J. DIFFERENTIAL GEOMETRY
**10** (1975) 435-466

# A THEOREM OF GÉOMÉTRIE FINIE

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## 1. Introduction

In a paper in the Kodaira *Festschrift* R. Thom [6] gave a proof, admittedly incomplete, of the following. Let $V^{2k} \subset P_{n+k}$ be a real compact embedded submanifold of real dimension $2k$ of a complex projective space of complex dimension $n + k$. Suppose there exists an everywhere dense subset $U \subset G_{n,k}$ of the Grassmannian of all complex projective subspaces of complex dimension $n$ of $P_{n+k}$, such that if $u \in U$ then $u \cap V^{2k}$ consists of exactly $m$ points, where $m$ is independent of $u$. Then $V^{2k}$ is an algebraic subvariety of $P_{n+k}$; the flat case is excluded. In this paper we give a complete and corrected statement and proof of this result. Moreover, we will allow $V^{2k}$ to have certain singularities.

By a *semi-real flat L* we mean the closure in $P_{n+k}$ of an affine subspace $L_0 \subset R^{2(n+k)}$, where we make the canonical identification $R^{2(n+k)} = C^{n+k} \subset P_{n+k}$. These may be classified in the following way. Apply a complex affine transformation (which is also a real affine transformation) to make $L_0$ pass through the origin. Let $I: C^{n+k} \to C^{n+k}$ be the multiplication by $i = \sqrt{-1}$. We call the real dimension of $L_0 \cap I(L_0)$ the *type* of $L$. If a complex projective transformation sends $L$ into a semi-real flat, then it preserves the type, so that a semi-real flat is classified up to a complex projective transformation by its dimension $j$ and its type $t$, where $0 \leq t \leq j$ with every such even $t$ possible. If $t = j$, then $L$ is a complex projective subspace, and if $t = 0$ then $L$ is complex projectively equivalent to the real projective space $P^j$ with its canonical embedding $P^j \subset P_j \subset P_{n+k}$. In the intermediate cases $0 < t < j$, $L$ is singular; in fact it is a kind of cone.

We say that a continuous map $f: X \to Y$ of topological spaces is *proper onto its image* if for every compact subset $A \subset f(X)$, $f^{-1}(A)$ is compact. We now state the main result.

**Theorem.** *Let $V \subset P_{n+k}$ be a compact subset. Suppose there exists a closed subset $S \subset V$ such that the closure of $V - S$ is all of $V$, and an immersion $f: M \to P_{n+k}$ of class $C^4$ of a differentiable manifold $M$ of (real) dimension $2k$ which maps $M$ onto $V - S$ and which is proper onto its image. Suppose further:*

1) *there exists an everywhere dense subset $T \subset G_{n+1,k-1}$ such that if $v \in T$*

Received October 5, 1973, and, in revised form, November 18, 1974. Research supported by the National Science Foundation under Grant GP-29321.

*then $v \cap S$ consists of finitely many points and $v$ is transversal to $f$; and*

2) *there exists an everywhere dense subset $U \subset G_{n,k}$ such that either*

a) *$u \cap V$ consists of exactly $m$ points for every $u \in U$, with $0 < m < \infty$ and $m$ independent of $u$, or else*

b) *$f^{-1}(u)$ consists of exactly $m'$ points for every $u \in U$, where $m'$ is independent of $u, 0 < m' < \infty$.*

*Then $V$ is a finite union of complex projective transforms of semi-real flats of dimension $2k$ and the closures in $P_{n+k}$ of analytic subvarieties of complex dimension $k$ of $P_{n+k} - S$.*

**Corollary 1.** *Suppose $V \subset P_2$ satisfies the hypothesis of the Theorem with $k = 1$. Then $V$ is a finite union of algebraic curves and complex projective transforms of the real projective plane $P^2$ with its canonical embedding in $P_2$.*

**Corollary 2.** *Let $M$ be a compact connected differentiable manifold, and $f: M \to P_{n+k}$ a $C^4$ embedding. Suppose that almost every projective subspace of $P_{n+k}$ of complex dimension $n$ meets $f(M)$ in exactly $m$ points, where $m$ is independent of the subspace, $0 < m < \infty$. Then $f(M)$ is either an algebraic variety of dimension $k$ or a complex projective transform of the real projective $2k$-space with its canonical embedding $P^{2k} \subset P_{2k} \subset P_{n+k}$.*

**Corollary 3.** *Suppose that $V$ satisfies the hypothesis of the Theorem, and in addition that $S$ has Hausdorff $(2k - 1)$-measure zero. Then $V$ is a finite union of complex projective transforms of $2k$-dimensional semi-real flats and algebraic varieties of complex dimension $k$.*

The corollaries are proved from the Theorem as follows. In the case of Corollary 1, Hypothesis 1) of the Theorem implies that $S$ is finite. Hence the closure in $P_2$ of every analytic subvariety of $P_2 - S$ is an analytic subvariety of $P_2$ by a theorem of Remmert and Stein [1], and is an algebraic variety by Chow's theorem. Any semi-real flat of real dimension 2 in $P_2$ has type 0 or 2, so is either a projective transform of $P^2 \subset P_2$ or a complex line. In the case of Corollary 2, the hypothesis of the Theorem is fulfilled with $S = 0$; the dimension of $M$ must be $2k$, for if it is smaller there exists an open set of $n$-planes which do not meet $f(M)$, and if it is greater there exists an open set of $n$-planes which meet $f(M)$ in infinitely many points. If $f(M)$ is a complex-projective transform of a semi-real flat, its type must be 0 or $2k$, for those of intermediate type are singular. If $f(M)$ is an analytic variety, it must be algebraic by Chow's theorem. In the case of Corollary 3, a theorem of Shiffman [5] asserts that the closure in $P_{n+k}$ of any analytic subvariety of $P_{n+k} - S$ is an analytic subvariety of $P_{n+k}$ and hence an algebraic variety by Chow's theorem. Whether the hypothesis of the Theorem is strong enough by itself to yield the conclusion of Corollary 3, the author does not know.

Let us note that finite unions of algebraic varieties and complex-projective transforms of semi-real flats satisfy the hypothesis of the Theorem, so that Corollary 3 gives a characterization.

We shall prove Corollary 1 directly in this paper, the proof occupying

§§ 2–7. Various parts of this proof are used to prove the general case of the Theorem in § 8. The essential part of the hypothesis of the Theorem, namely Hypothesis 2), is used only at the end of the argument, so to speak, so that much of what we prove in this paper is valid in general for real submanifolds of complex projective spaces. Notable are Propositions 2 and 12, which characterize locally those even-dimensional submanifolds of $P_{n+k}$ which are complex-projective transforms of semi-real flats, Proposition 4, concerning the singularity at the diagonal of the secant map, and Proposition 11 (generalized in § 8) on the intersection of linear spaces, near to a tangent linear space, with a submanifold.

## 2. A separation of cases

Let $f: M \to C^N$ be a $C^2$-immersion of a differentiable manifold into a complex number space. For each $p \in M$ let $\tau_p$ denote the tangent space to $M$ at $p$, and $T_p$ the space of real lines through the origin of $\tau_p$. Let $\pi: T(M) \to M$ denote the bundle of (real) lines through the origins of the tangent spaces of $M$. We call the map $l: T(M) \to G_{1,N-1}$, which assigns to each real tangent line the complex line in $P_N$ containing it, the *associated map* to $f$.

Let $t \in T(M)$, and let $c$ be a curve on $M$ through $p = \pi(t)$ tangent to $t$. Now suppose that $l(t)$ is not contained in $\tau_p$, so that $\tau_p$ and $l(t)$ span a real affine subspace $K(t)$ of $C^N = R^{2N}$ of dimension one more than that of $\tau_p$. If the curvature vector of $c$ at $p$ is not contained in $K(t)$, we call $t$ an *ordinary direction*. Note that by Meusnier's theorem the condition that the curvature vector lie in $K(t)$ is independent of the choice of $c$ tangent to $t$. If $l(t)$ is not contained in $\tau_p$ but the curvature vector of $c$ at $p$ *is* contained in $K(t)$, we call $t$ a *direction of type F*. If $T_p$ contains an ordinary direction, we say that $p$ is a *point of type O*. Note that the set of points of $M$ of type $O$ is open. If $p \in M$ is not a point of type $O$, then either $l(t) \subset \tau_p$ for every $t \in T_p$, in which case we say that $p$ is a *point of type C*, or else, for some $t \in T_p$, $l(t) \not\subset \tau_p$, and every $t \in T_p$ such that $l(t) \not\subset \tau_p$ is a direction of type $F$, in which case we say that $p$ is a *point of type F*. Note that if $p$ is a point of type $C$, then $\tau_p$ is a complex vector space in $C^N$. We have used the word "type" already in connection with semi-real flats, but we think that no confusion will arise from the two usages of the term. The property of being a point of type $C$ is clearly invariant under complex projective transformations of $C^N$. We show below (Proposition 1) that the properties of being a point of type $F$ or $O$ are also invariant. Consequently these notions make sense for points of submanifolds of $P_N$.

