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# GEODESIC FOLIATIONS BY CIRCLES

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## 1. Introduction

Smooth foliations by circles of compact three-manifolds have been completely analysed by D. B. A. Epstein in the paper [2]. Essentially, he shows that all such foliations arise as a decomposition of the manifold by the orbits of a smooth circle action. The theorem of this paper shows that the same is true of an arbitary smooth manifold, compact or not, with a foliation by circles satisfying a certain (rather strong) regularity condition.

It is known that not all foliations by circles arise as the orbits of some action by $S^1$; indeed, the paper [2] presents a foliated noncompact three-manifold as a counter-example to such a proposition. However, it is an open question whether or not such examples exist in the case of a foliated *compact* manifold of dimension greater than three.

A $C^r$ flow on a $C^r$ manifold $M$ is a $C^r$ action $\mu : \boldsymbol{R} \times M \to M$ of the additive reals on $M$. A $C^r$ flow without fixed points, each of whose orbits is compact, gives rise to a $C^r$ foliation of the manifold by circles. Further, any $C^r$ foliation by circles of a manifold $M$ gives rise to a $C^r$ flow on (a double cover of) $M$. The version of the theorem presented here is stated for flows; an equivalent version for circle foliations in terms of differential forms is readily obtainable (see § 2). The theorem is the following.

**Theorem.** *Let $\mu : \boldsymbol{R} \times M \to M$ be a $C^r$ action $(3 \leq r \leq \infty)$ of the additive group of real numbers with every orbit a circle, and $M$ a $C^r$ manifold. Then there is a $C^r$ action $\rho : S^1 \times M \to M$ with the same orbits as $\mu$ if and only if there exists some riemannian metric on $M$ with respect to which the orbits of $\mu$ are embedded as totally geodesic submanifolds of $M$.*

Finding some such metric given a circle action on $M$ is easy (see § 3); the proof of the converse requires a little more effort. The author wishes to thank David Epstein for his gentle encouragement and for his many helpful suggestions.

## 2. The invariant one-form

Suppose a riemannian metric exists on the manifold $M$ as in the theorem. At each point $m \in M$ choose a unit vector $T_m$ in the direction of the flow $\mu$.

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Then the vector field $T$ satisfies the relations $|T| = 1$ and $\nabla_T T = 0$ on $M$, where $\nabla$ is the Levi-Civita connection of the metric. Without loss of generality we may assume that the vector field $T$ generates the flow $\mu$. That is, $(d/dt)\mu_t(m)|_{t=0} = T_m$ where $\mu_t(m) = \mu(t, m)$.

**Lemma 2.1.** *Let $X \in T_m M$ and suppose that $X$ is orthogonal to $T$. Then the vector $\mu_{t*}X$ in the tangent space of $M$ at $p = \mu_t(m)$ for $t \in \boldsymbol{R}$ is orthogonal to $T_p$. That is, $\langle \mu_{t*}X, T\rangle = 0$ for all $t \in \boldsymbol{R}$.*

The proof of the lemma appears at the end of the section.

Thus the flow $\mu$ maps orthogonal vectors into orthogonal vectors for all time. Define a one-form $\alpha$ on $M$ by $\alpha_m(X) = \langle X, T\rangle_m$; then $\alpha(T) = 1$ and $L_T\alpha = 0$, where $L_T$ denotes Lie derivative with respect to the vector field $T$. This follows from Lemma 2.1 and the expression $(L_T\alpha) = \lim ((\mu_t{}^*\alpha)_m - \alpha_m)/t$ as $t \to 0$. In fact, we have a converse: given a vector field $Y$ on $M$ and a one-form $\beta$ with $\beta(Y) = 1$ and $L_Y\beta = 0$ let $Q_m = \{X \in T_m M : \beta(X) = 0\}$ and $P_m = \{X \in T_m M : X = cY, c \in \boldsymbol{R}\}$. Then the tangent bundle of $M$ splits: $TM = Q \oplus P$. Furthermore, a straightforward construction defines a riemannian metric on $M$ such that $Q_m$ is orthogonal to $P_m$ at each $m \in M$. The reverse argument to the proof of Lemma 2.1 (see below) then shows that with respect to this metric the trajectories of $Y$ are geodesics in $M$.

