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LOCAL PROPERTIES OF SMOOTH MAPS EQUIVARIANT WITH RESPECT TO FINITE GROUP ACTIONS

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1. Introduction

In this paper we prove that a smooth (C^∞) invariant function on a representation space of a finite group can be written as a smooth function of a (finite) set of generators for the algebra of invariant polynomials. This result can be used to study local properties of equivariant maps between manifolds with a finite group action. We show here that, generically, a smooth equivariant map between Z_2-manifolds of dimension two has a simple explicit form with respect to suitably chosen invariant coordinates near any point in the source and its image. The adjective "generic" will apply to a local property of maps in some space of maps (usually the space of smooth equivariant maps between manifolds with group action); a local property is generic if the set of maps having that property at each point of a given compact set in the source is an open dense set. Our manifolds and group actions will always be smooth.

In studying the singularities of smooth maps $f: M \to N$ between manifolds, one considers certain "singularity submanifolds" Σ^I of the jet spaces $J^r(M, N)$. These submanifolds were defined by Boardman [2], systematizing earlier proposals of Thom [8] and Whitney [12]. Thom [7] proved that the r-jet map $f^r = j^r f: M \to J^r(M, N)$ induced by f is generically transverse to these submanifolds. The transversality conditions translate into conditions on the partial derivatives of f, which can be used (as least in simple, *stable* cases) to put f locally into normal form (see, for example [12], [8], [4]).

In many situations giving rise to stability problems there are certain natural symmetries which must be preserved, so that a study of generic singularities of equivariant maps could be important to the solution of these stability problems. But such a study relates also to certain *nonstable* problems. Consider, for example, the smooth map

$$(x, y) \to (x^2 - y^2, 2xy) + R(x, y) ,$$

where the map R has order at least 3. This map has an unstable isolated

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singularity at the origin. One cannot in general make smooth local coordinate changes near the origin of the source and target so that in the new coordinates the map is $(x, y) \to (x^2 - y^2, 2xy)$ (though by a result of Stoïlow [6], [14] these maps are topologically equivalent at the origin). We will see however that if the given map is invariant with respect to the antipodal involution in the source (or, in other words, R is even), we can make smooth equivariant corrdinate changes to obtain the normal form $(x, y) \to (x^2 - y^2, 2xy)$. The point is that this map is stable at the origin in the space of smooth invariant maps.

One of the main obstacles in studying equivariant maps is the lack of a transversality theory for manifolds with group action and equivariant maps. We will see that an attempt to classify generic singularities of equivariant maps involves notions of transversality of their induced jet maps to certain algebraic subsets of the jet spaces, at singular points of these subsets. The case of equivariant maps between 2-dimensional manifolds with involution already presents some interesting features.

If M, N are manifolds with the action of a group G, and p is a point in M, then we denote by G_p the isotropy subgroup of p, and by $C_G^\infty(M, N)$ the space of smooth equivariant maps from M to N (with the C^∞ topology). Every point p in a Z_2-manifold of dimension 2 has a $(Z_2)_p$-invariant neighborhood which is equivalent to one of the three orthogonal Z_2-spaces of dimension 2:

T: the space $\boldsymbol{R}^2$ with involution $(x, y) \to (x, y)$ (the trivial involution);

R: the space $\boldsymbol{R}^2$ with involution $(x, y) \to (-x, y)$ (reflection in the y-axis);

A: the space $\boldsymbol{R}^2$ with involution $(x, y) \to (-x, -y)$ (the antipodal map).

In other words, one of these spaces serves as an invariant local coordinate system near p.

Theorem 1. *Let M, N be smooth Z_2-manifolds of dimension 2, and K a compact subset of M. There is an open dense set $\mathcal{O} \subset C_{Z_2}^\infty(M, N)$ such that if $f \in \mathcal{O}$, $p \in K$, then in some neighborhood of p, f can be expressed in one of the following normal forms (with respect to suitable $(Z_2)_p$-invariant local coordinates (x, y), (u, v) near p, $f(p)$ respectively).*

$(Z_2)_p$-neighborhood of p	$(Z_2)_p$-neighborhood of $f(p)$	Generic equivariant maps	
T	T	$u = x,$	$v = y$
		$u = x^2,$	$v = y$
		$u = -xy + x^3,$	$v = y$
R	R	$u = x,$	$v = y$
		$u = x,$	$v = y^2$
		$u = -xy + x^3,$	$v = y$

$(Z_2)_p$-neighborhood of p	$(Z_2)_p$-neighborhood of $f(p)$	Generic equivariant maps	
A	A	$u = x,$	$v = y$
T	A	$u = 0,$	$v = 0$
A	T	$u = x^2 \pm y^2,$	$v = 2xy$
T	R	$u = 0,$	$v = y$
		$u = 0$	$v = x^2 \pm y^2$
R	T	$u = x^2,$	$v = y$
		$u = -x^2y + x^4,$	$v = y$
R	A	$u = x,$	$v = xy$
		$u = x,$	$v = xy^2$
A	R	$u = x,$	$v = y^2$

Of course for given connected Z_2-manifolds M, N of dimension 2, not all these cases can occur. The maps listed are equivariantly stable at the origin: any sufficiently close smooth equivariant map can be put into the same form by making smooth equivariant local coordinate changes in the source and target. The normal forms for the case $T \to T$ were obtained by Whitney [13]; the singularities of maps $(x, y) \to (x^2, y)$, $(x, y) \to (-xy + x^3, y)$ at the origin are called a fold and cusp respectively. The case $T \to R$ is given by Morse theory [5] since the whole source plane must map into the y-axis (the fixed point set) of the target. In the case $A \to T$, the map $(x, y) \to (x^2 - y^2, 2xy)$ is just the complex map $z \to z^2$, while the map $(x, y) \to (x^2 + y^2, 2xy)$ is a "double fold" (equivariantly equivalent to the map $(x, y) \to (x^2, y^2)$); its singular set $(y = \pm x)$ and its image are illustrated in Fig. 1(a). Fig. 1(b) shows the singular set $(x = 0) \cup (y = 2x^2)$ and image of the map $(x, y) \to (-x^2y + x^4, y)$, for the case $R \to T$.