We must next examine the structure of semi-real flats and make some observations needed in the sequel. As in § 1, we will distinguish real and complex projective spaces by raised and lowered indices, so that $P^m$ will denote the real projective space of real dimension $m$ and $P_m$ the complex projective space of complex dimension $m$. The canonical inclusions $R^m \subset R^{m+j}$, $C^m \subset$

$C^{m+j}$, taking the subspace as that defined by the vanishing of the last $j$ coordinates, induce inclusions $P^{m-1} \subset P^{m+j-1}$ and $P_{m-1} \subset P_{m+j-1}$. Regarding a real $m$-tuple as a complex $m$-tuple gives a canonical inclusion $R^m \subset C^m$, which induces a canonical inclusion $P^{m-1} \subset P_{m-1}$.

Let $L \subset P_{n+k}$ be a semi-real flat, the closure in $P_{n+k}$ of a real affine subspace $L_0 \subset R^{2(n+k)} = C^{n+k} \subset P_{n+k}$, (the middle identification arising from the usual identification of $R^2$ with $C$). Apply a translation to bring $L$ to the origin. Let $s_1, \cdots, s_r$ be a complex basis for $L_0 \cap I(L_0)$, the latter regarded as a linear subspace of $C^{n+k}$, $I$ being multiplication by $i$ in $C^{n+k}$, and $2r = t$ the type of $L$, as previously defined. Extend $s_1, \cdots, s_r, Is_1, \cdots, Is_r$ to a real basis of $L_0$, $s_1, \cdots, Is_r, s_{r+1}, \cdots, s_{j-r}$, where $j$ is the real dimension of $L$. Next we claim that $s_1, \cdots, s_r, s_{r+1}, \cdots, s_{j-r}$ are linearly independent over the complex numbers. For, if

$$\sum_{l=1}^{j-r} (a_l + Ib_l)s_l = 0 \ , \qquad a_l, b_l \text{ real} \ ,$$

then

$$\sum a_l s_l = -I \sum b_l s_l \ ,$$

which implies that $\sum a_l s_l$ lies in $L_0 \cap I(L_0)$. Hence we must have $a_{r+1} = \cdots = a_{j-r} = 0$, since $s_1, \cdots, Is_r$ forms a real basis for $L_0 \cap I(L_0)$ and $s_1, \cdots, Is_r$, $s_{r+1}, \cdots, s_{j-r}$ a real basis for $L_0$. Multiplying by $i$ on the other hand gives

$$I \sum a_l s_l = \sum b_l s_l$$

which implies that $b_{r+1} = \cdots = b_{j-r} = 0$, by the same argument. Then since $s_1, \cdots, s_r$ forms a complex basis for $L_0 \cap I(L_0)$, we must have $a_l + ib_l = 0$ for all $l \leq r$. Hence all $a_l, b_l$ vanish, which establishes the claim. Apply a complex linear transformation to $C^{n+k}$ to bring $s_1, \cdots, s_r, s_{r+1}, \cdots, s_{j-r}$ to the first $j - r$ vectors of the standard basis $e_1, \cdots, e_{n+k}$ of $C^{n+k}$. We now say that the semi-real flat lies in standard position, and we have shown that any two semi-real flats of the same dimension and type are complex projectively equivalent. Since the type of a semi-real flat is the same as the type of its tangent spaces, and since a holomorphic transformation preserves the type of the tangent spaces of any real submanifold, we have completely classified semi-real flats. It is also apparent from the form of the basis that if $t = 0$ then $L$ is just $P^j \subset P_j \subset P_{n+k}$.

Using the canonical inclusions, we see that since $L$ contains $e_1, \cdots, e_{j-r}$ it contains $P^{j-r}$. Let $c$ be the intersection of $P_r$ with the hyperplane at infinity, so that $c$ is a complex projective space of dimension $r - 1$. Since $L$ contains $e_1, \cdots, e_r, Ie_1, \cdots, Ie_r$, it contains $P_r$, which is the complex $r$-plane spanned by $c$ and the origin. Since $e_1, \cdots, e_r, Ie_1, \cdots, Ie_r, e_{r+1}, \cdots, e_{j-r}$ form a real basis for $L \cap C^{n+k} \subset P_{n+k}$, $L$ consists of the locus of all complex $r$-planes through points of $P^{j-r}$ parallel to $P_r$, or equivalently, $L$ is the locus of all complex $r$-planes containing $c$ and a variable point of $P^{j-r}$. Thus we see that in case $0 < 2r < j$, $L$ is a kind of cone as previously asserted.

Let us now assume that $0 < 2r < j$. We must next determine which points of $L$ are singular points. Since $L$ is the closure of a real affine subspace of $R^{2(n+k)}$, all points of $L$ not lying at infinity are regular points. Let $p \in L - c$ lie in the hyperplane at infinity. The point $p$ lies in the complex $r$-plane $Q$ spanned by $c$ and some $q \in P^{j-r}$. Let $c' = c \cap P^r$, that is to say, the real locus of $c$. Then $c'$ spans $c$ in $P_{n+k}$, so that $Q$ is spanned by the real $r$-plane $Q'$ spanned in $P^{j-r}$ by $c'$ and $q$. Let us now apply a complex projective transformation to $P_{n+k}$ with real coefficients so that $P^{j-r}$ is sent into itself, so that $c'$ is sent into itself, and so that $Q'$ is sent into an $r$-plane which meets the hyperplane at infinity only in $c'$. Since $c'$ is preserved, $c$ must be preserved, and then since $P^{j-r}$ is preserved, $L$ must be mapped onto itself. Since $Q'$ meets the hyperplane at infinity only in $c'$, $Q$ now meets the hyperplane at infinity only in $c$. (The complex span of the intersection is the intersection of the complex spans.) Consequently, $p$ must have been sent to a point of $L$ not at infinity, in particular to a regular point. This proves that every point of $L - c$ is a regular point. Since holomorphic transformation preserves the type of the tangent space, the tangent space to $L$ at every point of $L - c$ is of type $t = 2r$.

We claim that all points of $c$ are singular points of $L$. For let $p \in c$ be an arbitrary point. Let $d \subset c'$ be a real $(r-2)$-plane such that the complex $(r-2)$-plane spanned by $d$ does not contain $p$. Make a complex projective transformation of $P_{n+k}$ with real coefficients in such a way that $P^{j-r}$ is sent into itself, so that $c'$ is taken to $P^{r-1}$ (and hence $c$ to $P_{r-1}$), and so that $d$ is taken to the intersection of $P^{r-1}$ with the hyperplane at infinity. The point $p$ is then moved to a point not at infinity. $L$ now consists of the locus of complex $r$-planes spanned by $P_{r-1}$ and a variable point of $P^{j-r} - P^{r-1}$. The real linear span of these in $R^{2(n+k)} = C^{n+k}$ is all of $C^{j-r}$, which has real dimension $2(j-r) > j$. Consequently $L$ cannot have a $j$-dimensional tangent space at $p$, proving that $p$ is a singular point and establishing the claim.

Next we must study the limiting tangent spaces at the singular locus $c$. Suppose $\{p_\nu\}$ is a sequence of points of $L$ approaching a point of $c$, which we may assume to be the point $p$ arbitrarily taken just above. We may assume that the $p_\nu$ do not lie at infinity. We may write

$$p_\nu = \sum_{l=1}^{r-1} a_{\nu l} e_l + \sum_{l=1}^{r-1} b_{\nu l} I e_l + (a_\nu + I b_\nu) \sum_{l=r}^{j-r} c_{\nu l} e_l ,$$

with the $a$'s, $b$'s and $c$'s real. The tangent space at $p_\nu$ can be found by differentiating this expression with respect to the parameters $a_{\nu l}, b_{\nu l}, a_\nu, b_\nu, c_{\nu l}$; it is spanned over the reals by

$$e_1, \cdots, e_{r-1}, Ie_1, \cdots, Ie_{r-1}, \sum c_{\nu l}e_l, I \sum c_{\nu l}e_l ,$$
$$(a_\nu + Ib_\nu)e_r, \cdots, (a_\nu + Ib_\nu)e_{j-r} .$$

It is just as well spanned by

$$e_1, \cdots, e_{r-1}, Ie_1, \cdots, Ie_{r-1}, f_\nu, If_\nu, z_\nu e_r, \cdots, z_\nu e_{j-r} ,$$

where

$$f_\nu = \frac{\sum c_{\nu l}e_l}{(\sum c_{\nu l}{}^2)^{1/2}} , \qquad z_\nu = \frac{a_\nu + Ib_\nu}{(a_\nu{}^2 + b_\nu{}^2)^{1/2}} .$$

Passing to a subsequence if necessary, we find the unit vectors $f_\nu$ converging to a unit vector $f$, and the complex numbers (of unit norm) $z_\nu$ to a complex number (of unit norm) $z$. Then $f$ has the form

$$f = \sum_{l=r}^{j-r} c_l e_l ,$$

and the limit of the tangent spaces at $p_\nu$ is a $j$-plane spanned by $e_1, \cdots, e_{r-1}$, $Ie_1, \cdots, Ie_{r-1}, f, If, ze_r, \cdots, ze_{j-r}$, which is a $j$-plane of type $t = 2r$. Thus we have shown that every limiting tangent space is a $j$-plane of type $t$. This concludes our remarks on the structure of semi-real flats.