In the formula $L_T\alpha = C_T(d\alpha) + d(C_T\alpha)$ where $d$ is the exterior derivative and $C_T$ is contraction by $T$, we have $d(C_T\alpha) = 0$, since $C_T\alpha = \alpha(T) = 1$. Whence $C_T(d\alpha) = L_T\alpha = 0$. Conversely, given a one-form $\beta$ and vector field $Y$ with $C_Y(d\beta) = 0$ and $\beta(Y) > 0$ it is easy to verify that $L_{Y'}\beta = 0$ and $\beta(Y') = 1$ where $Y' = Y/\beta(Y)$. We can summarise the above two paragraphs in the following

**Lemma 2.2.** *Let $T$ be a nonzero vector field on the manifold $M$. Then there exists a riemannian metric on $M$ so that the trajectories of $T$ are embedded as totally geodesic submanifolds if and only if there exists a one-form $\alpha$ on $M$ with $C_T(d\alpha) = 0$ and $\alpha(T) > 0$.*

Such one-forms arise naturally in the study of contact manifolds as defined by Boothby and Wang [1]. In this case, the manifold $M$ is assumed to have dimension $2n + 1$ with a globally defined one-form $\omega$ such that $\omega \wedge (d\omega)^n \neq 0$ on $M$ ($(d\omega)^n = d\omega \wedge \cdots \wedge d\omega$). On the subspace $V_x = \{X \in T_x M : C_X(d\omega) = 0\}$ we have $\omega \neq 0$; further, $V_x$ has dimension one and is complementary to the subspace of dimension $2n$ on which $\omega$ is zero. Let $Z_x$ be that element of $V_x$ for which $\omega(Z_x) = 1$. Then the vector field $Z$ and one-form $\omega$ satisfy the conditions of Lemma 2.2. Thus with a suitable metric on $M$ the trajectories of $Z$ are geodesics.

Indeed, in their paper [1] Boothby and Wang proved a special case of our theorem. They consider the case where the manifold $M$ is compact and the induced foliation of $M$ by the trajectories of $Z$ is *regular* in the sense of Palais [6]. That is, about each point $x$ of $M$ there is an open neighborhood $U$ of $x$ so that any nonempty intersection of a trajectory with $U$ is a connected set.

In this situation, each trajectory is closed and hence compact; so each orbit is a circle. They deduce that there is an *effective* circle action on $M$ with the same orbits as the $\boldsymbol{R}$-action generated by $Z$.

*Proof of Lemma* 2.1. Suppose $X_m \in T_mM$ is orthogonal to the flow. Let $V_0$ be a small open disc transverse to the flow through $m$ with cl $V_0$ compact (cl = closure), and $X_m$ tangent to $V_0$ at $m$. Furthermore, assume there are defined on $V_0$ coordinate functions $x^2, \cdots, x^n$ ($n = \dim M$) with $x^i(m) = 0$ and $(\partial/\partial x^n)_m = X_m$. Then there is an $\varepsilon > 0$ such that $V = \mu((-\varepsilon, \varepsilon) \times V_0)$ is the diffeomorphic image of the open set $(-\varepsilon, \varepsilon) \times V_0$ under $\mu$. Moreover, on $V$ we may define coordinate functions $y^1, \cdots, y^n$ as follows: for $p = \mu_t(q)$ ($q \in V_0$, $-\varepsilon < t < \varepsilon$), set $y^1(p) = t$ and $y^i(p) = x^i(q)$, $2 \le i \le n$. Then $(\partial/\partial y^1)_p = T_p$ and $(\partial/\partial y^i)_p = \mu_{t*}(\partial/\partial x^i)_q)$; in particular, if $p = \mu_t(m)$ then $(\partial/\partial y^n)_p = \mu_{t*}X_m$. Define the vector field $X$ on $V$ by $X = \partial/\partial y^n$.

Because our hypotheses imply (i) $\nabla_T T = 0$, (ii) $\langle T, T\rangle = 1$ and (iii) $0 = [T, X] = \nabla_T X - \nabla_X T$, we have $T\langle X, T\rangle = \langle \nabla_T X, T\rangle + \langle X, \nabla_T T\rangle = \langle \nabla_T X, T\rangle = \langle \nabla_X T, T\rangle = \frac{1}{2}X\langle T, T\rangle = 0$. That is, the inner product $\langle X, T\rangle$ is constant along the orbit of $T$ through $m$. In particular, we have $\langle \mu_{t*}X_m, T\rangle = 0$ for $-\varepsilon < t < \varepsilon$. So $\mu$ translates vectors orthogonal to the orbits of the flow into orthogonal vectors. This completes the proof. In general, the flow $\mu$ need not be metric-preserving.