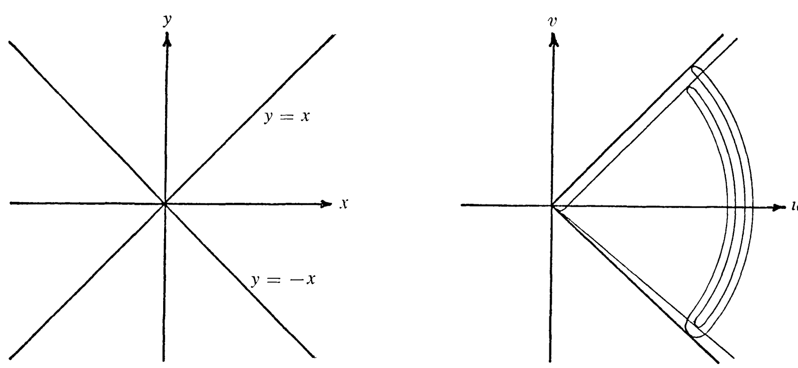

Fig. 1(a) The map $(x,y) \to (x^2 + y^2, 2xy)$

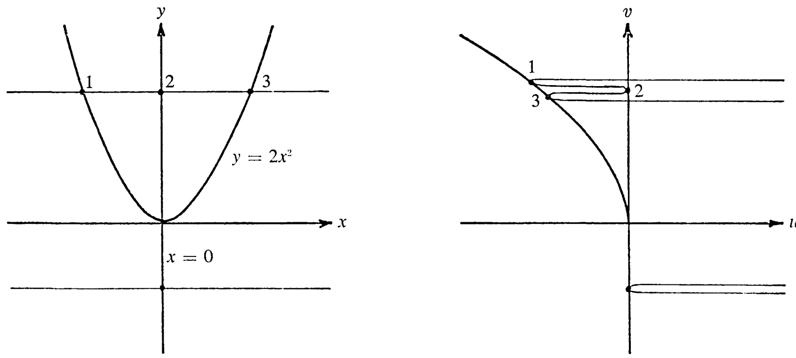

Fig. 1(b) The map $(x, y) \to (-x^2y + x^4, y)$

The theorem follows from the corresponding local theorem:

Theorem 2. *Take M, N orthogonal Z_2-spaces of dimension 2 in the statement of Theorem 1.*

In § 2 we prove the result, mentioned at the beginning, on smooth functions invariant under the action of a finite group:

Theorem 3. *Suppose the finite group G acts linearly on $\boldsymbol{R}^n$, and let $\phi_1, \cdots, \phi_K$ be a set of generators for the algebra of invariant polynomials. Then the germ at $0 \in \boldsymbol{R}^n$ of any invariant C^∞ function can be expressed as a C^∞ function of the $\phi_1, \cdots, \phi_K$.*

Some aspects of singularity theory are recalled in § 3. The generic singularities of equivariant maps between orthogonal Z_2-spaces of dimension 2 are described in § 4 in terms of general position of their induced jet maps to certain algebraic subsets of the jet spaces, and corresponding conditions on their partial derivatives. In § 5 these conditions and the result of Theorem 3, in the case of Z_2-actions, are used to obtain the normal forms of Theorem 1.

During the early part of the research for this paper, the author profitted from many conversations with Felice Ronga at the Institute for Advanced Study.

2. Smooth functions invariant under the action of a finite group

If a compact Lie group G acts linearly on $\boldsymbol{R}^n$, then there is a finite set of generators for the algebra of polynomials on $\boldsymbol{R}^n$ which are invariant under the action of G. In other words there is a finite set of invariant polynomials $\phi_1(X), \cdots, \phi_K(X)$ such that for any invariant polynomial $f(X) = f(x_1, \cdots, x_n)$ on $\boldsymbol{R}^n$ there is a polynomial $h(Y) = h(y_1, \cdots, y_K)$ such that

$$f(X) = h(\phi_1(X), \cdots, \phi_K(X))$$

(Weyl [9, p. 274]).

It is an open question whether any invariant C^∞ function f on $\boldsymbol{R}^n$ can be written as a C^∞ function of the K basic invariant polynomials $\phi_1, \cdots, \phi_K$, though an affirmative answer is known in certain cases[1]. If the map

$$\Phi = (\phi_1, \cdots, \phi_K): \boldsymbol{R}^n \to \boldsymbol{R}^K$$

has rank K on an open dense subset of $\boldsymbol{R}^n$, the result follows from a theorem of Glaeser on the composition of C^∞ functions [3] together with results of Whitney on C^∞ approximation by polynomials [10]. This applies, for example, to $0(n)$ acting on $\boldsymbol{R}^n \times \boldsymbol{R}^n \times \cdots \times \boldsymbol{R}^n$ (h copies) by the standard diagonal action, as long as $h \leq n$. In Theorem 3 we establish the result (locally) for the action of any *finite* group G.

Remark 1. The orbit space $\boldsymbol{R}^n/G$ is stratified by the subspaces corresponding to the sets of points of $\boldsymbol{R}^n$ of a given orbit type. The invariant map $\Phi: \boldsymbol{R}^n \to \boldsymbol{R}^K$ induces an embedding $\Phi^*: \boldsymbol{R}^n/G \to \boldsymbol{R}^K$ which is C^∞ on each stratum. The image of Φ is a semi-algebraic set: it is contained in the algebraic set of zeros of the ideal of algebraic relations among the $\phi_1, \cdots, \phi_K$, but is only a part of this algebraic set given by certain invariant inequalities involving the ϕ_i. Theorem 3 shows that $\phi_1, \cdots, \phi_K$ in some sense serve as smooth local coordinate functions at the point $0 \in \operatorname{Im} \Phi$ of the orbit space.

Remark 2. The result of Theorem 3 was first obtained in the case of Z_2 acting on the real line by Whitney [11], and in the case of the symmetric group on n symbols acting on $\boldsymbol{R}^n$ by permutation of the coordinates by Glaeser [3]. John Mather has informed the author that he has also obtained the result of Theorem 3, and it has recently been proved independently by Sandor Straus. Using Mather's division theorem it can, in fact, be proved for invariant functions rather than just germs, though we need only the local case.

The theorem is proved using the Malgrange preparation theorem [4]. We denote by $\mathscr{E}_n$ the ring of germs at 0 in $\boldsymbol{R}^n$ of C^∞ functions, and by $\mathscr{E}_n^*$ the ring of formal power series in n variables. The map $\Phi: \boldsymbol{R}^n \to \boldsymbol{R}^K$ defined by $\Phi(V) = (\phi_1(V), \cdots, \phi_K(V))$ induces a morphism $u: \mathscr{E}_K \to \mathscr{E}_n$ of differentiable algebras over $\boldsymbol{R}$, given by $u(h) = h \circ \Phi$.

The group G is a finite group of linear transformations $A_{(\alpha)}, \alpha = 1, \cdots, q$ (say with $A_{(1)}$ the identity). For $V = (x, y, \cdots, z) \in \boldsymbol{R}^n$, denote by $V_{(\alpha)} = (x_{(\alpha)}, y_{(\alpha)}, \cdots, z_{(\alpha)})$ the image $A_{(\alpha)}V$ of V under $A_{(\alpha)}$ (so in particular $V_{(1)} = V$).

Lemma 1. *The ring $\mathscr{E}_n^*$ of formal power series in n variables $(x, y, \cdots, z)$ is generated (as a module) over the subalgebra generated by the $\phi_1, \cdots, \phi_K$, by the polynomials*

$$x_{(1)}^{\alpha_1} y_{(1)}^{\beta_1} \cdots z_{(1)}^{\gamma_1} x_{(1)}^{\alpha_2} y_{(2)}^{\beta_2} \cdots z_{(2)}^{\gamma_2} \cdots x_{(q-1)}^{\alpha_{q-1}} y_{(q-1)}^{\beta_{q-1}} \cdots z_{(q-1)}^{\gamma_{q-1}}, \quad 0 \leq \alpha_i + \beta_i + \cdots + \gamma_i \leq q - i\,. \tag{1}$$

Once we have Lemma 1, we deduce from the preparation theorem (see [4,

[1]Since the acceptance of this paper, G. W. Schwarz has proved this result for compact Lie groups [Topology 14 (1975) 63–68].

