{X_1} Y_1(\varphi p)$ is independent of the choices of the local representatives X_1 and Y_1. Suppose Y vanishes on $\varphi^{-1} V$. Writing

$$Y_1 = \sum_\alpha a^\alpha \frac{\partial}{\partial x^{\alpha_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\alpha_k}}$$

we have

$$\begin{aligned} \mathscr{L}_{X_1} Y_1(\varphi p) &= \sum_\alpha (X_1(\varphi p) \cdot a^\alpha) \frac{\partial}{\partial x^{\alpha_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\alpha_k}}(\varphi p) \\ &\quad + \sum_i \sum_\alpha a^\alpha(\varphi p) \frac{\partial}{\partial x^{\alpha_1}} \otimes \cdots \otimes \left[X_1, \frac{\partial}{\partial x^{\alpha_i}} \right] \otimes \cdots \otimes \frac{\partial}{\partial x^{\alpha_k}}(\varphi p) . \end{aligned} \tag{5.1}$$

The second term on the right vanishes because a^α vanishes on $\varphi U_\varphi \cap V$, and the first vanishes by virtue of Lemma 2.1. Thus $\mathscr{L}_{X_1} Y_1(\varphi p)$ is independent of the choice of Y_1.

Now suppose that X vanishes on $\varphi^{-1} V$ and Y is arbitrary. Since $\mathscr{L}_{X_1} Y_1$ is invariant under diffeomorphisms, we may assume without loss of generality that $\varphi U_\varphi \cap V \subseteq i_{n_\varphi, n} \boldsymbol{R}^n \subseteq \boldsymbol{R}^{n_\varphi}$, where $n = n_{S,p}$ and V is chosen sufficiently small. From (5.1),

$$\mathscr{L}_{X_1} Y_1(\varphi p) = \sum_i \sum_\alpha a^\alpha(\varphi p) \frac{\partial}{\partial x^{\alpha_1}} \otimes \cdots \otimes \left[X_1, \frac{\partial}{\partial x^{\alpha_i}} \right] \otimes \cdots \otimes \frac{\partial}{\partial x^{\alpha_k}}(\varphi p) ,$$

where $a^\alpha(\varphi p) = 0$ for every α having some $\alpha_j > n$. Then

$$P\left[X_1, \frac{\partial}{\partial x^{\alpha_i}}\right](\varphi p) = \frac{\partial}{\partial x^{\alpha_i}} P X_1(\varphi p) \ ,$$

where P is the principal part projection. Since

$$P X_1|_{\varphi U_\varphi \cap V} = 0 \quad \text{and} \quad \frac{\partial}{\partial x^{\alpha_i}}(\varphi p) \in \varphi_* T_p S$$

for all of those α with $a^\alpha(\varphi p) \neq 0$, Lemma 2.1 implies $\mathscr{L}_{X_1} Y_1(\varphi p) = 0$. Thus $\mathscr{L}_{X_1} Y_1(\varphi p)$ is also independent of the choice of X_1.

Before continuing, we remark that when $X = 0$ in a neighborhood of p, then the above argument shows that

$$(5.2) \qquad (\mathscr{L}_{X_1} Z)\varphi p = 0$$

for any $Z \in \mathscr{T}_k(\boldsymbol{R}^{n_\varphi})$ satisfying $Z(\varphi p) \in \varphi_* \otimes^k T_p S$.

Having shown $\mathscr{L}_{X_1} Y_1(\varphi p)$ to be independent of the choices of X_1 and Y_1 for fixed φ, we now show coordinate invariance. For any $n \geq n_\varphi$ and any local representatives X_2 and Y_2 relative to $i_{n,n_\varphi} \circ \varphi$, and for any local extensions X_3 and Y_3 of $i_{n,n_\varphi *} \circ X_1 \circ i_{n,n_\varphi}{}^{-1}$ and $i_{n,n_\varphi *} \circ Y_1 \circ i_{n,n_\varphi}{}^{-1}$ in $\boldsymbol{R}^n$, X_3 and Y_3 are also local representatives relative to $i_{n,n_\varphi} \circ \varphi$, and so it follows that

$$\mathscr{L}_{X_2} Y_2(i_{n,n_\varphi} \circ \varphi)p = \mathscr{L}_{X_3} Y_3(i_{n,n_\varphi} \circ \varphi)p = i_{n,n_\varphi *} \mathscr{L}_{X_1} Y_1(\varphi p) \ .$$

Thus it is sufficient to consider only those $\theta \in \mathfrak{A}_S$ with $p \in U_\theta$ and $n_\theta = n_\varphi$. Let f be a connecting diffeomorphism defined in a neighborhood of θp, and let X_2 and Y_2 be local representatives of X and Y relative to θ in some (sufficiently small) neighborhood of θp. Then $f_* \circ X_2 \circ f^{-1}$ and $f_* \circ Y_2 \circ f^{-1}$ also are local representatives of X and Y relative to φ. Thus

$$f_* \mathscr{L}_{X_2} Y_2(\theta p) = \mathscr{L}_{f_* \circ X_2 \circ f^{-1}} f_* \circ Y_2 \circ f^{-1}(\varphi p) = \mathscr{L}_{X_1} Y_1(\varphi p) \ .$$