## 3. Necessity

Let $M$ be a riemannian manifold with metric tensor $g$. A vector field $X$ on $M$ which generates a one-parameter group of isometries of $M$ with respect to $g$ is known as a Killing vector field with respect to $g$. Such a vector field satisfies the condition $L_X g = 0$, where $L_X g$ is the Lie derivative of the tensor field $g$ with respect to $X$.

**Lemma 3.1.** *Let $M$ be a riemannian manifold with metric tensor $g$ and a nonzero Killing vector field $X$. Then there exists a metric $g'$ on $M$, conformal to $g$, such that $X$ remains a Killing vector field with respect to $g'$ and, in addition, we have $|X|' = 1$. Furthermore, with respect to $g'$ the trajectories of $X$ are geodesics with parametrisation by are-length.*

*Proof.* Define the function $f: M \to R$ by $f = (g(X, X))^{-1} = |X|^{-2}$. We may define the conformal metric $g'$ by the tensor $g' = fg$. Now $L_X f = (g(X, X))^{-2} \cdot L_X(g(X, X)) = (g(X, X))^{-2}(L_X g)(X, X) = 0$, thus $L_X(fg) = (L_X f)g + f(L_X g) = 0$ because $L_X g = 0$ by hypothesis. The flow generated by $X$ is isometric; in particular, the flow preserves the subspace of vectors orthogonal to $X$ with respect to $g'$. It follows from § 2 that the trajectories of $X$ are geodesics with parametrisation by are-length, as $|X|' = 1$.

Returning to the theorem, suppose we have $\rho: S^1 \times M \to M$, a smooth action of the circle group $S^1$ without fixed points. Identifying $S^1 = \boldsymbol{R}/\boldsymbol{Z}$, we may suppose $\rho$ defines a flow with derived vector field $T$. Choose any metric $g''$ on $M$ and define another metric by

$$g = \int (\rho^* g'') ,$$

where the integral is taken with respect to the invariant Haar measure on $S^1$. Then $g$ is invariant under the action $\varphi$; that is, $T$ is a Killing vector field with respect to $g$. Lemma 3.1 can now be applied to $T$ thus proving necessity in the theorem.

## 4. Sufficiency

Suppose that we are given a flow $\mu: \boldsymbol{R} \times M \to M$ with every orbit a circle, and that with respect to some riemannian metric on the manifold $M$ the orbits of $\mu$ are geodesics. Without loss of generality we may suppose parametrisation by arc-length. By Lemma 2.1 we see that the flow maps orthogonal vectors into orthogonal vectors.

Let $V_0$ be a small disc in $M$ transverse to the flow, with cl $V_0$ compact. Then there is an $\varepsilon > 0$ such that $\mu$ defines a homeomorphism of $[-\varepsilon, \varepsilon] \times \text{cl}\, V_0$ into $M$, which is a diffeomorphism on $(-\varepsilon, \varepsilon) \times V_0$. By a *flat neighborhood* in $M$ (resp. of a point $m$ in $M$) shall be meant an open subset $V$ of $M$ (resp. an open neighborhood $V$ of $m$) such that $V = ((-\varepsilon, \varepsilon) \times V_0)$ for some disc $V_0$ (resp. for some disc $V_0$ with $m \in V_0$). Let $\pi: V \to V_0$ be the projection map.

**Lemma 4.1.** *Let $V$ be a flat neighborhood in $M$. Let $\sigma_1: [0, 1] \to V$, $\sigma_2: [0, 1] \to V$ be smooth curves in $V$ orthogonal to the flow. If $\pi \circ \sigma_1 = \pi \circ \sigma_2$ and $\sigma_1(0) = \sigma_2(0)$, then $\sigma_1 = \sigma_2$.*

*Proof.* A straightforward application of the uniqueness of solutions of ordinary differential equations.

Following [2, p. 69], we define $\lambda: M \to \boldsymbol{R}$ by the conditions

$$\begin{aligned} &\text{i.} && \lambda x > 0 , \\ &\text{ii.} && \mu_t(x) \neq x \qquad \text{for } 0 < t < \lambda x , \\ &\text{iii.} && \mu_{\lambda x}(x) = x . \end{aligned}$$

The function $\lambda$ is invariant under the flow.