Cor. 4.4, p. 77]) that the monomials (1) generate $\mathscr{E}_n$ over the subalgebra of germs of differentiable functions of $\phi_1, \cdots, \phi_K$. Thus, if $f \in \mathscr{E}_n$ is invariant under G, we write f as a sum of products of a polynomial of the form (1) with the germ of a function of $\phi_1, \cdots, \phi_K$, and see, by averaging over G, that there exists $h \in \mathscr{E}_K$ with

$$f(x, y, \cdots, z) = h(\phi_1, \cdots, \phi_K) .$$

To prove Lemma 1 we use an explicit construction of a finite generating set for the invariants of a finite group due to Noether (see [9, p. 275]). Recall that for a polynomial $f(X) = f(x_1, \cdots, x_q)$ of q variables x_i, the *polarized polynomial* $D_{YX}f$ with respect to $Y = (y_1, \cdots, y_q)$ is defined by

$$D_{YX}f = \frac{\partial f}{\partial x_1} y_1 + \cdots + \frac{\partial f}{\partial x_q} y_q .$$

If $f(X) = f(x_1, \cdots, x_q)$ is a homogeneous polynomial of degree r in the q variables x_i, a multilinear form $F(X, Y, \cdots, Z)$ depending on r vectors $X, Y, \cdots, Z$ is obtained by *complete polarization*:

$$F(X, Y, \cdots, Z) = D_{XU} D_{YU} \cdots D_{ZU} f(U) .$$

Full polarization of the elementary symmetric polynomials $\sigma_1, \cdots, \sigma_q$ in q variables yields (up to multiples $i!$):

$$(2) \qquad \begin{aligned} \psi_1(U) &= \sum u_i , \\ \psi_2(U, V) &= \sum u_i v_j \qquad (i \neq j) , \\ &\cdots\cdots\cdots \\ \psi_q(U, V, \cdots, W) &= \sum u_i v_j \cdots w_k \qquad (i, j, \cdots, k \text{ are all distinct}). \end{aligned}$$

Now consider a finite group G of linear transformations $A_{(\alpha)}, \alpha = 1, \cdots, q$, acting on vectors $V = (x, y, \cdots, z) \in \boldsymbol{R}^n$ as before. Any invariant polynomial $f(V)$ can be written as

$$(3) \qquad f(V) = \frac{1}{q} \sum_{\alpha=1}^{q} f(V_{(\alpha)}) .$$

The right-hand side of (3), considered as a function of n vectors

$$(4) \qquad \begin{aligned} X &= (x_{(1)}, \cdots, x_{(q)}) , \\ Y &= (y_{(1)}, \cdots, y_{(q)}) , \\ &\cdots\cdots\cdots \\ Z &= (z_{(1)}, \cdots, z_{(q)}) , \end{aligned}$$

is invariant under the symmetric group on q symbols acting by permutation of the coordinates in each vector. It is thus expressible as a polynomial in the polarized elementary symmetric functions. We conclude that a finite set of generators $\phi_1, \cdots, \phi_K$ for the algebra of polynomials invariant under G is obtained by substituting the n vectors (4) for the arguments $U, V, \cdots, W$ in (2) in all possible combinations, including repitions.

Proof of Lemma 1. We must show that every monomial

$$x^\alpha y^\beta \cdots z^r = x^\alpha_{(1)} y^\beta_{(1)} \cdots z^r_{(1)}$$

is a sum of terms $\theta\phi$, where θ is a polynomial of the form (1), and ϕ is an invariant polynomial.

Consider the monomial

$$f(U) = f(u_1, \cdots, u_q) = u_1^{\alpha+\beta+\cdots+r} .$$

$f(U)$ is a sum of terms

$$(5) \qquad u_1^{\rho_1} u_2^{\rho_2} \cdots u_{q-1}^{\rho_{q-1}} \tau ,$$

where $0 \leq \rho_i \leq q - i$, and τ is a symmetric polynomial. Form the polarized polynomial

$$D^\alpha_{XU} D^\beta_{YU} \cdots D^r_{ZU} f = F(X, Y, \cdots, Z) ,$$

where $X, Y, \cdots, Z$ are the vectors of (4). Then

$$F(X, Y, \cdots, Z) = (\alpha + \beta + \cdots + \gamma)!\, x^\alpha y^\beta \cdots z^r$$

is a sum of terms formed by successive polarizations of terms (5) with respect to $Z, \cdots, Y, X$, that is, a sum of terms $\theta\phi$, where θ is a polynomial of the form (1) and ϕ an invariant polynomial.

3. Singularities of smooth maps

We recall that in studying the singularities of a smooth map $f: \boldsymbol{R}^n \to \boldsymbol{R}^m$, we classify the points p of $\boldsymbol{R}^n$ according to the rank of f at p: $p \in \Sigma^{i_1}(f)$ if the kernel of df_p, the differential of f at p, has dimension i_1. If $\Sigma^{i_1}(f)$ is a manifold, we can then consider the restriction $f \mid \Sigma^{i_1}(f): \Sigma^{i_1}(f) \to \boldsymbol{R}^m$, and define $\Sigma^{i_1,i_2}(f) = \Sigma^{i_2}(f \mid \Sigma^{i_1}(f))$. In general, for any decreasing sequence $I: i_1 \geq i_2 \geq \cdots \geq i_k$ of nonnegative integers, the set $\Sigma^I(f)$ is defined by induction: if $\Sigma^{i_1,\cdots,i_{k-1}}(f)$ is a manifold, then

$$\Sigma^I(f) = \Sigma^{i_k}(f \mid \Sigma^{i_1,\cdots,i_{k-1}}(f)) .$$

The subspaces $\Sigma^I(f)$ need not, of course, be manifolds, but they are for a dense set of smooth maps f. Consider the spaces $J^r = J^r(n, m)$ of r-jets at 0 of germs of C^∞ maps $f: \boldsymbol{R}^n \to \boldsymbol{R}^m$ with $f(0) = 0$, i.e., $J^r = \bigoplus_{k=1}^r L^k_s(\boldsymbol{R}^n; \boldsymbol{R}^m)$, where $L_s^k(\boldsymbol{R}^n; \boldsymbol{R}^m)$ denotes the space of symmetric k-multilinear maps from $\boldsymbol{R}^n$ to $\boldsymbol{R}^m$. A smooth map $f: \boldsymbol{R}^n \to \boldsymbol{R}^m$ induces the r-jet map $f^r = j^r f: \boldsymbol{R}^n \to J^r$, given at each point by the partial derivatives of orders 1 through r.