Finally we show that $\mathscr{L}_{X_1} Y_1(\varphi p) \in \varphi_* \otimes^k T_p S$. Choose φ so that $\varphi U_\varphi \subseteq i_{n_\varphi, n} \boldsymbol{R}^n$ and choose local representatives X_2 and Y_2 of X and Y relative to φ in a neighborhood of φp such that

$$X_2 = \sum_{i=1}^{n_\varphi} b^i \frac{\partial}{\partial x^i}$$

with $b^i = 0$ for $i > n$, and

$$Y_2 = \sum_\alpha a^\alpha \frac{\partial}{\partial x^{\alpha_i}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\alpha_k}}$$

with $a^\alpha = 0$ for any α having some $\alpha_j > n$. Then

$$\mathscr{L}_{X_1}Y_1(\varphi p) = \mathscr{L}_{X_2}Y_2(\varphi p) \in \varphi_* \otimes^k T_pS \ .$$

We may thus make the following definition.

Definition 5.3. Let $X \in \mathscr{X}(S)$ and $Y \in \mathscr{T}_k(S)$. Define $\mathscr{L}_XY \in \mathscr{T}_k(S)$ by setting

$$\mathscr{L}_XY(p) = \varphi_*{}^{-1} \circ \mathscr{L}_{X_1}Y_1 \circ \varphi(p)$$

for each $p \in S$, any $\varphi \in \mathfrak{A}_S$ with $p \in U_\varphi$, and any local representatives X_1 and Y_1 of X and Y relative to φ in a neighborhood of φp. When $Y \in \mathscr{X}(S)$, write $[X, Y] := \mathscr{L}_XY$. When $f \in C^\infty(S)$, set $\mathscr{L}_Xf := X \cdot f$.

For each $X \in \mathscr{X}(S)$, $\mathscr{L}_X$ is a type-preserving derivation in $\mathscr{T}(S)$ which commutes with Alt and Sym and satisfies

$$\begin{aligned} \mathscr{L}_XfY &= (X \cdot f)Y + f\mathscr{L}_XY \ , & f &\in C^\infty(S) \ , \\ [\mathscr{L}_X, \mathscr{L}_Z] &= \mathscr{L}_{[X,Z]} \ , & Z &\in \mathscr{X}(S) \ . \end{aligned}$$

For every $R \in \boldsymbol{C}^\infty$, $p \in S$, and every C^∞-diffeomorphism $f: S \to R$,

$$f_*\mathscr{L}_XY(p) = \mathscr{L}_{f_* \circ X \circ f^{-1}} f_* \circ Y \circ f^{-1}(fp) \ .$$

With the bracket product, $\mathscr{X}(S)$ is a $C^\infty(S)$-Lie module (cf. [10]).

We now consider Lie derivatives of covariant and mixed tensor fields. If $\mu \in \mathscr{F}^k_m(S)$, $X \in \mathscr{X}(S)$, $Y \in \otimes^k T_pS$, and if μ_1 and X_1 are local representatives of μ and X relative to $\varphi \in \mathfrak{A}_S$ in some neighborhood of φp, then an obvious candidate for $\mathscr{L}_X\mu(Y)$ is $(\varphi^*\mathscr{L}_{X_1}\mu_1)Y$. This, however, is not generally well-defined.

Example 5.4. Let $S_n \subseteq \boldsymbol{R}^2$ be the set

$$\{(x, k/2^n) \mid x \in [-1/2^n, 1/2^n], \ k \text{ odd and } 1 \le k < 2^n\} \ ,$$

and let $i: S \subsetneq \boldsymbol{R}^2$ be the set

$$\{(0, y) \mid y \in [0, 1]\} \cup \bigcup_{n=1}^{\infty} S_n \ .$$

Then $\partial/\partial x \in \mathscr{X}(\boldsymbol{R}^2)$ represents a vector field $X \in \mathscr{X}(S)$, and $i^*(xdy) = 0 \in \mathscr{F}^1(S)$. But

$$i^*\mathscr{L}_{\partial/\partial x}xdy = i^*dy \neq 0 \in \mathscr{F}^1(S) \ .$$

Lemma 5.5. *If $\mathscr{L}_{X_1}\mu_1(\varphi_*Y)$ is independent of the choice of μ_1, then for any other $\theta \in \mathfrak{A}_S$ with $p \in U_\theta$, any local representative μ_2 of μ relative to θ in a neighborhood of θp, and any connecting* map *f with $f \circ \varphi = \theta$ near p, we have*

$$f_* \mathscr{L}_{X_1}\mu_1(\varphi_* Y) = \mathscr{L}_{X_2}\mu_2(\theta_* Y) \in \theta_* \otimes^m T_p S \ .$$