Let $V, S, T, U, f: M \to P_{n+k}$ satisfy the hypothesis of the Theorem. We show in § 7, Proposition 11, for $k = n = 1$, and in § 8 for general $k, n$, that $M$ can contain no points of type $O$, so that all points are of type $C$ or $F$. Let $M_1$ be a connected component of $M$, and suppose that $M_1$ contains a point $p$ of type $F$. Then there must be some neighborhood $N$ of $p$ in $M_1$ all of whose points are of type $F$, for otherwise $p$ is a limit point of points of $M_1$ of type $C$, that is, points at which the tangent space is a complex $k$-plane, which would imply that the tangent space at $p$ is a complex $k$-plane. Hence the set of points of type $F$ is open in $M_1$. Let $N$ be the largest connected neighborhood of $p$ in $M_1$ consisting of points of type $F$. We show below (Proposition 2 for $k = 1$, and Proposition 12 for general $k$) that $f$ maps $N$ into a complex projective transform $Q$ of a semi-real flat of dimension $2k$ in $P_{n+k}$.

We claim that $N$ has no boundary points in $M_1$, so that $M_1 = N$. For suppose that $N$ has a boundary point $q \in M_1$. By continuity $f(q) \in Q$. If $f(q)$ is a regular point of $Q$, then by continuity the tangent space of $f$ at $q$ is the same as the tangent space of $Q$ at $f(q)$, which is not a complex $k$-plane, so that $q$ must be a point of type $F$. If $f(q) \in c$, the singular locus of $Q$, then the tangent space of $f$ at $q$ is not a complex $k$-plane, because, as we have shown, the limit of the tangent planes of $Q$ at a sequence of points of $Q$ converging to a point of $c$ is never a complex $k$-plane, so again $q$ must be a point of type $F$. But the set of points of type $F$ is open in $M$, which implies that $N$ contains a neighborhood of $q$, and hence $q$ is not a boundary point of $N$. This contradiction establishes the claim, and shows that every point of $M_1$ is of type $F$, and that $f(M_1) \subset Q$.

We next claim that in this case $f(M_1)$ is everywhere dense in $Q$. For suppose $Q - f(M_1)$ contains an open set $A$. Let $v'$ be a complex $(n + 1)$-plane transversal to $Q, c$, and to the hyperplane at infinity of $P_{n+k}$, which contains interior points in $Q$ of both $A$ and $f(M_1)$. By hypothesis we can find $v \in T$ close enough to $v'$ that $v$ is transversal to $Q, c$, and the hyperplane at infinity of $P_{n+k}$ and contains interior points in $Q$ of both $A$ and $f(M_1)$. Then $Q \cap v$ is a semi-real flat of dimension 2. $Q \cap v \cap S$ consists of only finitely many points by hypothesis, and $v \cap c$ consists of finitely many points by the transversality. Hence we can find an arc in $Q \cap v - S - c$ which joins a point of $f(M_1)$ to a point of $A$. This arc must contain a boundary point of $f(M_1)$ in $Q$. But since $V$ is compact and $f$ is proper onto its image, all boundary points of $f(M_1)$ in $Q$ must lie in $S$. This gives a contradiction, which proves the claim that $f(M_1)$ is everywhere dense in $Q$. Since $V$ is compact and $f(M_1) \subset V$, $Q \subset V$.

Suppose that $M_1$ is a connected component of $M$ which contains a point of type $C$. Then every point of $M_1$ must be of type $C$ because no points of type $O$ can occur, and if $M_1$ contained a point of type $F$ then every point of $M_1$ would be of type $F$ by the above. But the condition that every point of $M_1$ be of type $C$, that is, the condition that the tangent space of $f$ at every point of $M_1$ be a complex $k$-plane, implies that $f(M_1)$ is a complex-analytic immersed submanifold of $P_{n+k}$. Since $f$ is proper onto its image, and $f(M_1) \subset V$, and $V$ is compact, all limit points of $f(M_1)$ not lying in $f(M_1)$ must lie in $S$. Consequently $f(M_1)$ is a complex-analytic subvariety of $P_{n+k} - S$. We have now shown that $f(M)$ is dense in a union of semi-real flats of dimension $2k$ and complex analytic subvarieties of complex dimension $k$ of $P_{n+k} - S$. Since $V$ is the closure of $f(M)$ in $P_{n+k}$, $V$ *is* the union of these semi-real flats and the closures in $P_{n+k}$ of these complex-analytic subvarieties of $P_{n+k} - S$.

We next claim that $V$ is a *finite* union of such semi-real flats and such closures of analytic subvarieties. To prove this it suffices to show that $M$ has only finitely many components. Let $u'$ be an arbitrary complex $n$-plane in $P_{n+k}$, and suppose $M_1$ is a connected component of $M$ which is mapped by $f$ onto a complex analytic subvariety of $P_{n+k} - S$. Pass a complex $(n + 1)$-plane $v'$ through $u'$ and some point of $f(M_1)$. We may choose a complex $(n + 1)$-plane $v''$ arbitrarily close to $v'$ which meets $f(M_1)$ in at least one point and is transversal to $f$. By hypothesis we can find a $v \in T$ arbitrarily close to $v''$, which also meets $f(M_1)$. Since $v$ is transversal to $f$ by hypothesis, $v \cap f(M_1)$ is a complex-analytic curve, and the limit points of $v \cap f(M_1)$ not in $v \cap f(M_1)$ lie in the set $S \cap v$, which is finite by hypothesis. It follows from the theorems of Remmert-Stein and Chow that the closure $K$ of $v \cap f(M_1)$ is an algebraic curve. Now any complex $n$-plane contained in $v$ must meet $K$, which is to say must meet the closure of $f(M_1)$. Since $v$ can be taken arbitrarily close to $v'$, $v$

can be taken to contain a complex $n$-plane arbitrarily close to $u'$. It follows that $u'$ must meet the closure of $f(M_1)$, where $u'$ is an arbitrary complex $n$-plane. But let $u''$ be a complex $n$-plane in $v'' \in T$ which does not meet the finite set $v'' \cap S$. Since $S$ is compact, any $n$-plane sufficiently close to $u''$ will not meet $S$. Hence we can choose $u \in U$ which does not meet $S$. Since by what we have shown $u$ must meet the *closure* of the image under $f$ of any component of $M$ mapped by $f$ onto an analytic subvariety of $P_{n+k} - S$, $u$ must meet the image of any such component. If $M_1$ is any component of $M$ mapped by $f$ onto an everywhere dense subset of a projective transform of a semi-real flat $Q$, then clearly $u$ meets $Q$. Since $f(M_1)$ is closed in $P_{n+k} - S$, $u$ meets $f(M_1)$. Hence we have shown that $u$ meets the image under $f$ of any component of $M$. Now by hypothesis $u \cap f(M)$ is a finite set. If $M$ contained infinitely many components, then for some point $p \in u \cap f(M), f^{-1}(p)$ would be an infinite set. Since $f$ is proper onto its image, $f^{-1}(p)$ is compact and therefore contains a limit point. But since $f$ is an immersion, and therefore locally one-to-one, $f^{-1}(p)$ cannot contain a limit point. This contradiction proves that $M$ contains only finitely many connected components and thereby establishes the claim that $V$ is a finite union of projective transforms of semi-real flats and closures in $P_{n+k}$ of complex-analytic subvarieties of $P_{n+k} - S$.