Boardman [2] defined the singularity submanifolds Σ^I of J^r, where $I: i_1 \geq \cdots \geq i_k$ is a decreasing sequence of nonnegative integers with $k \leq r$. By Thom's transversality theorem [7], the r-jet map f^r of a smooth map $f: \boldsymbol{R}^n \to \boldsymbol{R}^m$ is generically transverse to the submanifolds Σ^I, so that $\Sigma^I(f) = (f^r)^{-1}(\Sigma^I)$ is a submanifold of $\boldsymbol{R}^n$ of codimension equal to the codimension of Σ^I in J^r. The $\Sigma^I(f)$ are the same as the subspaces given above.

In the low dimensions we work with in this paper, it is convenient to use a more naïve approach to the subspaces $\Sigma^I(f)$ due to Whitney [12]. Let $\nu = \inf(n, m)$, and $\Sigma_1^i \subset J^1 = L(\boldsymbol{R}^n, \boldsymbol{R}^m)$ be the subspace of linear maps of rank $\nu - i$. Then Σ_1^i is a submanifold of codimension $i(|n - m| + i)$, and $Cl(\Sigma_1^i) = \cup \{\Sigma_1^j \mid j \geq i\}$ (Cl = closure) is an algebraic set with singular set $Cl(\Sigma_1^{i+1})$. The 1-jet map f^1 of a smooth map $f: \boldsymbol{R}^n \to \boldsymbol{R}^m$ is generically transverse to the submanifolds Σ_1^i, so that $\Sigma^i(f) = (f^1)^{-1}(\Sigma_1^i)$ is a submanifold of $\boldsymbol{R}^n$ of codimension $i(|n - m| + i)$.

Let $\Sigma_2^i = (\pi_1^2)^{-1}(\Sigma_1^i)$, where $\pi_1^2: J^2 \to J^1$ is the natural projection. For a smooth map $f: \boldsymbol{R}^n \to \boldsymbol{R}^m$, to say that $f^1(p) \in \Sigma_1^i$ and f^1 is transverse to Σ_1^i at p is to say that the 2-jet $f^2(p)$ of f at p belongs to a certain subset Σ_{*2}^i of Σ_2^i. The subset Σ_{*2}^i splits into manifolds $\Sigma_{*2}^i = \Sigma_{*2}^{i,0} \cup \Sigma_{*2}^{i,1} \cup \cdots$ defined as follows: $f^2(p) \in \Sigma_{*2}^{i,j}$ if the kernel of $d(f \mid \Sigma^i(f))_p$ has dimension j. Then $\Sigma^{i,j}(f) = (f^2)^{-1}(\Sigma_{*2}^{i,j})$. We likewise define submanifolds $\Sigma_{*3}^{i,j,k}$ of J^3, and so on.

We now consider Theorem 2 in the case $M = N = T$, the trivial Z_2-space of dimension 2, studied by Whitney [13]. Let Ω be an open subset of T, and K a compact subset of Ω. We first note that the subspace $\mathcal{O}_1 \subset C^\infty(\Omega, T)$ of maps of rank at least 1 throughout K is open and dense. In fact the subspace Σ_1^2 of linear maps of rank zero in $J^1 = L(\boldsymbol{R}^2, \boldsymbol{R}^2)$ has codimension 4, so that the 1-jet map of a smooth map generically misses it.

Now suppose $f \in \mathcal{O}_1$, so that taking Ω sufficiently small and changing variables in the source and target we can assume $f(x, y) = (u(x, y), y)$. The 1-jet map is given by the matrix of partial derivatives

$$f^1 = \begin{pmatrix} u_x & u_y \\ 0 & 1 \end{pmatrix}.$$

Let p be a singular point of f, i.e., $f^1(p) \in \Sigma_1^1$, or equivalently $u_x(p) = 0$. Then f^1 is transverse to Σ_1^1 at p if and only if $u_{xx}(p) \neq 0$ or $u_{xy}(p) \neq 0$. Generically $\Sigma^1(f) = (u_x = 0)$ is a 1-manifold (here $(u_x = 0)$ denotes the solution set of $u_x = 0$). In this case $f^2(p) \in \Sigma_{*2}^{1,0}$ if and only if $u_{xx}(p) \neq 0$; $\Sigma^{1,0}(f) = (f^2)^{-1}(\Sigma_{*2}^{1,0})$ is the set of regular points of $f \mid (u_x = 0)$. On the other hand, the conditions on the partial derivatives of f at p for $f^2(p) \in \Sigma_{*2}^{1,1}$ are $u_x(p) = 0$ and $u_{xx}(p) = 0$. Hence f^2 is transverse to $\Sigma_{*2}^{1,1}$ at this point p if and only if the matrix

$$\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xxx} & u_{xxy} \end{pmatrix}$$

has rank 2 at p; i.e., $u_{xy}(p)u_{xxx}(p) \neq 0$. Since $\Sigma^{1,1}_{*2}$ has codimension 2 in J^2, $\Sigma^{1,1}(f)$ consists of isolated points in $(u_x = 0)$.

Let $\mathcal{O} \subset C^\infty(\Omega, T)$ be the set of maps f which can be written near any point of K as $f(x, y) = (u(x, y), y)$, where if $p \in K$ is such that $u_x(p) = u_{xx}(p) = 0$, then $u_{xy}(p) \neq 0$ and $u_{xxx}(p) \neq 0$ (we will call such singularities "Whitney singularities"). We conclude from the transversality theorem that $\mathcal{O}$ is open and dense.

4. Generic singularities of equivariant maps

In this section we begin the proof of Theorem 2. There are nine cases depending on the choice of M, N among the three orthogonal Z_2-spaces A, R, T defined in § 1. In each case we let Ω be an invariant open neighborhood of the origin in M, and K a compact subset of Ω. We will describe an open dense set $\mathcal{O}$ of maps $f \in C^\infty_{Z_2}(\Omega, N)$ in terms of general position of their induced jet maps to certain algebraic subsets of the jet spaces, and corresponding conditions on their partial derivatives (throughout K). The case $T \to T$ has already been considered in § 3. Theorem 2 in the case $T \to R$ is given by Morse theory, while the case $T \to A$ is trivial: there is only one equivariant map $f(x, y) = 0$.

$R \to T$. We first note that the set $\mathcal{O}_1 \subset C^\infty_{Z_2}(\Omega, T)$ of maps of rank at least one throughout K is open and dense. It is clearly open. Given $f(x, y) = (u(x, y), v(x, y))$ in $C^\infty_{Z_2}(\Omega, T)$, consider $f' = (u', v')$ defined by

$$u' = u + \alpha_1 y , \qquad v' = v + \beta_1 y .$$

By Sard's theorem α_1, β_1 can be chosen arbitrarily small so that $(f')^1$ avoids Σ^2_1 on $(x = 0)$. Now consider the equivariant map $f'' = (u'', v'')$ given by

$$u'' = u' + \alpha_2 y + \gamma x^2 , \qquad v'' = v' + \beta_2 y + \delta x^2 .$$

Again using Sard's theorem, $\alpha_2, \beta_2, \gamma, \delta$ can be chosen arbitrarily small so that $(f'')^1$ now avoids Σ^2_1 throughout K. This establishes the denseness of $\mathcal{O}_1$.