Proof. First suppose $n_\varphi \leq n_\theta$. Without loss of generality assume that f is an imbedding (Axiom A2′). Let $X_3 \in \mathscr{X}(\boldsymbol{R}^{n_\theta})$ be a local extension of $f_* \circ X_1 \circ f^{-1}$ near θp. Then $W|_{\theta U_\theta} = 0$, where $W := X_2 - X_3$. Let $Z \in \mathscr{T}_k(\boldsymbol{R}^n)$ satisfy $Z(\theta p) = \theta_* Y$. Then

$$(5.6) \qquad \mathscr{L}_W \mu_2(\theta_* Y) = \mathscr{L}_W \langle \mu_2, Z \rangle \theta p - \langle \mu_2, \mathscr{L}_W Z \rangle \theta p \ .$$

From remark (5.2) it follows that both terms on the right side vanish. Thus

$$\mathscr{L}_{X_2}\mu_2(\theta_* Y) = \mathscr{L}_{X_3}\mu_2(\theta_* Y) = f_*(\mathscr{L}_{X_1}(f^*\mu_2)(\varphi_* Y)) \ .$$

By hypothesis, however,

$$\mathscr{L}_{X_1}(f^*\mu_2)(\varphi_* Y) = \mathscr{L}_{X_1}\mu_1(\varphi_* Y) \ .$$

Thus $f_*(\mathscr{L}_{X_1}\mu_1(\varphi_* Y)) = \mathscr{L}_{X_2}\mu_2(\theta_* Y)$.

Now assume that φ is tangential at p. Then $\mathscr{L}_{X_1}\mu_1(\varphi_* Y) \in \varphi_* \otimes^m T_p S$, and so $f_*(\mathscr{L}_{X_1}\mu_1(\varphi_* Y)) \in \theta_* \otimes^m T_p S$.

The case $n_\varphi > n_\theta$ follows similarly. q.e.d.

Lemma 5.5 then reduces the question of well-definedness of $\mathscr{L}_X\mu(Y)$ to that of whether for some φ and X_1, $\mathscr{L}_{X_1}\mu_1(\varphi_* Y)$ is independent of the choice of μ_1.

Theorem 5.6. *Each of the following conditions is sufficient for the well-definedness of* $\mathscr{L}_X\mu(Y)$:

(i) $Y \in (\otimes^k T_p S)^\vee$, *where* $p = \otimes^k \tau Y$.

(ii) X *has a local flow in some neighborhood of* p.

(iii) $X(p) = 0$.

(iv) *The structural dimension of*

$$\{q \in S / n_{S,q} = n_{S,p}\} \subseteq S$$

with the induced structure is $n_{S,p}$.

Proof. (i) Let $\tilde{Y} \in \mathscr{T}_k(\boldsymbol{R}^{n_\varphi})$ be a local representative of a tensor field in $\mathscr{T}_k(S)$ near φp such that $\tilde{Y}(\varphi p) = \varphi_* Y$. If μ_1 represents $0 \in \mathscr{F}^k_m(S)$ near φp, then

$$(5.7) \quad \mathscr{L}_{X_1}\mu_1(\varphi_* Y) = \mathscr{L}_{X_1}\mu_1(\tilde{Y}(\varphi p)) = \mathscr{L}_{X_1}(\mu_1 \circ \tilde{Y})(\varphi p) - \mu_1(\mathscr{L}_{X_1}\tilde{Y}(\varphi p)) \ .$$

From the previous results on Lie derivatives of contravariant tensor fields it follows that each term on the right side of (5.7) vanishes.

(ii) Let Φ be a local flow of X. Define $\gamma(t) = (\otimes^k \Phi_{t*})Y, |t|$ sufficiently small. Then γ is a C^∞-curve in $\otimes^k TU_\varphi$. Thus $\dot{\gamma}(0) \in T_Y \otimes^k TS$, and

$$\mathscr{L}_{X_1}\mu_1(\varphi_* Y) = (\varphi_{**}\dot{\gamma}(0)) \cdot \mu_1 = 0 \ ,$$

the last equality following from Lemma 2.1.

(iii) Let φ be tangential at p. Then $\varphi_* T_p S = T_{\varphi p}\boldsymbol{R}^{n_\varphi}$, and consequently $(\otimes^k \varphi_*) \otimes^k T_p S = \otimes^k T_{\varphi p}\boldsymbol{R}^{n_\varphi}$. Let X_1 be a local representative of X relative to φ, and let Φ be a local flow of X_1 near φp. Since $X(p) = 0$, $X_1(\varphi p) = 0$ and $\Phi_{t*}(T_{\varphi p}\boldsymbol{R}^{n_\varphi}) \subseteq T_{\varphi p}\boldsymbol{R}^{n_\varphi}$. Thus, if $\gamma(t) := \otimes^k \Phi_{t*}(\varphi_* Y)$, $Y \in \otimes^k T_p S$, then $\gamma(t) \in \otimes^k T_{\varphi p}\boldsymbol{R}^{n_\varphi}$. It follows that $\dot\gamma(0) \in (\otimes^k \varphi_*)_* T_Y \otimes^k TS$ with the same result as in (ii).