To complete the proof of the Theorem, then, it remains only to show that property of being a point of type $O$ and the property of being a point of type $F$ are invariant under complex projective transformations of $P_{n+k}$; that any immersion of a $2k$-dimensional connected manifold, all of whose points are of type $F$ in $P_{n+k}$, is an immersion into a $2k$-dimensional semi-real flat; and that, under the hypothesis of the theorem, points of type $O$ cannot occur.

## 3. The rank of the associated map

Let $f: M \to P_N$ be a $C^2$ immersion with $M$ of dimension $h$, and let $\tau_p$ denote the tangent space at $p \in M, T_p$ the space of real lines through the origin of $\tau_p$, $\pi: T(M) \to M$ the bundle of real lines through the origins of the tangent spaces of $M$, and $l: T(M) \to G_{1,N-1}$ the associated map, as before. Let $T^G(M)$ consist of those $t \in T(M)$ such that $l(t)$ does not lie in $\tau_p$, and let $T_p^G = T^G(M) \cap T_p$. Note that every $t \in T^G(M)$ is either an ordinary direction or a direction of type $F$.

**Proposition 1.** a) *$l$ restricted to $T_p^G$ is one-to-one, and has rank $h - 1$ provided $T_p^G$ is nonempty.*

b) *$l$ has rank $2h - 2$ at $t \in T^G(M)$ if and only if $t$ is a direction of type $F$, and rank $2h - 1$ if and only if $t$ is an ordinary direction.*

**Note.** We have defined the notions of an ordinary direction and a direction of type $F$ only for immersions in $C^N$. However, the associated map and hence the rank of the associated map are complex-projective-invariant notions. Hence, if we prove the proposition under the assumption that $f(M) \subset C^N \subset$

$P_N$, it will follow that the notions of being a direction of type $F$ or an ordinary direction and hence the notions of being a point of type $O$ or $F$ are invariant under complex projective transformations, and hence well-defined for maps $f: M \to P_N$.

*Proof of Proposition* 1. That $l$ restricted to $T_p^G$ is one-to-one is almost obvious. If $t, t' \in T_p^G$ and $l(t) = l(t')$, then $t$ and $t'$ lie in the same complex line. If $t \neq t'$, this complex line must lie in $\tau_p$, contradicting the assumption that $l(t)$ not be contained in $\tau_p$.

For the remainder, by the above note we need only consider a $t \in T_p^G$ with $f(p) \in C^N \subset P_N$. Let $x_1, \cdots, x_h$ be local coordinates in a neighborhood of $p$ in $M$ so chosen that $t$ is along $\partial/\partial x_1$. In a neighborhood of $p$ the immersion $f$ is represented by

$$(z_1(x_1, \cdots, x_h), \cdots, z_N(x_1, \cdots, x_h)) ,$$

where $z_i$ are the complex coordinates in $C^N$. Now introduce homogeneous coordinates $w_0, \cdots, w_N$ in $C^N$ so that $z_i = w_i/w_0$. In these homogeneous coordinates $f$ is represented by

$$Y(x_1, \cdots, x_h) = (1, z_1(x_1, \cdots, x_h), \cdots, z_N(x_1, \cdots, x_h)) .$$

Hence we may represent $l$ by the complex bivector

$$l\left(\frac{\partial}{\partial x_1} + s_2\frac{\partial}{\partial x_2} + \cdots + s_h\frac{\partial}{\partial x_h}\right) = Y \wedge (Y_1 + s_2 Y_2 + \cdots + s_h Y_h) ,$$

where the subscripts on $Y$ denote partial derivatives. Note that $x_1, \cdots, x_h, s_2, \cdots, s_h$ form a local coordinate system on $T(M)$. Differentiating we obtain ($Y$, $Y_j$ and $Y_{jk}$ henceforth being evaluated at $p$)

$$\begin{aligned} \Omega_0 &\equiv l(t) = Y \wedge Y_1 , \\ \Omega_j &\equiv \partial l/\partial s_j|_t = Y \wedge Y_j , \qquad 2 \leq j \leq h , \\ \Omega_{h+1} &\equiv \partial l/\partial x_1|_t = Y \wedge Y_{11} , \\ \Omega_{h+j} &\equiv \partial l/\partial x_j|_t = Y_j \wedge Y_1 + Y \wedge Y_{1j} , \qquad 2 \leq j \leq h . \end{aligned}$$

Now these bivectors are to be regarded as ordinary vectors in the space of homogeneous coordinates of the projective space containing $G_{1,N-1}$. The kernel of the Jacobian of the projection mapping of this space of homogeneous coordinates into the projective space at any point $x$ is the complex line through the origin and $x$. Consequently to find the rank of $l$ it suffices to find the dimension of the vector space spanned over the reals by $\Omega_0, i\Omega_0, \Omega_2, \cdots, \Omega_{2h}$ modulo that spanned by $\Omega_0$ and $i\Omega_0$.

We claim that $\Omega_0, i\Omega_0, \Omega_2, \cdots, \Omega_h, \Omega_{h+2}, \cdots, \Omega_{2h}$ are linearly independent over the reals. For if there are real numbers $\xi_l$ such that

$$\xi_0\Omega_0 + \xi_1 i\Omega_0 + \sum_{j=2}^{h}\xi_j\Omega_j + \sum_{j=2}^{h}\xi_{h+j}\Omega_{h+j} = 0 ,$$

then multiplication by $Y$ gives

$$Y \wedge Y_1 \wedge \sum_{j=2}^{h}\xi_{h+j}Y_j = 0 .$$

Now $Y$ has leading entry nonzero and the $Y_k$ have leading entry 0. Hence $Y_1$ and $\sum \xi_{h+j}Y_j$ must be linearly dependent over the complex numbers, which is to say that they lie in the same complex line. But since $t \in T^G(M)$ the complex line containing $\partial f/\partial x_1$ meets $\tau_p$ only in a real line, which says that $Y_1$ and $\sum \xi_{h+j}Y_j$ are already linearly dependent over the reals. But since $f$ is an immersion, $Y_1, Y_2, \cdots, Y_h$ are linearly independent over the reals. Hence $\xi_{h+2} = \cdots = \xi_{2h} = 0$. This gives

$$0 = \xi_0\Omega_0 + \xi_1 i\Omega_0 + \sum_{j=2}^{h}\xi_j\Omega_j = Y \wedge \left(\xi_0 Y_1 + \xi_1 iY_1 + \sum_{j=2}^{h}\xi_j Y_j\right) .$$

Again, since $Y$ has leading entry nonzero and the $Y_k$ leading entry zero, this says that $\xi_0 Y_1 + \xi_1 iY_1 + \sum \xi_j Y_j = 0$. But again $iY_1$ lies outside the real linear span of $Y_1, Y_2, \cdots, Y_h$, and the latter are linearly independant. Hence $\xi_0 = \xi_1 = \xi_2 = \cdots = \xi_h = 0$, which proves the claim that $\Omega_0, i\Omega_0, \Omega_2, \cdots, \Omega_h, \Omega_{h+2}, \cdots, \Omega_{2h}$ are linearly independent over the reals.

This has two consequences. First, since in particular $\Omega_0, i\Omega_0, \Omega_2, \cdots, \Omega_h$ are linearly independent, the restriction of $l$ to $T_p^G$ has rank $h - 1$ at $t$, which proves Part a) of Proposition 1. Secondly, it shows that the rank of $l$ at $t$ is at least $2h - 2$. The rank of $l$ is then exactly $2h - 2$ if and only if there exist real numbers $\eta, \xi_l$ not all zero such that

$$\eta\Omega_{h+1} + \xi_0\Omega_0 + \xi_1 i\Omega_0 + \sum_{j=2}^{h}\xi_j\Omega_j + \sum_{j=2}^{h}\xi_{h+j}\Omega_{h+j} = 0 .$$

Multiplication by $Y$ leads to $\xi_{h+2} = \cdots = \xi_{2h} = 0$, as before, which implies that

$$Y \wedge \left(\eta Y_{11} + \xi_0 Y_1 + \xi_1 iY_1 + \sum_{j=2}^{h}\xi_j Y_j\right) = 0 ,$$

which implies as before that

$$\eta Y_{11} + \xi_0 Y_1 + \xi_1 iY_1 + \sum_{j=2}^{h}\xi_j Y_j = 0 .$$

But since $Y_1, iY_1, Y_2, \cdots, Y_h$ are linearly independent over the reals, this is the condition that $Y_{11}$, which is the curvature vector of a curve on $M$ tangent to $t$, (we ignore the first component of $Y_{11}$, which is zero) lie in the linear span of $l(t)$ and $\tau_p$. If the condition does not hold, then the rank of $l$ is $2h - 1$ and $t$ is an ordinary direction. This proves Part b) and completes the proof of the Proposition.