**Proposition 4.2,** [2, § 5]. *The function $\lambda: M \to \boldsymbol{R}$ giving the period of a point is lower semi-continuous. If $W \subset M$, then the set of points of continuity of $\lambda | W$ is open in the induced topology on $W$.*

We now use an idea basically due to Montgomery (see [4, p. 224]). We define the sets $B_1, B_2 \subset M$ as follows

$$B_1 = \{x \in M : \lambda \text{ is not continuous at } x\} ,$$
$$B_2 = \{x \in B_1 : \lambda | B_1 \text{ is not continuous at } x\} .$$

Each of $B_1, B_2$ is invariant. Furthermore, $B_1$ (resp. $B_2$) is closed and has null interior as a subspace of $M$ (resp. $B_1$). $M - B_2$ has a countable number of connected components each of which is an invariant open subset of $M$.

**Lemma 4.3.** *Let $U$ be an open connected set in $M$, and $f: U \to \boldsymbol{R}$ a continuous, invariant real-valued map such that $\mu_{fm}(m) = m$ for all $m \in U$. Then $f$ is a constant map.*

*Proof.* Fix $x \in U$. Let $V = \mu((-\varepsilon, \varepsilon) \times V_0)$ be a flat neighborhood of $x$ in $M$. Then on $V, \lambda \geq 2\varepsilon$. Choose another neighborhood $W$ of $x, W = \mu((-\varepsilon, \varepsilon) \times W_0), x \in W_0 \subset V_0$ such that for $y \in W$ we have $|fx - fy| < \varepsilon$. For $p' \in W$, by taking a smaller neighborhood if need be, we may further suppose that there exists an orthogonal curve $\sigma: [0, 1] \to W$ with $\sigma(0) = x$ and $\sigma(1) = p$, where $p$ and $p'$ lie on the same connected component of an orbit in $W$. Now $\mu_{fx} \circ \sigma$ is orthogonal and its image is contained in $W$; furthermore, it is easy to see that $\pi \circ \sigma = \pi \circ (\mu_{fx} \circ \sigma)$ where $\pi: W \to W_0$ is projection. Since $\sigma(0) = x = \mu_{fx} \circ \sigma(0)$ we may apply Lemma 4.1 to obtain $\sigma = \mu_{fx} \circ \sigma$. In particular, $\mu_{fx} \circ \sigma(1) = \mu_{fx}(p) = p$. Clearly $fp = k_1 \lambda p$ where $k_1$ is an integer; similarly, we have $fx = k_2 \lambda p$. As $|fx - fp| < \varepsilon$ and $\lambda p \geq 2\varepsilon$ we obtain $|k_1 - k_2| < \frac{1}{2}$, which implies $k_1 = k_2$. Whence $fx = fp = fp'$. As $p' \in W$ was arbitary and $U$ is connected, the lemma is proved.

**Corollary 4.4.** *Let $U$ be a connected component of $M - B_1$. Then $\lambda | U = c$, a constant.*

Define $C_1 = \{x \in M: \lambda$ is unbounded in any neighborhood of $x\}$. $C_1$ is a closed invariant subset of $M$. Furthermore, we have $C_1 \subset B_1$ as the function $\lambda$ is locally constant on $M - B_1$. In the proof we assume $C_1$ is nonempty and prove a contradiction.

**Proposition 4.5.** *Let $D$ be a connected component of $M - C_1$. If $U \subset D$ is a component of $M - B_1$ with $\lambda | U = c$, then $\mu_c | D =$* id.