(iv) Assume without loss of generality that $\varphi \in \mathfrak{A}_S$ is tangential at p. Because of (iii), we need only consider the case $X(p) \neq 0$. We may then assume that X_1 is constant in some neighborhood of φp, say $X_1 = \partial/\partial x^1$ (by composing φ with an appropriate straightening diffeomorphism). Writing

$$\mu_1 = \sum a^{\alpha,\beta} dx^{\alpha_1} \otimes \cdots \otimes dx^{\alpha_k} \otimes \frac{\partial}{\partial x^{\beta_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\beta_m}},$$

we have

$$\mathscr{L}_{X_1}\mu_1 = \sum \frac{\partial a^{\alpha,\beta}}{\partial x^1} dx^{\alpha_1} \otimes \cdots \otimes dx^{\alpha_k} \otimes \frac{\partial}{\partial x^{\beta_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\beta_m}}.$$

If μ_1 represents $0 \in \mathscr{F}^k_m(S)$ relative to φ, then $a^{\alpha,\beta}(\varphi q) = 0$ for every α and β, and every $q \in S$ such that $n_{S,q} = n_\varphi$. Condition (iv) and Lemma 2.1 now imply $(\partial a^{\alpha,\beta}/\partial x^1)(\varphi p) = 0$ for all α, β. Thus $\mathscr{L}_{X_1}\mu_1(\varphi p) = 0$.

Definition 5.8. Define $\mathfrak{L}(S) \subseteq \mathscr{X}(S)$ to be the set of vector fields X such that $\mathscr{L}_X\mu$ is well-defined for every element μ in the bi-graded algebra $\mathscr{F}^{\bullet}_{\bullet}(S) := \{\mathscr{F}^m_k(S) \mid k, m \geq 0\}$.

Since $\mathscr{L}_X Y$ is well-defined for every $Y \in \mathscr{T}(S)$, $X \in \mathfrak{L}(S)$ if and only if $\mathscr{L}_X\phi$ is well-defined for every $\phi \in \mathscr{F}(S)$. Since $\mathscr{F}(S)$ is locally generated by $\mathscr{F}^1(S)$, $X \in \mathfrak{L}(S)$ if and only if $\mathscr{L}_X\phi$ is well-defined for every $\phi \in \mathscr{F}^1(S)$.

Theorem 5.9. *$\mathfrak{L}(S)$ is a Lie submodule of $\mathscr{X}(S)$. For each $X \in \mathfrak{L}(S)$, $\mathscr{L}_X$ is a type-preserving derivation in $\mathscr{F}^{\bullet}_{\bullet}(S)$ which commutes with* Alt, Sym *and every contraction C^i_j. For any $X, X' \in \mathfrak{L}(S)$,*

$$[\mathscr{L}_X, \mathscr{L}_{X'}] = \mathscr{L}_{[X,X']}$$

(as derivations on $\mathscr{F}^{\bullet}_{\bullet}(S)$). If $R \in \boldsymbol{C}^\infty$, $f \in C^\infty(S, R)$, and if $\mu \in \mathscr{F}^m_k(S)$ and $\nu \in \mathscr{F}^m_k(R)$ satisfy $\nu \circ f_* = f_* \circ \mu$, and if $X \in \mathscr{X}(S)$ and $Y \in \mathscr{X}(R)$ satisfy $Y(fp) = f_* Xp$ for all $p \in S$, then $\mathscr{L}_Y\nu \circ f_* = f_* \circ \mathscr{L}_X\mu$.

The proof is elementary and we omit it.

Proposition 5.10. *Let D be a type-preserving derivation on $\mathscr{F}^{\bullet}_{\bullet}(S)$ which commutes with contractions. Then there are unique $X \in \mathfrak{L}(S)$ and $\mu \in \mathscr{F}^1_1(S)$ such that $D = \mathscr{L}_X + \mu$. (Recall the parlance of* [7, p. 30].)

Proof. There is a unique $X \in \mathscr{X}(S)$ such that $D|_{\mathscr{F}^0(S)} = X$. As in the case of manifolds it follows that D is local. To show that $X \in \mathfrak{L}(S)$, let $p \in S$ and let $\varphi \in \mathfrak{A}_S$ be tangential at p. Let $\nu = \sum a_i dx^i$ be a local representative of $0 \in \mathscr{F}^1(S)$ relative to φ near φp. Then $a_i(\varphi p) = 0$ for $i = 1, \cdots, n_\varphi$. Let X_1 be a local representative of X relative to φ near φp. Then

$$0 = D(\varphi^*\nu)p = \left(\sum_{i=1}^{n_\varphi} (X \cdot a_i \circ \varphi)\varphi^* dx^i\right)p = (\varphi^* \mathscr{L}_{X_1}\nu)p \ .$$

Thus $\mathscr{L}_X\omega$ is well-defined for every $\omega \in \mathscr{F}^1(S)$, and it follows that $X \in \mathfrak{L}(S)$.