Now suppose $f \in \mathcal{O}_1$. Near any point p outside $(x = 0)$ we can treat f as in the case $T \to T$ considered in § 3. On the other hand, taking Ω a sufficiently small invariant neighborhood of a point p on $(x = 0)$, and changing variables equivariantly in the source and target, we can assume $f(x, y) = (u(x, y), y)$ with $u(-x, y) = u(x, y)$. Note that equivariance dictates the form of f: since $(x, y) \to (y, x)$ is not an equivariant coordinate change, we cannot write f in the form $(x, v(x, y))$. The 1-jet map f^1 of $f = (u, y)$ is given by the matrix of partial derivatives

$$f^1 = \begin{pmatrix} u_x & u_y \\ 0 & 1 \end{pmatrix} .$$

Recall from § 3 that the 3-jet of a map $f(x, y) = (u(x, y), y)$ at a singular point generically belongs either to the submanifold $\Sigma^{1,0}_{*3}$ of J^3 given by $u_x = 0$, $u_{xx} \neq 0$, or to the submanifold $\Sigma^{1,1,0}_{*3}$ of J^3 given by $u_x = 0$, $u_{xx} = 0$, $u_{xy}u_{xxx} \neq 0$. But in our equivariant situation, along $(x = 0)$ 3-jets which do not lie in $\Sigma^{1,0}_{*3}$ can belong only to the frontier of $\Sigma^{1,1,0}_{*3}$, i.e., the cone

$$C: u_x = 0 , \quad u_{xx} = 0 , \quad u_{xy}u_{xxx} = 0 .$$

Fig. 2 shows the subspace of J^3 spanned by the u_{xx}-, u_{xy}-, and u_{xxx}-coordinates. Since u is even in x, the image of $(x = 0)$ in this subspace lies along the u_{xx}-axis, so that the 3-jet of f at a point p along $(x = 0)$ with $u_{xx}(p) = 0$ is trapped by equivariance at the origin of the cone C. But by a small equivariant deformation we can make the tangent map of f^3 at p span a plane in general position with respect to the tangent cone of C at the origin, i.e., a plane which intersects the cone only at its vertex (as shown in Fig. 2). In fact the tangent map of $(u_{xx}, u_{xy}, u_{xxx})$ is given by the matrix of partial derivatives

$$\begin{pmatrix} u_{xxx} & u_{xxy} \\ u_{xxy} & u_{xyy} \\ u_{xxxx} & u_{xxxy} \end{pmatrix} .$$

Since u_{xxx}, u_{xyy}, u_{xxxy} vanish at p by equivariance, the image of the tangent map is the linear space spanned by the vectors

$$(0, u_{xxy}(p), u_{xxxx}(p)) , \qquad (u_{xxy}(p), 0, 0) ;$$

this space is a plane in general position with respect to the tangent cone of C if $u_{xxy}(p) \neq 0$ and $u_{xxxx}(p) \neq 0$. It is easy to check that such points p are isolated on $(x = 0)$, and that nearby singular points are of the usual Whitney type.

Given $f = (u, y) \in C^\infty_{Z_2}(\Omega, T)$ there is an equivariant map arbitrarily close to f displaying singularities of the above type on $(x = 0) \cap K$, and singularities of Whitney type elsewhere in K. In fact, let

$$u' = u + \alpha_1 x^2 + \beta_1 x^2 y + \gamma_1 x^4 .$$

By Sard's theorem we can choose $\alpha_1, \beta_1, \gamma_1$ arbitrarily small so that at each point p of $(x = 0)$:

(6) If $u'_{xx}(p) = 0$, then $u'_{xxy}(p) \neq 0$ and $u'_{xxxx}(p) \neq 0$.

Then let

$$u'' = u' + \alpha_2 x^2 + \beta_2 x^2 y + \gamma_2 x^4 + \delta x^6 \ .$$

Again using Sard's theorem we can choose $\alpha_2, \beta_2, \gamma_2, \delta$ arbitrarily small so that (6) still holds along $(x = 0) \cap K$, and at at any point p outside $(x = 0)$, if $u''_x(p) = 0 = u''_{xx}(p)$, then $u''_{xy}(p) \neq 0$ and $u''_{xxx}(p) \neq 0$. We have now shown the following:

Lemma 2. *Let Ω be an invariant open neighborhood of the origin in R, and K a compact subset of Ω. Let $\mathcal{O}$ be the set of maps $f \in C^\infty_{Z_2}(\Omega, T)$ which can be written in invariant coordinates near any point p of K as $f(x, y) = (u(x, y), y)$, where if $p \in (x = 0)$ then u satisfies* (6), *and elsewhere in K, f displays only Whitney singularities. Then $\mathcal{O}$ is open and dense.*

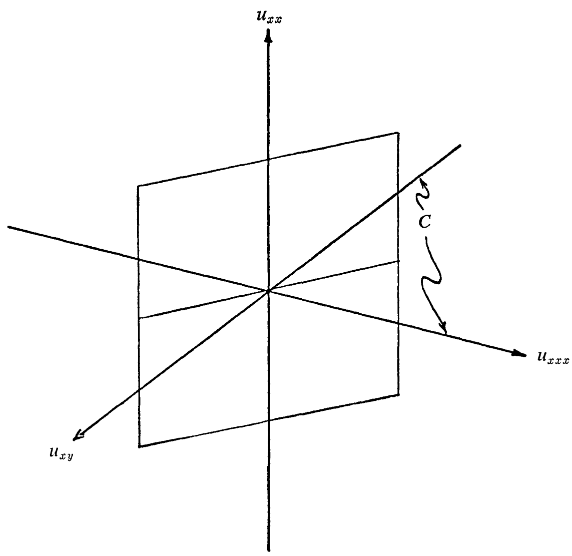

Fig. 2

$A \to T$. For any $f(x, y) = (u(x, y), v(x, y))$ in $C^\infty_{Z_2}(\Omega, T)$, the coordinate functions u and v are both even in (x, y). The 1-jet map is given by the matrix of partial derivatives

$$f^1 = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} ,$$

so that $f^1(0) = 0$. In the space of 1-jets J^1, the closure of Σ^1_1 is a quadratic cone $u_x v_y - u_y v_x = 0$. Recall that in the case $T \to T$ the 1-jet of a smooth map generically hits this cone only in nonsingular points (i.e., avoids the vertex Σ^2_1), and is transverse to the cone at these points. But in the case $A \to T$ equivariance traps $f^1(0)$ at the vertex of the cone. That f^1 be transverse to the cone at the origin means that the tangent map of f^1 at $0 \in A$ spans a plane L in general position with respect to the tangent cone C at the vertex of $Cl(\Sigma^1_1)$;

i.e., $L \cap C$ contains either 2 linearly independent directions or the zero vector only.