*Proof.* $D$ is an open invariant subset of $M$. Fix $m \in D$ and let $A \subset D$ be the orbit of $\mu$ through $m$. Let $V = \mu((-\varepsilon, \varepsilon) \times V_0)$ be a flat neighborhood of $m$ in $D$, so $\lambda \geq 2\varepsilon$ on $V$ and cl $V_0$ is compact. Because $\lambda$ is locally bounded on $D$, we may assume that $\lambda \leq \Lambda$ on $V$, $\Lambda \in \boldsymbol{R}$. Additionally, it can be supposed that the disc $V_0$ is sufficiently small to ensure that the orbit $A$ intersects $V_0$ in only the single point $m$. We define the Poincaré map $S: V_1 \to V_0$ for some smaller disc $V_1 \subset V_0$. For more detail the reader is referred to [2, §§ 4, 5]. Essentially, there exists a neighborhood $V_1$ of $m$ in $V_0$ such that the map $f: V_1 \to \boldsymbol{R}$, given by the conditions

$$\begin{aligned} &\text{i.} \quad fx > 0 \,, \\ &\text{ii.} \quad \mu_t(x) \notin V_0 \qquad \text{for } 0 < t < fx \,, \\ &\text{iii.} \quad \mu_{fx} \in V_0 \,, \end{aligned}$$

is well-defined and $C^r$ on $V_1$. The Poincaré map $S: V_1 \to V_0$ is defined by $S_x = \mu_{fx}(x)$. The point $m \in V$ is invariant under $S$. Let $N = [\Lambda/(2\varepsilon) + 1]$.

We define by induction neighborhoods $V_i$ of $m$ in $V_0$ such that $SV_{i+1} \subset V_i$ $(1 \leq i \leq N!)$. Because $\lambda \geq 2\varepsilon$ on the open invariant set orb $V_0$ (where orb $V_0 = \{y \in M : y = \mu_t(x)$ for $t \in \boldsymbol{R}, x \in V_0\}$) and because $\lambda \leq \Lambda$ here, it is easy to show that for each point $x \in V_q$, where $q = N!$, $S^r x = x$ for some $r$, $1 \leq r \leq N$. Hence $S^q = \mathrm{id}$ on $V_q$. We obtain an open neighborhood $W$ of $m$ in $V_0$ which is invariant under $S$ by putting $W = \bigcap_{i=1}^q S^i V_q$. The set orb $W \subset D$ is invariant, connected and open in $M$. Define the function $g: W \to \boldsymbol{R}$ by

$$g(x) = \sum_{i=1}^{q} (f \circ S^i x) .$$

Then $g$ is continuous and invariant under $S$. Thus it may be extended continuously to a function $g$ on all of orb $W$, invariant under $\mu$ and agreeing on $W$. Because $S^q = \mathrm{id}$ on $W$ we have $\mu_{gx}(x) = x$ for every $x \in$ orb $W$. By Lemma 4.3, $g$ must be constant on orb $W$. As the set $M - B_1$ is open and dense in $M$, some component $U$ of $M - B_1$ intersects orb $W$ nontrivially. Let $\lambda | U = c$. It is easy to see that $g = kc$ on $W$, where $k$ is some integer, and thus $g = kc$ on orb $W$. The transformation $\mu_c |$ orb $W$ is periodic and is the identity on the interior set $U$ orb $W$. By a theorem of Newman [5], $\mu_c |$ orb $W = \mathrm{id}$. Straightforward use of a covering of $D$ by flat neighborhoods and the fact that $D$ is connected completes the proof of the proposition.

**Corollary 4.6.** *For $D$ as above, $\mu_c | \mathrm{cl}\, D = \mathrm{id}$, and if $x \in \mathrm{cl}\, D$ then we have $k_x \lambda x = c$ where $k_x \geq 1$ is an integer. Furthermore, $\mu_{c*}: T_x M \to T_x M$ is the identity for each $x \in \mathrm{cl}\, D$.*

**Corollary 4.7.** *For $D$ as above we have* $\mathrm{bdy}\, D = \mathrm{bdy}\, (\mathrm{cl}\, D)$; *that is,* $\mathrm{int}\, (\mathrm{cl}\, D) = D$.

It will be useful to consider the action $\mu$ on the component $D$ of $M - C_1$, where $\mu_c | D = \mathrm{id}$ as above. Define another metric $g''$ on $M$ by

$$g'' = c^{-1} \int_0^c (\mu_t{}^* g) dt .$$

It follows from Corollary 4.6 that on cl $D$ the flow is isometric with respect to $g''$. It will be convenient to work with the $g''$-metric only for the remainder of the proof.