Now $K: = D - \mathscr{L}_X$ is a type-preserving derivation on $\mathscr{F}^{\bullet}_{\bullet}(S)$ which commutes with contractions and vanishes on $\mathscr{F}^0(S)$, i.e., is $C^\infty(S)$-linear. From Proposition 4.5 it follows that

$$K|_{\mathscr{F}^1(S)} = \kappa\mu$$

for a uniquely determined $\mu \in \mathscr{F}^1_1(S)$. Since $\mathscr{F}(S)$ is generated by $\mathscr{F}^0(S)$ and $\mathscr{F}^1(S)$, it follows that $D|_{\mathscr{F}(S)} = \mathscr{L}_X + \kappa\mu$.

The standard argument with coordinates shows that K is punctual. From Proposition 4.12 it follows then that K is completely determined by $\kappa\mu$ and its point-wise actions $K|_{\check{T}_pS}$, $p \in S$. Since K commutes with contractions,

$$0 = KC^1_1(v^* \otimes v) = C^1_1(\kappa\mu_p(v^*) \otimes v + v^* \otimes Kv)$$

for every $v^* \in T_pS^*$ and $v \in \check{T}_pS$. It follows that $K|_{\check{T}S} = \mu|_{\check{T}S}$. Thus with the traditional abuse of notation we may write $D = \mathscr{L}_X + \mu$.

6. Exterior differentiation and differential forms

If $\omega \in \mathscr{F}^k_{\mathrm{alt}}(S)$, $\varphi \in \mathfrak{A}_S$, and ρ is a local representative of ω relative to φ defined in some neighborhood V of φU_φ, then $\varphi^* d\rho \in \mathscr{F}^{k+1}_{\mathrm{alt}}(U_\varphi)$. This form on U_φ, however, is not always uniquely determined by ω. For example, consider the space S of Example 4.8 and let $\omega = i^*(z dx \wedge dy)$. Then $\omega = 0$, but $i^*(dz \wedge dx \wedge dy) \neq 0$. This phenomenon motivates the following definition.

Definition 6.1. Let $\omega \in \mathscr{F}^{k-1}_{\mathrm{alt}}(S)$ and $\theta \in \mathscr{F}^k_{\mathrm{alt}}(S)$. Then θ is an exterior differential of ω if for every $p \in S$ there exist $\varphi \in \mathfrak{A}_S$ with $p \in U_\varphi$ and representatives ω_φ and θ_φ of ω and θ relative to φ in a neighborhood of p such that $d\omega_\varphi = \theta_\varphi$. For each $k \geq 1$ we denote the set of exterior differentials of $0 \in \mathscr{F}^{k-1}_{\mathrm{alt}}(S)$ by $\mathfrak{m}^k(S)$, and we define $\mathfrak{m}^0(S) = \{0\}$ and $\mathfrak{m}(S) = \{\mathfrak{m}^k(S) \mid k \geq 0\}$.

Note that $\mathfrak{m}^k(S)$, $k \geq 0$, is local, that is, if $\mu|_U \in \mathscr{F}^k_{\mathrm{alt}}(S)$ satisfies $\mu|_U \in \mathfrak{m}^k(U)$ for every element U of some open cover of S, then $\mu \in \mathfrak{m}^k(S)$.

Proposition 6.2. (i) *If θ is a differential of ω, then $r\theta$ is a differential of $r\omega$ for all $r \in \boldsymbol{R}$. If θ_i is a differential of $\omega_i \in \mathscr{F}^{k_i}_{\mathrm{alt}}(S)$, $i = 1, 2$, then $\theta_1 + \theta_2$ is a differential of $\omega_1 + \omega_2$, and $\theta_1 \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge \theta_2$ is a differential of $\omega_1 \wedge \omega_2$. If θ_1 and θ_2 are differentials of $\omega \in \mathscr{F}^{k-1}_{\mathrm{alt}}(S)$, then $\theta_1 - \theta_2 \in \mathfrak{m}^k(S)$.*

(ii) *For each k, $\mathfrak{m}^k(S) \neq \emptyset$. $\mathfrak{m}^1(S) = \{0\}$. $\mathfrak{m}(S)$ is a homogeneous ideal in $\mathscr{F}_{\mathrm{alt}}(S)$. If $\mu \in \mathfrak{m}^k(S)$, then the set of differentials of μ is $\mathfrak{m}^{k+1}(S)$. If θ is a dif-*

ferential of $\omega \in \mathscr{F}^{k-1}_{\mathrm{alt}}(S)$, *then the set of differentials of* ω *is* $\theta + \mathfrak{m}^k(S)$, *and the set of differentials of* θ *is* $\mathfrak{m}^{k+1}(S)$.

(iii) *Let* $f \in C^\infty(S, S')$, *and let* θ *be a differential of* $\omega \in \mathscr{F}_{\mathrm{alt}}(S')$. *Then* $f^*\theta$ *is a differential of* $f^*\omega$. *In particular*, $f^*\mathfrak{m}(S') \subseteq \mathfrak{m}(S)$.

Proof. The proof of (i) follows directly from the definition and the properties of the exterior differential operator.