It can be checked that this situation is equivalent to looking at the space of quadratic forms $a\xi^2 + 2b\xi\eta + c\eta^2$ in 2 variables (ξ, η), and considering the linear subspace L spanned by

$$u''(0) = \tfrac{1}{2}(u_{xx}(0), u_{xy}(0), u_{yy}(0)) , \qquad v''(0) = \tfrac{1}{2}(v_{xx}(0), v_{xy}(0), v_{yy}(0)) ,$$

in relation to the quadratic cone $C: b^2 - ac = 0$. Given $f = (u, v) \in C^\infty_{Z_2}(\Omega, T)$ we can add quadratic terms with arbitrarily small coefficients to u and v so that L is a plane in general position with respect to C. In this case singular points of f nearby the origin are of Whitney type. Planes in general position with respect to C are shown in Fig. 3.

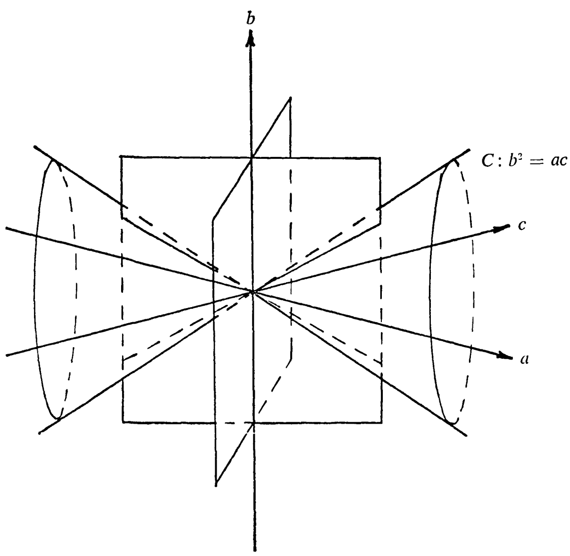

Fig. 3

Lemma 3. *Let Ω be an invariant open neighborhood of the origin in A, and K a compact subset of Ω. Let $\mathcal{O} \subset C^\infty_{Z_2}(\Omega, T)$ be the set of maps $f = (u, v)$ such that the quadratic forms $u''(0)$, $v''(0)$ span a plane in general position with respect to the cone $b^2 - ac = 0$, and f has only Whitney singularities throughout $K - \{0\}$. Then $\mathcal{O}$ is open and dense.*

$R \to R$, $A \to A$, $A \to R$. In these cases only singularities of Whitney type occur generically, and equivariance restricts their positions.

In the case $R \to R$ first note that the subset $\mathcal{O}_1 \subset C^\infty_{Z_2}(\Omega, R)$ of maps f of rank at least one throughout K is open and dense. In fact given $f = (u, v)$, the map $f' = (u', v')$ defined by

$$u' = u + \alpha x + \gamma xy\ , \qquad v' = v + \beta y$$

for suitable arbitrarily small α, β, γ has rank at least one at every point. If $f \in \mathcal{O}_1$, and Ω is a sufficiently small invariant neighborhood of a point on $(x = 0)$, then we can assume either $f(x, y) = (x, v(x, y))$, with $v(-x, y) = v(x, y)$, or $f(x, y) = (u(x, y), y)$ with $u(-x, y) = -u(x, y)$. In the former case, by an equivariant deformation of the form

$$v' = v + \alpha y + \beta y^2 + \gamma x^2 y + \delta y^3\ ,$$

with $\alpha, \beta, \gamma, \delta$ arbitrarily small, we can arrange that $v'_y(p)$ and $v'_{yy}(p)$ are not both zero for $p \in (x = 0)$, and that $f' = (x, v')$ displays only Whitney singularities elsewhere. In the latter case $u_{xx} = 0$ throughout $(x = 0)$, but by a deformation

$$u' = u + \alpha x + \beta xy + \gamma x^3 + \delta x^5\ ,$$

$\alpha, \beta, \gamma, \delta$ aritrarily small, we arrange that $u_x(p)$ and $u_{xy}(p)u_{xxx}(p)$ are not both zero for $p \in (x = 0)$, and that $f' = (u', y)$ displays only Whitney singularities elsewhere in Ω.

In the case $A \to A$, if $f(x, y) = (u(x, y), v(x, y))$ is in $C^\infty_{Z_2}(\Omega, A)$, then u, v are both odd in (x, y). Since $\Sigma^1_1 \subset J^1$ has codimension one, we can add linear terms with arbitrarily small coefficients to u, v, making f nonsingular at the origin. By a subsequent equivariant deformation the map can be made to display only Whitney singularities elsewhere.

On the other hand, in the case $A \to R$ an equivariant map $f = (u, v)$ must be singular at the origin. Let $\mathcal{O}$ be the set of maps $f \in C^\infty_{Z_2}(\Omega, R)$ which can be written, in suitable invariant coordinates near the origin, as $f(x, y) = (x, v(x, y))$ with $v_{yy}(0) \neq 0$ ($v_y(0) = 0$ since $v(-x, -y) = v(x, y)$, and which displays only Whitney singularities elsewhere in K. Then $\mathcal{O}$ is open and dense.

$R \to A$. Again the set $\mathcal{O}_1 \subset C^\infty_{Z_2}(\Omega, A)$ of maps of rank at least one throughout K is open and dense, and we can assume $f \in \mathcal{O}_1$ is of the form $f(x, y) = (x, v(x, y))$ with $v(-x, y) = -v(x, y)$. Then 1-jet map of f is given by the matrix of partial derivatives

$$f^1 = \begin{pmatrix} 1 & 0 \\ v_x & v_y \end{pmatrix}\ .$$

Along $(x = 0)$, the 3-jet of f belongs only to the frontier of $\Sigma^{1,1,0}_{*3}$, i.e., the cone

$$C\colon v_y = 0\ , \quad v_{yy} = 0\ , \quad v_{xy}v_{yyy} = 0\ .$$

The image of $(x = 0)$ under the 3-jet map lies along the v_{xy}-axis of the $(v_y, v_{yy}, v_{xy}, v_{yyy})$-coordinate subspace of J^3. In this equivariant situation, general position of f^3 with respect to C at points p on $(x = 0)$ means that either $v_{xy}(p) \neq 0$ or $v_{xyy}(p) \neq 0$: In either case the image of the tangent map of f^3

at p contains a direction which does not lie in the (v_{xy}, v_{yyy})-plane. Fig. 4 shows the space of $(v_{yy}, v_{xy}, v_{yyy})$-coordinates and the image of the tangent map in general position at a point with $v_{xy} = 0$.