Since $\mu$ is isometric on the open set $D$, it commutes with the exponential map there. For $p \in M$, $r > 0$ set $B'_r(p) = \{X \in T_p M : |X|'' < r\}$ and define $B_r(p) = \exp B'_r(p)$. If $p \in D$, then there exists some $r > 0$ such that $B_r(p) \subset D$ and $B_r(p)$ is the diffeomorphic image of the ball $B'_r(p)$ in $T_p M$. Thus $\mu_t \circ \exp_p | B'_r(p) = \exp_q \circ \mu_{t*} | B'_r(p)$ for $q = \mu_t(p)$ and all time $t$; in particular, the set $B_r(p) = \mu_{\lambda p} B_r(p)$, so that the action of $\mu_{\lambda p}$ in a neighborhood of $p$ is linear with respect to geodesic coordinates at $p$.

It follows from Proposition 4.2 that if $m \in (B_1 - B_2) \cap D$ then there exists a neighborhood $W$ of $m$ in $D$ such that $\lambda | B_1 \cap W$ is continuous. By choosing some smaller neighborhood if necessary, we can suppose $\lambda | B_1 \cap W$ is constant. (Because $\mu_c | \mathrm{cl}\, D = \mathrm{id}$ and $\lambda$ is locally bounded below, we may first suppose that $\lambda | B_1 \cap W$ takes only a finite set of values. Then, since $\lambda$ is continuous on this set, we can easily find a (smaller) neighborhood $W'$ of $m$ so that $\lambda | B_1 \cap W'$ is constant.) Suppose $\lambda m = c/k$, $k \geq 1$ an integer. Then the transformation $\mu_{\lambda m *} \colon T_m M \to T_m M$ is such that every vector is either fixed or has period $k$. Using the diffeomorphism $B_r(m) = \exp B'_r(m)$ it is easy to see that if $k = 1$ then $\lambda$ would be continuous at $m$, whence $k \geq 2$; thus the fixed point set of $T_m M$ (with respect to $\mu_{\lambda m *}$) has codimension at least one. Denote this set by $H'(m)$ and define $H(m) = \exp_m H'(m)$. Thus $\mu_{\lambda m} | H(m) \cap B_r(m) = \mathrm{id}$ and the only fixed points of $B_r(m)$ under the transformation $\mu_{\lambda m}$ are contained in $B_r(m) \cap H(m)$. (Note that $B_r(m) \cap B_1$ possibly includes points of $B_2$.)

Define $C_2 = \{x \in C_1 \colon \lambda | C_1$ *is* continuous at $x\}$. By Proposition 4.2, $C_2$ is an open subset of $C_1$ (with respect to the relative topology). Let $p \in \mathrm{bdy}\, D \cap C_2$ where $D$ is some component of $M - C_1$. ($\mathrm{bdy}\, D \subset C_1$ because points of $\mathrm{bdy}\, D$ are not interior in $M - C_1$.) Then there exists a neighborhood $W$ of $p$ in $M$ such that $\lambda | W \cap \mathrm{bdy}\, D$ is continuous and, as before, we may suppose that $\lambda$ is constant there.

**Lemma 4.8.** $\lambda | \mathrm{bdy}\, D \cap W = c$.

*Proof.* Most of the work in the proof of this lemma arises because $\mathrm{bdy}\, D$ need not *a priori* be a smoothly embedded submanifold of $M$.

Without loss of generality, the point $p$ is arcwise accessible from $D$; that is, there is some (regular) arc lying in $D \cup \{p\}$ having $p$ as an endpoint. Such points are obviously dense in the boundary (see, for example, [4, p. 119]). With a slight abuse of notation, denote some such arc by $[q, p]$ with $[q, p)$ contained in $W \cap D$.

It is well-known that given any compact set $A \subset M$ there exists an $s > 0$ such that for each $x \in A$ the ball $B_s(x)$ is convex and such that if the vector $X \in T_y M$, $y \in \mathrm{bdy}\, B_s(x)$ is tangent to the sphere $\mathrm{bdy}\, B_s(x)$ then the geodesic $\exp tX$ does not penetrate the ball $B_s(x)$ near $y$ (see, for example, [3, § 9.4]). Setting $A = [q, p]$ we let $s > 0$ as above; we may further suppose that $B_s(q) \subset D$ and that if $x \in [q, p]$ then $\mathrm{cl}\, B_s(x) \subset W$. Then there exists some $y \in [q, p)$ such that $\mathrm{bdy}\, D \cap \mathrm{cl}\, B_s(y) \neq \emptyset$ and $\mathrm{bdy}\, D \cap \mathrm{cl}\, B_s(y) \subset \mathrm{bdy}\, B_s(y)$. Let $z \in \mathrm{bdy}\, D \cap \mathrm{bdy}\, B_s(y)$. If $X \in T_z M$ is tangent to the sphere $\mathrm{bdy}\, B_s(y)$ then the geodesic $\exp tX$ lies outside of $B_s(y)$ near $z$; furthermore, if $X$ is not tangent to this sphere, then the geodesic $\exp tX$ or $\exp(-tX)$, $t > 0$ penetrates the ball $B_s(y)$ for some positive distance. Note that as the radius $s$ varies over lesser values such points $z$ will be arbitrary near $p$, and that $\lambda p = \lambda z$.