(ii) For each k, $0 \in \mathscr{F}^k_{\mathrm{alt}}(S)$ is an element of $\mathfrak{m}^k(S)$. Let $p \in S$ and assume that φ is tangential at p. Let $\theta \in \mathfrak{m}^1(S)$, and θ_φ and ω_φ be local representatives of θ and 0 relative to φ such that $\theta_\varphi = d\omega_\varphi$. Lemma 2.1 implies $d\omega_\varphi(\varphi p) = 0$. Thus $\theta(p) = 0$, and it follows that $\mathfrak{m}^1(S) = \{0\}$. Let $\omega \in \mathscr{F}^k_{\mathrm{alt}}(S)$ and $\mu \in \mathfrak{m}^m(S)$, let ω_φ and μ_φ be representatives of ω and μ, and let ζ_φ be a representative of $0 \in \mathscr{F}^{m-1}_{\mathrm{alt}}(S)$ relative to φ with $\mu_\varphi = d\zeta_\varphi$. Then

$$\varphi^*(\zeta_\varphi \wedge \omega_\varphi) = 0 \in \mathscr{F}^{k+m-1}_{\mathrm{alt}}(U_\varphi)\ , \qquad \varphi^*(\zeta_\varphi \wedge d\omega_\varphi) = 0 \in \mathscr{F}^{k+m}_{\mathrm{alt}}(U_\varphi)\ .$$

Thus

$$\varphi^*(\mu_\varphi \wedge \omega_\varphi) = \varphi^*(d(\zeta_\varphi \wedge \omega_\varphi) - (-1)^{m-1}\zeta_\varphi \wedge d\omega_\varphi) = \varphi^* d(\zeta_\varphi \wedge \omega_\varphi)\ .$$

It follows that $\mu \wedge \omega \in \mathfrak{m}^{k+m}(S)$. Thus $\mathfrak{m}(S)$ is an ideal in $\mathscr{F}_{\mathrm{alt}}(S)$ and is evidently homogeneous. Since $d\mu_\varphi = dd\zeta_\varphi = 0$, $0 \in \mathscr{F}^{m+1}_{\mathrm{alt}}(S)$ is a differential of μ. From (i) it now follows that the set of differentials of $\mu \in \mathfrak{m}^m(S)$ is $\mathfrak{m}^{m+1}(S)$. Similarly, if $\omega \in \mathscr{F}^{k-1}_{\mathrm{alt}}(S)$ has a differential θ, then the set of all differentials of ω is $\theta + \mathfrak{m}^k(S)$. Since $0 \in \mathfrak{m}^{k+1}(S)$ is a differential of θ, $\mathfrak{m}^{k+1}(S)$ is the set of differentials of θ.

(iii) Let $p \in S$, and let $\omega_{\varphi'}$ and $\theta_{\varphi'}$ be local representatives of ω and θ relative to $\varphi' \in \mathfrak{A}_{S'}$, where $f(p) \in U_{\varphi'}$, and $d\omega_{\varphi'} = \theta_{\varphi'}$. Let $\varphi \in \mathfrak{A}_S$ with $p \in U_\varphi$ and $f(U_\varphi) \subseteq U_{\varphi'}$, and let F be a local C^∞-extension of $\varphi' \circ f \circ \varphi^{-1}$. Then $F^*\omega_{\varphi'}$ and $F^*\theta_{\varphi'}$ are local representatives of $f^*\omega$ and $f^*\theta$ in a neighborhood of $\varphi(p)$, and $dF^*\omega_{\varphi'} = F^* d\omega_{\varphi'}$. Thus $f^*\theta$ is a differential of $f^*\omega$.

Definition and Corollary 6.3. Let $\mathscr{D}^k(S)$ be the submodule of $\mathscr{F}^k_{\mathrm{alt}}(S)$ of forms having differentials, and let $\mathscr{D}(S) := \{\mathscr{D}^k(S) \mid k \geq 0\}$. Part (ii) of the previous proposition shows that if θ is a differential, then $\theta \in \mathscr{D}(S)$, and in particular, $\mathfrak{m}(S) \subseteq \mathscr{D}(S)$. Define $\mathscr{A}(S) = \mathscr{D}(S)/\mathfrak{m}(S)$, i.e., $\mathscr{A}(S) = \{\mathscr{A}^k(S) \mid k \geq 0\}$, where $\mathscr{A}^k(S) = \mathscr{D}^k(S)/\mathfrak{m}^k(S)$. The elements of $\mathscr{A}^k(S)$ are called *differential k-forms*. For each $k \geq 0$ define $d : \mathscr{D}^k(S) \to \mathscr{A}^{k+1}(S)$ by $d\omega = \theta + \mathfrak{m}^{k+1}(S)$, where θ is any differential of ω. Then d satisfies

(i) $d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^k \omega \wedge d\omega'$, $\omega \in \mathscr{D}^k(S)$, $\omega' \in \mathscr{D}(S)$.