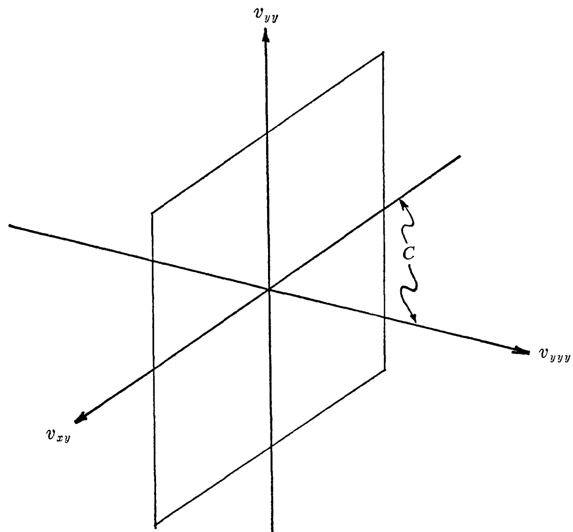

Fig. 4

Lemma 4. *Let Ω be an invariant open neighborhood of the origin in R, and K a compact subset of Ω. Let $\mathcal{O}$ be the set of maps $f \in C^\infty_{Z_2}(\Omega, A)$ which can be written in invariant coordinates near any point p of $(x = 0) \cap K$ as $f(x, y) = (x, v(x, y))$ with $v_{xy}(p), v_{xyy}(p)$ not both zero, and which display only Whitney singularities elsewhere in K. Then $\mathcal{O}$ is open and dense.*

In fact, given $f(x, y) = (x, v(x, y))$ in $C^\infty_{Z_2}(\Omega, A)$ we can choose $\alpha, \beta, \gamma, \delta$ arbitrarily small so that with

$$v' = v + \alpha xy + \beta xy^2 + \gamma xy^3 + \delta x^3 y \ ,$$

the map $f' = (x, v')$ satisfies the conditions of the lemma.

We remark that $f(x, y) = (x, v(x, y))$ can be written as $f(x, y) = (x, xV(x, y))$ where the map $g(x, y) = (x, V(x, y))$ belongs to $C^\infty_{Z_2}(\Omega, R)$. Hence Lemma 4 can, in fact, be deduced from the discussion of the case $R \to R$ above.

5. Normal forms for equivariant maps

In this section we complete the proof of Theorem 2, using the generic conditions of § 4 to obtain explicit normal forms. Only the cases $A \to T$, $R \to T$, and $R \to A$ will be discussed. The cases $T \to A$, $T \to R$ have already been completed, while the remaining cases can be handled in the same way as

$R \to T$ below (see also [13], [4] for the Whitney case $T \to T$). It will suffice, in each case, to consider an equivariant map f taking 0 into 0, and find a normal form for f in an invariant neighborhood of the origin. Again Ω denotes an invariant open neighborhood of the origin in the source.

$A \to T$. If $f(x, y) = (u(x, y), v(x, y))$ is a map in $C^\infty_{Z_2}(\Omega, T)$, then by Lemma 3 the quadratic forms $u''(0)$, $v''(0)$ generically span a plane in general position with respect to the cone $b^2 - ac = 0$ in the space of quadratic forms $a\xi^2 + 2b\xi\eta + c\eta^2$ in 2 variables (ξ, η). We can write

$$u(x, y) = x^2u_1(x, y) + 2xyu_2(x, y) + y^2u_3(x, y) ,$$
$$v(x, y) = x^2v_1(x, y) + 2xyv_2(x, y) + y^2v_3(x, y) ,$$

where u_i, v_i, $i = 1, 2, 3$, are smooth functions which are even in (x, y). Then

$$u''(0) = (u_1(0), u_2(0), u_3(0)) , \qquad v''(0) = (v_1(0), v_2(0), v_3(0)) .$$

In the following we use the fact that in the space of quadratic forms in 2 variables, the way in which the linear subspace spanned by 2 quadratic forms intersects the cone $b^2 - ac = 0$ is invariant under the natural action of $GL(2, \boldsymbol{R}) \times GL(2, \boldsymbol{R})$ on pairs of quadratic forms (see [1, § 3]).

We begin with some preliminary transformations used to rewrite f in the form

$$u = x^2 \pm y^2 + R(x, y) , \qquad v = 2xy , \tag{7}$$

where $R(-x, -y) = R(x, y)$, and R is of order at least 4. First, making linear coordinate changes in the source and target, we can assume $v''(0)$ is a vector outside the cone $b^2 = ac$. Then changing coordinates near the origin of the source as in the proof of the Morse lemma [5, Lemma 2.2], we put f in the form $u = x^2u_1 + 2xyu_2 + y^2u_3$, $v = x^2 - y^2$, and, with a subsequent linear change of variables, in the form $u = x^2u_1 + 2xyu_2 \pm y^2u_3$, $v = 2xy$, with $u_1(0)$, $u_3(0) > 0$. The $+$ or $-$ sign occurs according as the plane spanned by $u''(0)$, $v''(0)$ intersects the cone $b^2 = ac$ in 2 linearly independent directions, or in the point 0 alone. Further equivariant coordinate changes

$$x' = x(u_1(x, y)/u_3(x, y))^{1/4} , \qquad y' = y(u_3(x, y)/u_1(x, y))^{1/4}$$

in the source and

$$u' = (u - u_2(0)v)u_1(0)^{-1/2}u_3(0)^{-1/2} , \qquad v' = v$$

in the target, now put f in the form (7).

The polynomials $x^2 + y^2$, $x^2 - y^2$, $2xy$ generate the algebra of polynomials in (x, y) invariant under the antipodal map, so with f in the form (7) we use Theorem 3 to write

$$u = x^2 \pm y^2 + S(x^2 + y^2, x^2 - y^2, 2xy)$$

near the origin, where S is a smooth function of order at least 2. Consider the case $u = x^2 - y^2 + S$; the $+$ case is similar. We will use the algebraic relationship among the invariant polynomials $x^2 + y^2$, $x^2 - y^2$, $2xy$ to write

$$\begin{aligned} &S(x^2 + y^2, x^2 - y^2, 2xy) \\ &\qquad = \theta(x^2 - y^2 + S, 2xy) + (x^2 + y^2)\phi(x^2 - y^2 + S, 2xy) , \end{aligned} \tag{8}$$

where θ and ϕ are smooth functions (of orders at least 2 and 1 respectively). With S in this form, the equivariant coordinate changes

$$\begin{aligned} x' &= x(1 + \phi)^{1/2} , & y' &= y(1 - \phi)^{1/2} , \\ u' &= u - \theta(u, v) , & v' &= v(1 - \phi^2(u, v))^{1/2} \end{aligned}$$

near the origin of the source and target, put f in the normal form: $u' = x'^2 - y'^2$, $v' = 2x'y'$.

It remains to establish (8). Consider the equality

$$(x^2 - y^2 + S)^2 + (2xy)^2 = (x^2 + y^2)^2 + 2(x^2 - y^2)S + S^2 . \tag{9}$$

Let

$$\begin{aligned} f(\xi, \eta, \zeta) &= \xi^2 + 2\eta S(\xi, \eta, \zeta) + S^2(\xi, \eta, \zeta) , \\ g(\xi, \eta, \zeta) &= \eta + S(\xi, \eta, \zeta) , \\ h(\xi, \eta, \zeta) &= \zeta . \end{aligned}$$

By the preparation theorem [4, Cor. 4.4, p. 77], the ring $\mathscr{E}_3$ of germs at 0 of C^∞ functions in 3 variables (ξ, η, ζ) is generated, over the subalgebra of germs of C^∞ functions of f, g, h, by $1, \xi$. Hence there exist $\Theta, \Phi \in \mathscr{E}_3$ such that

$$S(\xi, \eta, \zeta) = \Theta(f, g, h) + \xi\Phi(f, g, h) .$$

Put $\xi = x^2 + y^2$, $\eta = x^2 - y^2$, $\zeta = 2xy$. Then $f = g^2 + h^2$ by (9), so that $S(x^2 + y^2, x^2 - y^2, 2xy)$ takes the form (8).