There are a metric ball $B_r(z) \subset W$ with center $z$ and an $a > 0$ such that if $x \in B_r(z)$ then the ball $B_a(x)$ is convex. Thus the open set $B_a(z) \cap B_s(y)$ is convex and contained within $D \cap W$. We may distinguish two cases:

1. $z$ is approximated by points of $B_1 - B_2$ in $B_a(z) \cap B_s(y)$,
2. case 1 does not occur.

Consider case 1. Let $m_1$ be an element of $B_1 - B_2$ in $B_a(z) \cap B_s(y)$ with associated fixed-point set $H(m_1)$ (see the paragraph following Corollary 4.7). Recall that the flow, when restricted to $D$, preserves the metric and consequently maps geodesics into geodesics whilst preserving their parametrisation. In particular, if $g: [0, 1] \to \operatorname{cl} D$ is a geodesic with $g(0) = x \in \operatorname{bdy} D$ and $g(0, 1] \subset D$, then for each integer $k$ we have $\mu_{k\lambda x} \circ g: [0, 1] \to \operatorname{cl} D$ is a geodesic on $(0, 1]$ and, by continuity, it must be geodesic at $\mu_{k\lambda x} \circ g(0) = x$. Now, if the set $H(m_1) \cap B_a(m_1) \cap B_a(z)$ intersects $\operatorname{bdy} D$ then it contains an open subset of $\operatorname{bdy} D$, which is impossible. For otherwise, there is some $w \in \operatorname{bdy} D \cap B_a(m_1)$, $w \notin H(m_1)$ with a geodesic $\exp tX$ in $B_a(m_1)$, $X$ tangent to $M$ at $m_1$, such that $\exp t_0 X = w$ and $\exp tX \in D \cap B_a(m_1)$ for $0 < t < t_0$. Thus the point $\exp t_0 X$ is fixed under the transformation $\mu_{\lambda m_1}$. By the definition of $H(m_1)$ we have $t_0 X \in H'(m_1)$ which contradicts the hypothesis that $w \notin H(m_1)$.

Furthermore, $H(m_1) \cap \operatorname{cl} B_a(m_1) \cap B_a(z)$ is closed in $B_a(z)$, and is therefore bounded away from $z$. Thus we may choose $m_2 \in (B_1 - B_2) \cap (B_a(z) \cap B_s(y))$ strictly nearer $z$ than $m_1$ so that $m_2 \notin H(m_1)$ but $m_2 \in B_a(m_1)$. (Because $B_a(m_1)$, $B_a(z)$ are convex and $z \in B_a(m_1)$.) Proceeding inductively, we may find $m_i \in (B_1 - B_2) \cap (B_a(z) \cap B_s(y))$ strictly nearer $z$ than $m_{i-1}$ with $m_i \notin H(m_j)$, $1 \leq j < i$ but with $m_i \in B_a(m_j)$, $1 \leq j < i$. By the definition of $H(m_j) \subset B_a(m_j)$ we have $\lambda m_i \neq \lambda m_j$ for $1 \leq j < i$. But, by hypothesis, $\lambda$ is bounded away from zero in $W$ and $B_a(z) \subset W$. Moreover, $\mu_c | \operatorname{cl} D = \mathrm{id}$. Hence there is only a finite number of values for $\lambda | D \cap B_a(z)$. In particular, $\lambda | B_1 \cap D \cap B_a(z)$ takes only finitely many values; but this contradicts the construction of our sequence $\{m_i\}$. Thus case 1 cannot occur.