Because $d\mathfrak{m}^k(S) = 0 \in \mathscr{A}^{k+1}(S)$, d factors through the $\boldsymbol{R}$-linear map $\boldsymbol{d}: \mathscr{A}(S) \to \mathscr{A}(S)$; $\boldsymbol{d}(\omega + \mathfrak{m}^k(S)) := d\omega$ for $\omega \in \mathscr{D}^k(S)$. Then $\boldsymbol{d}$ satisfies

(ii) $\boldsymbol{d}\mathscr{A}^k(S) \subseteq \mathscr{A}^{k+1}(S)$,

(iii) $\boldsymbol{d}(\omega \wedge \omega') = \boldsymbol{d}\omega \wedge \omega' + (-1)^k \omega \wedge \boldsymbol{d}\omega'$, $\omega \in \mathscr{A}^k(S)$, $\omega' \in \mathscr{A}(S)$,

(iv) $\boldsymbol{dd}\omega = 0 \in \mathscr{A}^{k+2}(S)$, $\omega \in \mathscr{A}^k(S)$,

(v) If $f \in C^\infty(S, S')$, then $f^* : \mathscr{F}_{\mathrm{alt}}(S') \to \mathscr{F}_{\mathrm{alt}}(S)$ induces a map $\mathscr{A}(S') \to$

$\mathscr{A}(S)$, also denoted by f^*, and $\boldsymbol{d} \circ f^* = f^* \circ \boldsymbol{d}$.

Proposition 6.4. *A sufficient condition for $\omega \in \mathscr{F}^k_{\text{alt}}(S)$ to belong to $\mathscr{D}^k(S)$ is that the support of ω have a paracompact neighborhood in S. Thus $\mathscr{D}(S) = \mathscr{A}_{\text{alt}}(S)$ if S is paracompact.*

Proof. Every $\omega \in \mathscr{F}^k_{\text{alt}}(S)$ has differentials locally. The proof follows by routine use of partitions of unity.

Remarks 6.5. Whether $\mathscr{D}(S) = \mathscr{F}_{\text{alt}}(S)$ for arbitrary S is an open question. Neither the k-forms of § 4 nor the differential k-forms of this section coincide with any of the three notions of differential forms introduced in [14]. In Example 4.8, zdx represents $0 \in \mathscr{F}^1(S)$, hence $0 \in \mathscr{A}^1(S)$, but $[zdx]_{(0,0,0)} \notin \mathscr{I}^1_3(S)$.

We now establish analogs of some classical identities involving $\mathscr{L}_X$, i_X, and $\boldsymbol{d}$. The main lemma is the following.

Lemma 6.6. *Let $X \in \mathfrak{L}(S)$, $X_i \in \mathscr{X}(S)$, $i = 1, \cdots, k$, and $\mu \in \mathfrak{m}^k(S)$. Then $\mu(X_1 \wedge \cdots \wedge X_k) = 0$, $i_X\mu \in \mathfrak{m}^{k-1}(S)$, and $\mathscr{L}_X\mu \in \mathfrak{m}^k(S)$.*

Proof. Let $p \in S$, $\varphi \in \mathfrak{A}_S$ be tangential at p, and let Y, Y_i be local representatives of X, X_i relative to φ in some neighborhood U of φp. Let ν be a local representative of $0 \in \mathscr{F}^{k-1}_{\text{alt}}(S)$ with respect to φ in some neighborhood U of φp, without loss of generality, such that $\mu|_{\varphi^{-1}U} = \varphi^* d\nu$. Since $\nu(Z_1, \cdots, Z_{k-1}) = 0$ for all tangent vectors $Z_i \in T_{\varphi p}\boldsymbol{R}^{n_\varphi}$, we have

$$\begin{aligned}\mu(p)(X_1^p \wedge \cdots \wedge X_k^p) &= d\nu(\varphi p)(Y_1(\varphi p) \wedge \cdots \wedge Y_k(\varphi p)) \\ &= \sum_{j=1}^{k} (-1)^{j-1} Y_j(\varphi p) \cdot \nu(Y_1 \wedge \cdots \wedge \hat{Y}_j \wedge \cdots \wedge Y_k) \\ &= 0\end{aligned}$$

by Lemma 2.1.

Since $\varphi^*\nu = 0$, then $\varphi^* i_Y \nu = i_X \varphi^* \nu = 0$, and

$$0 = \mathscr{L}_X \varphi^* \nu = \varphi^* \mathscr{L}_Y \nu = \varphi^*(d i_Y \nu + i_Y d\nu) ,$$

where $\varphi^* d i_Y \nu \in \mathfrak{m}^k(\varphi^{-1}U)$. Thus

$$i_X \mu|_{\varphi^{-1}U} = i_X \varphi^* d\nu = \varphi^* i_Y d\nu \in \mathfrak{m}^k(\varphi^{-1}U) .$$

It follows that $i_X\mu \in \mathfrak{m}^k(S)$.

Finally, $\mathscr{L}_X\mu|_{\varphi^{-1}U} = \varphi^* \mathscr{L}_Y d\nu = \varphi^* d \mathscr{L}_Y \nu$. Since $\varphi^* \mathscr{L}_Y \nu = 0$, $\varphi^* d \mathscr{L}_Y \nu \in \mathfrak{m}^k(\varphi^{-1}U)$. It follows that $\mathscr{L}_X\mu \in \mathfrak{m}^k(S)$. q.e.d.