## 4. Surfaces of type $F$

**Proposition 2.** *Let $f: M \to P_N$ be an immersion of class $C^3$ of a connected real surface. Then $f(M)$ lies in a complex projective transform of a semi-real flat of dimension 2 and type 0 if and only if $M$ consists of points of type $F$.*

**Remarks.** Such a surface will be called an *$F$-surface* or a *surface of type $F$*. Note that any $F$-surface is then complex-projectively equivalent to a portion of the real projective plane with its canonical embedding $P^2 \subset P_2 \subset P_N$.

*Proof of Proposition* 2. By a "*circle*" in $P_1$ we mean a euclidean circle lying on the Riemann sphere, the latter being canonically identified with $P_1$. By a "circle" in $P_N$ we mean a "circle" lying on some projective line in $P_N$; the definition is good because any (complex) projective transformation of $P_1$ takes "circles" to "circles". It follows that any projective transformation of $P_N$ takes "circles" to "circles". The canonical identification $R^{2N} = C^N \subset P_N$ being made, the "circles" of $P_N$, other than those lying in the hyperplane at infinity, are the euclidean circles in $R^{2N}$ which lie in complex lines, and the straight lines of $R^{2N}$ (each of which lies in a unique complex line).

Now consider the standard semi-real flat $P^2 \subset P_2 \subset P_N$. Ignore the points at infinity. Through each point of $P^2$ and in each direction there passes a euclidean line, contained in a unique complex line. If we apply a complex projective transformation to $P_N$ this euclidean line will be sent to some "circle". This "circle", together with its tangent lines and curvature vectors, is contained in the image complex line. It follows that any complex projective transform of $P^2$ consists of points of type $F$, which proves the forward implication of Proposition 2. Note that a general complex projective transform of $P^2 \subset P_2$ contains a 2-parameter family of circles. According to the Italians, a surface in $R^4$ containing a 2-parameter family of conics is a Steiner surface, i.e., a projection of the Veronese surface in $R^5$. Such a surface is by no means flat.

The proof of the converse of Proposition 2 parallels the argument of [3, §§ 4–6]. We first show that $f(M)$ lies in a complex 2-plane. Let $f: M \to P_N$ be a $C^3$-immersion of a surface with all points of type $F$. Since the tangent planes are 2-dimensional and none complex lines, $T^G(M) = T(M)$, and the associated map $l: T(M) \to G_{1,N-1}$ has rank 2 everywhere by Proposition 1, b). For any $t \in T(M)$, $l^{-1}(l(t))$ is therefore an embedded curve in $T(M)$, and such a curve is nowhere tangent to the fibre of $T(M)$ and meets no fibre of $T(M)$ in more than one point by Proposition 1, a). Consequently the projection of such a curve into $M$ gives an embedded curve in $M$. Such a curve we call an *s-curve*.

Let $x \in M$, let $W_x$ denote the set of all points of $M$ which are joined to $x$ by $s$-curves, and let $y \in W_x$ with $f(y) \neq f(x)$. We claim that $W_x$ contains a neighborhood of $y$. For, consider $l(T_x)$, which is a curve in $G_{1,N-1}$. Then $l^{-1}(l(T_x))$ contains a surface $S$ which meets both $T_x$ and $T_y$. We claim that $S$ is not tangent to $T_y$. For, $S$ is fibred by the curves $l^{-1}(l(t))$, $t \in T_x$, which are not tangent to $T_y$. Hence, if $S$ were tangent to $T_y$ there would be a curve $Q$ in $S$ transversal to the curves $l^{-1}(l(t))$, $t \in T_x$, and tangent to $T_y$. Apply a complex projective transformation to $P_N$ to bring $f(x)$ to the origin of $C^N \subset P_N$ and $\tau_x$ to the real plane spanned by $\partial/\partial x_1$ and $\partial/\partial x_2$, where $z_j = x_j + iy_j$ are the coordinate functions of $C^N$. Now the $s$-curve containing $x$ and $y$ is unique, for otherwise we would have two distinct complex lines in $P_N$ meeting in two distinct points. We may assume that the tangent vector to this unique $s$-curve is $\partial/\partial x_1$. The curve $f\pi(Q)$ can be represented in the real coordinates $x_1, y_1, \cdots, x_N, y_N$ by

$$\begin{aligned} f\pi Q(\theta) = {} & a(\theta)(\cos\theta, 0, \sin\theta, 0, 0, \cdots, 0) \\ & + b(\theta)(0, \cos\theta, 0, \sin\theta, 0, \cdots, 0) , \end{aligned}$$

so that

$$\begin{aligned} d/d\theta|_{\theta=0} f\pi Q = {} & a'(0)(1, 0, \cdots, 0) + b'(0)(0, 1, 0, \cdots, 0) \\ & + a(0)(0, 0, 1, 0, \cdots, 0) + b(0)(0, 0, 0, 1, 0, \cdots, 0) . \end{aligned}$$

This must vanish in order for $Q$ to be tangent to $T_y$; but this is impossible unless $a(0) = b(0) = 0$, which contradicts the assumption that $f(x) \neq f(y)$. Hence the surface $S$ is not tangent to $T_y$, as claimed. It follows that $\pi(S)$ contains a neighborhood of $y$, and since $\pi(S) \subset W_x$, we have established the claim that $W_x$ contains a neighborhood of $y$.

From this it follows that every point of $M$ has a neighborhood whose image under $f$ is contained in a projective subspace of $P_N$ of two complex dimensions. For, given $y \in M$, choose $x \in M$ on a connected component of an $s$-curve through $y$ such that $f(x) \neq f(y)$. Then a neighborhood of $y$ is contained in $W_x$. But $W_x$ is mapped by $f$ into the complex 2-plane spanned by the tangent 2-plane of $f$ at $x$. This complex 2-plane is also spanned by the tangent space of $f$ at $y$, and is therefore uniquely determined by $y$. It follows by analytic continuation, $M$ being assumed connected, that $f(M)$ lies in a complex 2-plane.

We may now assume that $f: M \to P_2$. Let $t, u \in T(M)$. Then we say that $t \equiv u$ if there is a curve $C$ joining $t$ and $u$ in $T(M)$ such that $l(C) = l(t)$. This is an equivalence relation, and since $l$ has rank 2 everywhere the set $M^*$ of equivalence classes forms a differentiable manifold of dimension 2. The map $l: T(M) \to G_{1,1}$ induces a map $l^*: M^* \to G_{1,1}$, which is an immersion. We call $l^*$ or $l^*(M^*)$ the *dual surface*. Note that $G_{1,1} = P_2^*$, the dual projective space.

As we have seen, $l$ embeds any fibre $T_p$ of $T(M)$ in $P_2^*$, and $l(T_p)$ lies on the dual surface. We shall call $l^{*-1}(l(T_p))$ a *c-curve*. We claim that every $l(T_p)$ is a "circle" in $P_2^*$. To show this we bring $p$ to the origin of $C^2 \subset P_2$ as above, so that the tangent plane at $p$ is spanned by $\partial/\partial x_1$ and $\partial/\partial x_2$. Using complex homogeneous coordinates as in the proof of Proposition 1 we find that for each tangent vector $t(\theta) = \cos\theta\, \partial/\partial x_1 + \sin\theta\, \partial/\partial x_2$, $l(t(\theta))$ is spanned over the complex numbers by

$$O = (1, 0, 0) \; , \qquad t_\theta = (1, \cos\theta, \sin\theta) \; ,$$

so that $l(t(\theta))$ is represented by the complex bivector

$$O \wedge t_\theta = (0, -\sin\theta, \cos\theta) \; .$$

In suitably chosen nonhomogeneous coordinates in $P_2^*$, $l(t(\theta))$ is then represented by $(0, -\tan\theta)$, which is a parametric representation of the real line, which establishes the claim.