Consider case 2. That is, $z$ is not approximated by points of $B_1 - B_2$ in $B_a(z) \cap B_s(y)$. But since $B_1 - B_2$ is open and dense in $B_1$, for some smaller value of $a$ we also have that $B_1 \cap (B_a(z) \cap B_s(y)) = \emptyset$. Thus $B_a(z) \cap B_s(y) \subset U$, where $U$ is some component of $M - B_1$, $U \subset D$ and $\lambda | U = c$. By Corollary 4.6, if $x \in \operatorname{bdy} D \cap W$ then $\lambda x = kc$ where $k \geq 1$ is an integer. Consider the case $k \geq 2$.

In $T_z M$ denote by $F$ the one-codimensional hyperplane of vectors tangent to the sphere $\operatorname{bdy} B_s(y)$. $F$ partitions $T_z M$ into two complementary open half-spaces $E^-$ and $E^+$ where $E^+$ consists of vectors $X$ such that the geodisic $\exp tX$, $t > 0$, penetrates the ball $B_s(y)$ for some positive distance. Restricting attention to the ball $B_a(z)$, for each vector $X \in E^+$ and integer $j$ the curve $\mu_{j\lambda z} \circ \exp tX$ (for small $t > 0$) is a geodesic in $D$; consequently, $\mu_{j\lambda z} \circ \exp tX = \exp_z \circ\, t(\mu_{j\lambda z *} X)$. Since $\mu_c | U = \mathrm{id}$, each vector $X$ in $E^+$ has period $k$ with respect to $\mu_{\lambda z *}: T_z M \to T_z M$. If there is $t_0 > 0$ such that $\exp t_0 X \in \operatorname{bdy} D \cap B_a(z)$ where $X \in E^+$, then the vector $t_0 X$ would be fixed under $\mu_{\lambda z *}$ (as $\lambda | W \cap \operatorname{bdy} D$ is constant) which contradicts the fact that $X$ has period $k \geq 2$. Thus $\exp_z$ maps $E^+ \cap B'_a(z)$ diffeomorphically into $U \subset D$.

**Lemma 4.9.** *Let $V$ be a vector space with $F \subset V$ a one-codimensional hyperplane, and $T: V \to V$ a linear transformation of finite period such that* *each point of the open half-spaces $E^-$, $E^+$ of $V$ determined by $F$ has least period (with respect to $T$) strictly greater than one. Then the orbit of $E^+$ under successive transformations by $T$ includes $E^-$.*

*Proof.* It is sufficient to show that the orbit of each point $v \in E^-$ intersects $E^+$ nontrivially. If none of $vT, vT^2, \cdots, vT^{k-1}$ (where $k$ is the period of $T$) are in $E^+$ then the invariant vector $v + vT + \cdots + vT^{k-1}$ is in $E^-$ as well, but all invariant vectors are contained in $F$. Thus at least one $vT^j \in E^+$.

In the case $k \geq 2$ the lemma applies to $\mu_{\lambda z*}: T_zM \to T_zM$. Thus $\exp B'_a(z) \subset$ cl $U$ and bdy $D \cap B_a(z) \subset$ cl $D$. But this contradicts Corollary 4.7. Therefore $k = 1, \lambda | \text{bdy}\, D \cap W = c$ and the proof of Lemma 4.8 is complete.

Let $x \in C_2$. There exists a neighborhood $W$ of $x$ in $M$ such that $\lambda | W \cap C_1$ is continuous. Furthermore, Lemma 4.8 shows that for each point $p \in$ bpy $D_i$ $\cap\, W$, where $D_i$ is a component of $M - C_1$ with $\mu_{c_i} | D_i = \text{id}$ (as in Proposition 4.5), we have $\lambda p = c_i$. Define the function $h: W \to R$ by

$$hq = \begin{cases} c_i & \text{if } q \in \text{cl}\, D \cap W\,, \\ \lambda q & \text{if } q \in W \cap C_1\,. \end{cases}$$

The function $h$ is clearly continuous, and $\mu_{hq}q = q$ for all $q \in W$. By Lemma 4.3, $h$ is constant on $W$, which implies $\lambda$ is bounded in a neighborhood of $x$. But this contradicts the hypothesis that $C_1$ is nonempty, because $C_2$ is dense in $C_1$. Thus $\lambda$ is bounded on $M$. Proposition 4.5 then implies that $\mu$ has period $c$ on $M$. Evidently, this proves the theorem.

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