Of course it is not generally true that $\langle \omega(p), X_p \rangle$ is single-valued for $\omega \in \mathfrak{m}(S)$ and $X_p \in \wedge T_pS$. From Lemma 6.6 and Proposition 4.12, however, we have the following.

Proposition 6.7. *Let $X \in \mathscr{X}^k(S)$ and $\omega \in \mathscr{A}^k(S)$. Then $\langle \omega, X \rangle \in C^\infty(S)$. In other words,* (6.8) $\mathfrak{m}^k(S)(p) := \{\mu(p) \mid \mu \in \mathfrak{m}^k(S)\} \subseteq (\wedge \check{T}_p S)^{\perp}$, *for all* $p \in S$.

The following example shows that the inclusion in (6.8) can be proper.

Example 6.9. Let S be the space of Example 4.8. Then $\wedge^2 \check{T}_{(0,0,0)} S = 0$. On the other hand,

$$\mathfrak{m}^2(S) = \operatorname{span}_{C^\infty(S)} \{i^*dx \wedge dz, i^*dy \wedge dz\} .$$

Thus $\mathfrak{m}^2(S)(0,0,0) \subsetneq (\wedge^2 T_{(0,0,0)} S)^* = (\wedge^2 \check{T}_{(0,0,0)} S)^{\perp}$.

In view of Lemma 6.6, the following proposition is routine.

Theorem 6.9. *Let* $X, Y \in \mathfrak{L}(S)$. *Then as operators on* $\mathscr{A}(S)$, $\mathscr{L}_X, i_X$, *and* $\boldsymbol{d}$ *satisfy the following identities*:

(i) $\mathscr{L}_X = i_X \circ \boldsymbol{d} + \boldsymbol{d} \circ i_X$,

(ii) $\boldsymbol{d} \circ \mathscr{L}_X = \mathscr{L}_X \circ \boldsymbol{d}$,

(iii) $i_{[X,Y]} = \mathscr{L}_X \circ i_Y - i_Y \circ \mathscr{L}_X$.

If $X_i \in \mathscr{X}(S)$, $i = 0, \cdots, k$, *then*

(iv)
$$\begin{aligned} \boldsymbol{d}\omega(X_0 \wedge \cdots \wedge X_k) &= \sum_{i=0}^{k} (-1)^i X_i \cdot \omega(X_0 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j] \wedge X_0 \wedge \cdots \wedge \hat{X}_i \wedge \\ &\qquad \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_k) . \end{aligned}$$

As in the classical case, we have the following.

Proposition 6.10. *Every derivation* D *on* $\mathscr{A}(S)$ *of degree* 0 *which commutes with* $\boldsymbol{d}$ *is equal to* $\mathscr{L}_X$ *for some* $X \in \mathfrak{L}(S)$.

Finally, we have the following singular version of Stokes' identity. As stated, the theorem is far from the best possible, but we shall defer a more careful examination of the facts until later.

Proposition 6.11. *Let* Δ^n *be the standard closed n-simplex and let* $\sigma: \Delta^n \to S$ *be of class* C^∞. *Then for every* $\mu \in \mathfrak{m}^n(S)$,

$$\int_{\Delta^n} \sigma^* \mu = 0 . \tag{6.12}$$

Thus for each $\omega \in \mathscr{A}^n(S)$ *and each* $\theta \in \mathscr{F}^n_{\mathrm{alt}}(S)$ *with* $\theta + \mathfrak{m}^n(S) = \omega$,

$$\int_{\Delta^n} \sigma^* \omega := \int_{\Delta^n} \sigma^* \theta$$

is well-defined. For every $\omega \in \mathscr{A}^{n-1}(S)$,

$$\int_{\Delta^n} \sigma^* d\omega = \int_{\partial\Delta^n} \sigma^* \omega \ . \tag{6.13}$$

Proof. By using a simplicial subdivision of Δ^n, the proof is reduced to the special case $\sigma(\Delta^n) \subseteq U_\varphi$, where $\varphi \in \mathfrak{A}_S$ satisfies the following: there exists a local representative ν of 0 relative to φ in a neighborhood of φU_φ such that $\mu|_{U_\varphi} = \varphi^* d\nu$. Then $\sigma^*\mu = \sigma^*\varphi^* d\nu = (\varphi \circ \sigma)^* d\nu$ is exact, yielding (6.12).

Now suppose that $\theta \in \mathscr{F}^{n-1}_{\text{alt}}(S)$ has a local representative ϕ relative to φ in a neighborhood of φU_φ and that θ satisfies $\theta + \mathfrak{m}^{n-1}(S) = \omega$. Then

$$\int_{\Delta^n} \sigma^* d\omega = \int_{\Delta^n} \sigma^* d\theta = \int_{\Delta^n} \sigma^* \varphi^* d\phi = \int_{\Delta^n} (\varphi \circ \sigma)^* d\phi \ ,$$

which by the classical singular Stokes' identity is

$$\int_{\partial\Delta^n} (\varphi \circ \sigma)^* \phi = \int_{\partial\Delta^n} \sigma^* \theta = \int_{\partial\Delta^n} \sigma^* \omega \ .$$

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