For each $x' \in M^*$ let $U_{x'}$ denote the set of those points of $M^*$ which are joined to $x'$ by $c$-curves. We claim that $U_{x'} - \{x'\}$ is open in $M^*$. For consider the $s$-curve $\pi(l^{-1}(x'))$ of $M$ and let $B = \pi^{-1}(\pi(l^{-1}(x')))$. By definition $U_{x'} = l(B)$. To show that $U_{x'} - \{x'\}$ is open it therefore suffices to show that $l$ has rank 2 on $B - l^{-1}(x')$. So let $t \in B - l^{-1}(x')$ be arbitrary, and $p = \pi(t)$. Choose local coordinates $x, y$ on $M$ in a neighborhood of $p$ such that $\pi(l^{-1}(x'))$ is defined by $x = 0$, $t = \partial/\partial x$, with $y = 0$ at $p$. It follows that we can represent a general point of $B$ in a neighborhood of $t$ by

$$t(s, y) = \frac{\partial}{\partial x}\bigg|_{(0,y)} + s\frac{\partial}{\partial y}\bigg|_{(0,y)} \; ,$$

so that $s$ and $y$ provide local coordinates on $B$. Using the notation and method of the proof of Proposition 1 we find that

$$l(t(s, y)) = Y \wedge (Y_1 + sY_2) \; , \qquad \Omega_0 \equiv l(t(0, 0)) = Y \wedge Y_1 \; ,$$
$$\Omega_2 \equiv \partial l/\partial s|_{(0,0)} = Y \wedge Y_2 \; , \qquad \Omega_4 \equiv \partial l/\partial y|_{(0,0)} = Y_2 \wedge Y_1 + Y \wedge Y_{12} \; .$$

But as a special case of the computation there we find that $\Omega_0, i\Omega_0, \Omega_2$ and $\Omega_4$ are linearly independent over the reals, which shows that $l$ has rank 2 on $B$ at $t$ and establishes the claim.

**Lemma 3.** *Let $\varphi: M \to P_2$ be a $C^2$ immersion of a surface in a complex projective plane.*

a) *Let $x \in M$, and $W \subset M$ be an open subset each point of which can be joined to $x$ by a curve which is mapped by $\varphi$ into a "circle" in $P_2$. Then a complex projective transform of $\varphi(W)$ is contained in a semi-real flat of dimension 2.*

b) *If two of these "circles" having distinct tangent lines at x are euclidean straight lines in* $R^4 = C^2 \subset P_2$, *then* $\varphi(W)$ *is already contained in a semi-real flat of dimension* 2.

*Proof.* If $x, W$ are as in Part a), and the tangent plane to $\varphi$ at $x$ is a complex line, then all the "circles" on $M$ passing through $x$ must lie in this complex line, so in particular $\varphi(W)$ lies in this complex line. Therefore we may henceforth assume that the tangent plane to $\varphi$ at $x$ is not a complex line; we may also assume that $W$ is nonempty. If all the "circles" in question are mutually tangent at $x$, then $\varphi(W)$ lies in the complex line spanned by their common tangent, in which case the lemma is proved. So we can assume that two of these circles have distinct tangent lines at $x$. Take as line at infinity in $P_2$ a complex line which meets two of the "circles" in question but does not pass through $\varphi(x)$. These two "circles" are now euclidean straight lines in $R^4 = C^2 \subset P_2$. To complete the proof of the lemma it suffices now to prove that $\varphi(W)$ lies in a semi-real flat of dimension 2.

Let us next recall some generalities about a surface in $R^4$. Suppose $p$ is a point of such a surface, and $C$ a curve on the surface through $p$ with unit tangent vector $t$. The orthogonal projection of the curvature vector of $C$ at $p$ into the normal plane at $p$ depends only on $t$. Hence we have a map $\mathcal{N}: \Sigma_p \to N_p$ from the circle of unit tangent vectors at $p$ to the normal plane at $p$. The properties of $\mathcal{N}$ may be found conveniently in [2]. In particular the image of $\mathcal{N}$ is an ellipse covered twice, which may degenerate to a line segment or a point. Consider now our surface $\varphi$, which maps a neighborhood of $x$ into $R^4$. Since there are two distinct real lines lying on it passing through $x$, the ellipse $\mathcal{N}(\Sigma_x)$ must pass through the origin of $N_x$ twice, which implies that $\mathcal{N}(\Sigma_x)$ degenerates, and hence is contained in a real line $L$ through the origin of $N_x$. Now let $C$ be any "circle" lying on $M$ and passing through $x$, which is not a straight line. The complex line containing $C$ is spanned by its tangent and curvature vectors at $x$. Consequently the orthogonal projection of this complex line in $N_x$ is $L$, since the tangent space of $\varphi$ at $x$ is not a complex line. Hence the complex line is contained in the 3-space spanned over the reals by $L$ and the tangent plane of $\varphi$ at $x$. Now this 3-space contains only one complex line passing through $\varphi(x)$, since any two distinct complex lines passing through a point of $R^4$ span $R^4$ over the reals. Hence all the "circles" lying on $M$ and passing through $x$, which are not straight lines, lie in a single complex line. If there are infinitely many such actual circles, then the tangent plane of $\varphi$ at $x$ *is* this complex line, a situation already ruled out. Consequently there are only finitely many such circles, and hence none, which implies that the "circles" lying in $M$ and joining $x$ to the points of $W$ are straight lines. These lines lie in the tangent plane of $\varphi$ at $x$, of course, so that all of $\varphi(W)$ lies in that semi-real flat which is the closure in $P_2$ of the tangent space of $\varphi$ at $x$. This completes the proof of Lemma 3.

Now apply this lemma to our immersion $l^*: M^* \to P_2^*$. If $y$ is an arbitrary point of $M^*$, choose a $c$-curve on $M^*$ passing through $y$, and $x$ a point on this $c$-curve such that $l^*(x) \neq l^*(y)$. As we have already shown, the set of points of $M^*$ which can be joined to $x$ by $c$-curves, $U_x$, is a neighborhood of $y$. The $c$-curves are mapped by $l^*$ to "circles". From Lemma 3 we conclude that $l^*(U_x)$ is contained in a projective transform of a semi-real flat of dimension 2, so that every point of $M^*$ has a neighborhood mapped by $l^*$ into a projective transform of a semi-real flat of dimension 2. Since $T(M)$, and hence $M^*$, are connected, we conclude by analytic continuation that $l^*(M^*)$ lies in a single projective transform of a semi-real flat of dimension 2. This cannot be a complex line; for if so, by duality all the complex lines spanned by real tangent lines of $f$ in $P_2$ have a common point. But two such complex lines spanned by two real lines tangent at a point of $M$ have only that point in common. It follows that $f(M)$ is a single point, contradicting the assumption that $f$ be an immersion. Hence some projective transform of $l^*(M^*)$ lies in a semi-real flat of dimension 2 and type 0. Applying the adjoint complex projective transformation to $P_2$, we have $l^*(M^*)$ actually lying in such a semi-real flat, which we can take to be $P^{2*} \subset P_2^*$.

Now consider $P^2 \subset P_2$. As we have already shown, its points are all of type $F$; its dual surface consists of those complex lines whose real locus is a line, i.e., $P^{2*} \subset P_2^*$, and by its homogeneity and isotropy every real line of $P^{2*}$ is the $c$-curve corresponding to some point of $P^2$. This enables us to establish a map $M \to P^2$, by which we assign to every point $p \in M$ the point $p'$ in $P^2$ such that $l(T_p) = l(T_{p'})$. The intersection of the complex lines $l(T_p)$ is $f(p)$, the intersection of the complex lines $l(T_{p'})$ is $p'$. Hence $f(p) = p'$, which proves that $f(M) \subset P^2$. This completes the proof of Proposition 2. We remark finally that $l^*(M^*)$ is a surface of type $F$ and its $s$-curves are the same as its $c$-curves.

## 5. The secant map

Let $f: M \to C^2 = R^4$ be an embedding of class $C^4$ of a real surface. Let $\pi: T(M) \to M$ denote the bundle of unoriented (real) tangent lines of $M$, as before. Consider

$$S(M) = (M \times M - \Delta) \cup T(M) ,$$

where $\Delta = \{(x, x)\}$ is the diagonal. It is shown in [4, pp. 1333–1337] that $S(M)$ has a differentiable structure compatible with the canonical differentiable structure on $M \times M - \Delta$, and in which $T(M)$ with its canonical differentiable structure is an embedded submanifold (this is done by blowing up the diagonal in $M \times M$). Moreover, the mapping $L'$ from $S(M)$ into the Grassmann manifold of all (real) lines in $R^4$ defined by

$$L'(x, y) = \text{the real line joining } f(x) \text{ and } f(y) \text{ in } R^4,\ (x, y) \in M \times M - \Delta ,$$
$$L'(t) = t \text{ realized as a real line in } R^4,\ t \in T(M) ,$$

is differentiable of class $C^3$. (Actually in the construction of [4], $T(M)$ is the bundle of *oriented* tangent lines, and $S(M)$ is a manifold with boundary $T(M)$. But if, in the treatment of [4], one interprets the $\xi_i$'s as homogeneous coordinates in a (real) projective space and ignores the inequalities $X_i\xi_i \geq 0$, one obtains the construction desired here.) The map $\varpi: S(M) \to M \times M$ defined by

$$\varpi(x, y) = (x, y) , \qquad (x, y) \in M \times M - \Delta ,$$
$$\varpi(t) = (\pi(t), \pi(t)) , \qquad t \in T(M)$$

is called the *canonical projection.*

For any real line $Q$ in $R^4 = C^2$ let $\lambda(Q)$ denote the (unique) complex line containing $Q$. Let $L = \lambda L': S(M) \to P_2^*$. Note that $L$ restricted to $T(M)$ gives $l$, the associated map previously defined. Let $\chi: S(M) \to S(M)$ be defined by

$$\chi(x, y) = (y, x) , \qquad (x, y) \in M \times M - \Delta ,$$
$$\chi(t) = t , \qquad t \in T(M) .$$

Let us take note of some dimensions and ranks: $S(M)$ and $P_2^*$ have (real) dimension 4, $T(M)$ has dimension 3; if $t \in T(M)$ is an ordinary direction for the map $f$, then by Proposition 1b) $l$ has rank 3 at $t$; hence $L$ has rank at least 3 at $t$. But $L$ cannot have rank 4 at $t$ because $L\chi = L$ and the set of fixed points of $\chi$ is $T(M)$, which implies that $L$ cannot be one-to-one on any neighborhood of $t$. Hence $L$ has rank exactly 3 at $t$.