$R \to T$. By Lemma 2, a map $f \in C^\infty_{Z_2}(\Omega, T)$ is generically of the form $f(x, y) = (u(x, y), y)$ with $u(-x, y) = u(x, y)$ in suitable invariant coordinates near 0, where if $u_{xx}(0) = 0$ then $u_{xxy}(0) \neq 0$ and $u_{xxxx}(0) \neq 0$.

First suppose $u_{xx}(0) \neq 0$. By Theorem 3 or [11], there is a C^∞ map $U(\xi, \eta)$ defined near 0 so that $u(x, y) = U(x^2, y)$. Then $U(0) = 0$ and $U_\xi(0) \neq 0$. By the preparation theorem, the ring $\mathscr{E}_2$ of germs at 0 of C^∞ functions in 2 variables (ξ, η) is identical to the subalgebra of germs of C^∞ functions of $U(\xi, \eta), \eta$. Hence there exists $\Phi \in \mathscr{E}_2$ such that $\xi = \Phi(U(\xi, \eta), \eta)$, so that $x^2 = \Phi(u(x, y), y)$. We check that

$$u' = \Phi(u, v) , \qquad v' = v$$

is a coordinate change near 0 in the target; in these coordinates f takes the normal form $(x, y) \to (x^2, y)$.

Now suppose $f(x, y) = (u(x, y), y)$, with

$$u_{xx}(0) = 0 , \quad u_{xxy}(0) \neq 0 , \quad u_{xxxx}(0) \neq 0 .$$

Again there exists $U \in \mathscr{E}_2$ with $u(x, y) = U(x^2, y)$, so that

$$(10) \qquad U_\xi(0) = 0 , \quad U_{\xi\xi}(0) \neq 0 , \quad U_{\xi\eta}(0) \neq 0 .$$

By the preparation theorem there exist $\Phi, \Psi \in \mathscr{E}_2$ such that

$$(11) \qquad \xi^2 = \Phi(U(\xi, \eta), \eta) + \xi\Psi(U(\xi, \eta), \eta) ,$$

so that

$$x^4 = \Phi(u(x, y), y) + x^2\Psi(u(x, y), y) .$$

Clearly $\Phi(0) = \Psi(0) = 0$. Using (10), (11), we check that the following are equivariant coordinate changes at the origin of the source and target respectively:

$$\begin{aligned} x' &= x , & y' &= \Psi(u(x, y), y) , \\ u' &= \Phi(u, v) , & v' &= \Psi(u, v) . \end{aligned}$$

In these new coordinates (dropping primes), f takes the normal form $(x, y) \to (-x^2y + x^4, y)$.

$R \to A$. By Lemma 4, $f \in C^\infty_{Z_2}(\Omega, A)$ is generically of the form $f(x, y) = (x, v(x, y))$ near 0, where $v_{xy}(0)$, $v_{xyy}(0)$ are not both zero. Since $v(-x, y) = -v(x, y)$, there is a C^∞ function $V(\xi, \eta)$ with $v(x, y) = xV(x^2, y)$.

If $v_{xy}(0) \neq 0$, so that $v_\eta(0) \neq 0$, then the equivariant coordinate change $x' = x$, $y' = V(x^2, y)$ at 0 in the source puts f in the normal form: $u = x'$, $v = x'y'$.

When $v_{xy}(0) = 0$, $v_{xyy}(0) \neq 0$, i.e., $V_\eta(0) = 0$, $V_{\eta\eta}(0) \neq 0$, we can argue as in Whitney [13, § 15]. Since $V_{\eta\eta}(0) \neq 0$, we can solve the equation $V_\eta(x^2, y) = 0$ near the origin using the implicit function theorem, obtaining an even function $y = \psi(x)$. Consider the equivariant coordinate changes

$$\begin{aligned} x' &= x , & y' &= y - \psi(x) , \\ u' &= u , & v' &= v - uV(u^2, \psi(u)) \end{aligned}$$

near the origin of the source and target (respectively). In the new coordinates $u' = x'$ and $v' = x'\Phi(x', y')$, where

$$\Phi(x', y') = V(x'^2, y' + \psi(x')) - V(x'^2, \psi(x')) .$$

Then $\Phi(x', 0) = 0$, $\Phi_{y'}(x', 0) = 0$, but $\Phi_{y'y'}(0, 0) \neq 0$, so that using Taylor's formula we can write $\Phi(x', y') = y'^2\Psi(x'^2, y')$ with $\Psi(0) \neq 0$. A further equivariant change of variables

$$x^* = x' , \qquad y^* = y'[\Psi(x'^2, y')]^{1/2}$$

in the source puts f in the normal form: $u' = x^*$, $v' = x^*y^{*2}$. This completes the proof of Theorem 2.

References

[1] V. I. Arnol'd, *Singularities of smooth mappings,* Uspehi Mat. Nauk. **23** (1968) 3–44=Russian Math. Surveys **23** (1968) 1–43.

[2] J. M. Boardman, *Singularities of differentiable maps,* Inst. Hautes Études Sci. Publ. Math. No. 33 (1967) 21–57.

[3] G. Glaeser, *Fonctions composées différentiables,* Ann. of Math. **77** (1963) 193–209.

[4] B. Malgrange, *Ideals of differentiable functions,* Oxford University Press, London, 1966.

[5] J. Milnor, *Morse theory,* Annals of Math. Studies, No. 51, Princeton University Press, Princeton, 1963.

[6] S. Stoïlow, *Sur les transformations continues et la topologie des fonctions analytiques,* Ann. Sci. École Norm. Sup. **45** (1928) 347–382.

[7] R. Thom, *Un lemme sur les applications différentiables,* Bol. Soc. Mat. Mexicana **1** (1956) 59–71.

[8] ——, *Les singularités des applications différentiables,* Ann. Inst. Fourier (Grenoble) **6** (1955–56) 43–87.

[9] H. Weyl, *The classical groups,* Princeton University Press, Princeton, 1946.

[10] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets,* Trans. Amer. Math. Soc. **36** (1934) 63–89.

[11] ——, *Differentiable even functions,* Duke Math. J. **10** (1943) 159–160.

[12] ——, *Singularities of mappings of Euclidean spaces,* Sympos. Internac. Topologia Algebraica, Mexico, 1958.

[13] ——, *On singularities of mappings of Euclidean spaces.* I: *Mappings of the plane into the plane,* Ann. of Math. **62** (1955) 374–410.

[14] G. T. Whyburn, *Topological analysis,* Princeton University Press, Princeton, 1958.

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