**Definition.** Let $N_1, N_2$ be 4-dimensional differentiable manifolds, $Q$ an embedded submanifold, and $\varphi: N_1 \to N_2$. We say that $\varphi$ is a *fold with center* $Q$ at $p \in Q$ if there exist $C^1$ local coordinates $x_1, \cdots, x_4$ in a neighborhood of $p$, and $y_1, \cdots, y_4$ in a neighborhood of $\varphi(p)$ such that in this neighborhood $Q$ is defined by $x_1 = 0$ and $\varphi$ has the form

$$y_1 = x_1^2 , \quad y_i = x_i , \quad i = 2, 3, 4 .$$

**Proposition 4.** *Let $t_0 \in T(M)$ be an ordinary direction. Then $L$ is a fold with center $T(M)$ at $t_0$.*

The proof is based on the following criterion.

**Lemma 5.** *Let $Q, \varphi: N_1 \to N_2$ be as above and differentiable of class $C^3$. Then $\varphi$ is a fold with center $Q$ at $p \in Q$ if $Q$ has dimension* 3, *and*

1) *there exists a neighborhood $U$ of $p$ in $N_1$ such that $\varphi$ has rank* 3 *at each point of $U \cap Q$ and the restriction of $\varphi$ to $U \cap Q$ has rank* 3, *and*

2) *there exists a $C^2$ real-valued function $\psi$ defined in a neighborhood of $\varphi(p)$ in $N_2$ such that the directional derivatives of $\psi \circ \varphi$ along $Q$ at $p$ vanish; and there exists an embedded $C^2$ curve $C(s)$ on $N_1$ with $C(0) = p$, whose tangent vector at $p$ lies in the kernal of $\varphi_*$, but such that*

$$\left.\frac{d^2\psi\circ\varphi\circ C}{ds^2}\right|_0 \neq 0 .$$

*Proof of Lemma* 5. Suppose $C(s)$ and $\psi$ are given with conditions 1) and 2) satisfied. Choose $C^3$ local coordinates $x_1'', \cdots, x_4''$ valid in a neighborhood $V \subset U$ of $p$ and vanishing at $p$, and $C^3$ local coordinates $y_1, \cdots, y_4$ valid in a neighborhood of $\varphi(V)$ and vanishing at $\varphi(p)$, such that $Q \cap V$ is defined by $x_1'' = 0$ and $\varphi(Q \cap V)$ is defined by $y_1 = 0$. Since $\varphi | V \cap Q$ has rank 3, we must have

$$0 \neq |(\partial y_i/\partial x_j''|_p)|_{2\leq i,j\leq 4} .$$

Hence

$$x_1' = x_1'' , \quad x_i' = y_i \circ \varphi , \quad i = 2, 3, 4 ,$$

is a valid $C^3$ change of local coordinates in a neighborhood $W$ of $p$. In these coordinates the map $\varphi$ has the form

$$y_1 = y_1(x_1', \cdots, x_4') , \quad y_i = x_i' , \quad i = 2, 3, 4 .$$

Since $\varphi$ has rank 3 on $Q \cap W$, we must have

$$\partial y_1/\partial x_1'|_q = 0 , \qquad \text{for all } q \in Q \cap W ,$$

and since $y_1 \equiv 0$ on $Q \cap W$,

$$\partial y_1/\partial x_k'|_q = 0 , \quad k = 2, 3, 4 , \qquad \text{for all } q \in Q \cap W .$$

Since the tangent to $C$ at $p$ lies in the kernal of $\varphi_*$, we must have

$$dy_1/ds|_0 = 0 , \quad dy_i/ds|_0 = dx_i'/ds|_0 = 0 , \quad i = 2, 3, 4 .$$

By assumption

$$\partial\psi/\partial y_i|_{\varphi(p)} = 0 , \qquad i = 2, 3, 4 .$$

Using these relations, we obtain

$$\frac{d^2\psi}{ds^2} = \sum_{i,j} \frac{\partial^2\psi}{\partial y_i \partial y_j}\frac{dy_i}{ds}\frac{dy_j}{ds} + \sum_i \frac{\partial\psi}{\partial y_i}\frac{d^2 y_i}{ds^2} = \frac{\partial\psi}{\partial y_1}\frac{d^2 y_1}{ds^2}$$

at 0, which is nonzero at $s = 0$ by assumption. Hence at $p$

$$0 \neq \frac{d^2 y_1}{ds^2} = \sum_{i,j} \frac{\partial^2 y_1}{\partial x_i' \partial x_j'}\frac{dx_i'}{ds}\frac{dx_j'}{ds} + \sum_i \frac{\partial y_1}{\partial x_i'}\frac{d^2 x_i'}{ds^2} = \frac{\partial^2 y_1}{\partial x_1'^2}\left(\frac{dx_1'}{ds}\right)^2 ,$$

so that $\partial^2 y_1/\partial x_1'^2 \neq 0$ at $p$. Since

$$y_1 \equiv \partial y_1/\partial x_1' \equiv 0 \qquad \text{on } Q \cap W ,$$

by elementary methods we have

$$y_1(x_1', \cdots, x_4') = x_1'^2 g(x_1', \cdots, x_4') ,$$

with $g$ of class $C^1$ in some neighborhood of $Q \cap W$ and

$$g(0, \cdots, 0) = \tfrac{1}{2}\partial^2 y_1/\partial x_1'^2|_p \neq 0 .$$

Let

$$x_1 = x_1' g^{1/2}(x_1', \cdots, x_4') , \quad x_i = x_i' , \quad i = 2, 3, 4 .$$

This is a valid $C^1$ change of local coordinates in some neighborhood of $p$, since

$$\partial x_1/\partial x_1'|_p = g^{1/2}(0, \cdots, 0) \neq 0 .$$

In these coordinates the map $\varphi$ takes the form

$$y_1 = x_1^2 , \quad y_i = x_i , \quad i = 2, 3, 4 .$$

This completes the proof of Lemma 5.

*Proof of Proposition* 4. Let $t_0$ be an ordinary direction. By continuity there is a neighborhood of $t_0$ in $T(M)$ consisting of ordinary directions, and hence, by the remarks just preceeding the definition of "fold", $L$ satisfies Condition 1) of Lemma 5. We proceed to construct $\psi$ and $C$ satisfying Condition 2).

Let $x(s)$ be a $C^3$ embedded curve on $M$ parametrized by arc length such that $x(0) = \pi(t_0)$ and such that $t_0$ is the tangent line of $x$ at 0. Let $C(s)$ be the curve on $S(M)$ defined by

$$C(s) = \begin{cases} (x(s), x(-s)) \in M \times M - \Delta , & s \neq 0 , \\ t_0 \in T(M) , & s = 0 . \end{cases}$$

That $C(s)$ is embedded and $C^2$ follows from properties of the $S(N)$-construction established in [4]. Since $LC(s) = LC(-s)$, the tangent to $C$ at 0 lies in the kernal of $L_*$.

Since $t_0$ is an ordinary direction, the tangent space to $f$ at $\pi(t_0)$ and $L(t_0)$ span a hyperplane $G$ of $R^4$. Choose a point $P \neq f\pi(t_0)$ on the real line through $f\pi(t_0)$ perpendicular to $G$. For each complex line $y \in P_2^*$ let $\psi(y)$ denote the euclidean distance in $R^4$ from $y$ to $P$. Clearly $\psi$ is infinitely differentiable on some neighborhood of $L(t_0)$ in $P_2^*$; $\psi$ may be computed as follows. For each $y \in P_2^*$ near $L(t_0)$ let $n(y)$ denote the unit vector along the unique line through $P$ meeting $y$ perpendicularly and oriented from $P$ toward $y$. (Note that $n(L(t_0))$

is perpendicular to $G$.) Take $P$ as the origin, and let $X$ be the position vector of a point on $y$. Then $\psi(y) = X \cdot n(y)$.

For each $t \in T(M)$ let $X(t) = f\pi(t)$. If $t(u)$ is any curve on $T(M)$ with $t(0) = t_0$, we have

$$(d\psi \circ L)/du|_{u=0} = dX/du|_0 \cdot n(0) + X(0) \cdot dn/du|_0 \ .$$

But the first term on the right-hand side of the equation vanishes because $(dX/du)(0)$ is a tangent vector of $M$ at $\pi(t_0)$, while $n(0)$ is normal to $f$ at $\pi(t_0)$. And the second term on the right vanishes because $dn/du$ is perpendicular to $n$ since $n$ is a unit vector, and $X(0)$ is a multiple of $n(0)$. Hence the directional derivatives of $\psi \circ L$ are zero along $T(M)$ at $t_0$.

Now let $X(s) = fx(s)$. Since the kernal of $L_*$ contains the tangent of $C$ at 0, we must have

$$dn/ds|_0 = 0 \ .$$

Consequently, since $n$ is a unit vector,

$$0 = (d^2 n \cdot n)/ds^2|_0 = 2\, d^2n/ds^2|_0 \cdot n(0) + 2(dn/ds|_0)^2 \ ,$$

which implies that $d^2n/ds^2|_0 \cdot n(0) = 0$. Hence, using the fact that $X(0)$ is a multiple of $n(0)$, we obtain

$$\left.\frac{d^2\psi \circ L \circ C}{ds^2}\right|_0 = \left.\frac{d^2X}{ds^2}\right|_0 \cdot n(0) + 2\left.\frac{dX}{ds}\right|_0 \cdot \left.\frac{dn}{ds}\right|_0 + X(0) \cdot \left.\frac{d^2n}{ds^2}\right|_0 = \left.\frac{d^2X}{ds^2}\right|_0 \cdot n(0) \ .$$

But this last cannot vanish because $d^2X/ds^2(0)$ is the curvature vector of $fx(s)$ at 0, and this cannot lie in $G$ because $t_0$ is an ordinary direction. Hence $L, \psi, C$ satisfy Condition 2) of Lemma 5, from which we obtain the conclusion of Proposition 4.

## 6. Maps proper onto their images

Let $f: N_1 \to N_2$ be a differentiable map of differentiable manifolds. We say that $p_1 \in N_1$ is a *good point* of $f$ if $f(p_1)$ has an open neighborhood $X \subset N_2$ such that $f^{-1}(X) = U_1 \cup \cdots \cup U_m$, where each $U_i$ is open and is embedded diffeomorphically by $f$ onto $f(U_1)$, with the $U_i$ pairwise disjoint. Note that every point of $U_i$ is then a good point.

**Lemma 6.** *Suppose $f: N_1 \to N_2$ is proper onto its image, and $p \in N_2$. Let $W$ be a neighborhood of $f^{-1}(p)$ in $N_1$. Then there exists a neighborhood $C$ of $p$ in $N_2$ such that $f^{-1}(C) \subset W$.*

*Proof.* Suppose not. Then there exists a sequence $p_\nu \to p$ such that $f^{-1}(p_\nu) \not\subset W$. Since $f$ is proper onto its image $f^{-1}(\bigcup_\nu \{p_\nu\} \cup \{p\})$ is compact. Choose $q_\nu \in f^{-1}(p_\nu) - W$; then $\{q_\nu\}$ has a convergent subsequence $q_\mu \to q$, so that by continuity $f(q_\mu) \to f(q)$. But $f(q_\mu) \to p$. Hence $q \in f^{-1}(p)$ and $q_\mu$ is eventually inside $W$, a contradiction.

**Lemma 7.** *Suppose* $f: N_1 \to N_2$ *is a differentiable map of differentiable manifolds, which is proper onto its image, and* $q \in N_1$ *a point at which* $f$ *is an immersion*, i.e., *such that the rank of* $f$ *at* $q$ *equals the dimension of* $N_1$. *Then there exists a good point of* $f$ *arbitrarily close to* $q$ *in* $N_1$.

*Proof.* Let $V$ be an arbitrary neighborhood of $q$ in $N_1$, where $q$ is a point at which $f$ is an immersion; we will show that $V$ contains a good point of $f$. Let $V' \subset V$ be a neighborhood of $q$, and $C$ a cubical coordinate neighborhood of $f(q)$ such that $f(V') \subset C$ and such that $f(V')$ appears as a linear space in $C$ parallel to a side of $C$ of the same dimension, with the boundary of $f(V')$ lying in the boundary of $C$. Let $\pi_C: C \to f(V')$ denote orthogonal projection in $C$ onto $f(V')$. Now $f^{-1}(C)$ is an open subset of $N_1$ of dimension equal to that of $f(V')$. Consequently by Sard's theorem there is a point $q_1 \in V'$ such that $f(q_1)$ is a regular value of $\pi_C f \mid f^{-1}(C)$. At each point of $f^{-1}(f(q_1))$, $f$ is then an immersion. Hence $f^{-1}(f(q_1))$ is finite, for otherwise $f^{-1}(q_1)$ would contain a limit point, at which point $f$ could not be an immersion. Let $f^{-1}(f(q_1)) = \{q_1, \cdots, q_n\}$, and $W_i$ be a neighborhood of $q_i$ such that $W_i$ is mapped by $\pi_C f$ diffeomorphically onto its image, with the $W_i$ pairwise disjoint. By Lemma 6 there exists a neighborhood $C'$ of $f(q_1)$ such that $f^{-1}(C') \subset W_1 \cup \cdots \cup W_n$. Let $p_1$ be a point of $f^{-1}(C') \cap V'$ such that $f^{-1}(f(p_1))$ consists of the smallest number of points for all $p \in f^{-1}(C') \cap V'$.

We claim that $p_1$ is a good point of $f$. For since $f^{-1}(f(p_1)) \subset W_1 \cup \cdots \cup W_n$, it is finite: $f^{-1}(f(p)) = \{p_1, \cdots, p_m\}$, and after some renaming we can take $p_i \in W_i$. By Lemma 6 there exists a neighborhood $C'' \subset C'$ of $f(p_1)$ such that $f^{-1}(C'') \subset W_1 \cup \cdots \cup W_m$. Now $f(W_i)$ must contain an open neighborhood $D_i \subset f(V')$ of $f(V') \cap C''$, $1 \leq i \leq m$, for otherwise we could find a point $p \in f^{-1}(C'') \cap V'$ with $f^{-1}(f(p))$ consisting of strictly fewer than $m$ points. Let $D = D_1 \cap \cdots \cap D_m$, and $U_i' = (\pi_C f)^{-1}(D) \cap W_i$. Then $U_i'$ is open, and $f$ must map $U_i'$ diffeomorphically onto $D$, $1 \leq i \leq m$. For if $x \in U_i'$, let $y \in W_i$ be the unique point such that $f(y) = \pi_C f(x)$. Then $\pi_C f(y) = \pi_C f(x)$, which implies that $x = y$, showing that $f$ maps $U_i'$ *into* $D$. And $f$ maps $U_i'$ *onto* $D$; for let $x \in D$, and $y \in W_i$ be the unique point such that $f(y) = x$. Then $\pi_C f(y) = x$, which implies that $y \in (\pi_C f)^{-1}(D) \cap W_i = U_i'$. By Lemma 6 there exists an open neighborhood $X$ of $f(p_1)$ in $N_2$ such that $f^{-1}(X) \subset U_1' \cup \cdots \cup U_m'$. Let $U_i = f^{-1}(X) \cap U_i'$. Then the $U_i$ are open and $f^{-1}(X) = U_1 \cup \cdots \cup U_m$, $f$ maps each $U_i$ diffeomorphically onto $f(U_1) = X \cap D$, and the $U_i$ are pairwise disjoint because $U_i \subset W_i$ and the $W_i$ are pairwise disjoint. This completes the proof.

## 7. Ordinary directions

Let $V \subset P_2$ be a compact subset, $S \subset V$ a finite set of points, and $f: M \to P_2$ a $C^4$ immersion of a surface which is proper onto its image, with $f(M) = V - S$. We will suppose that there is an ordinary direction $t_1 \in T(M)$. Let $W$ be an arbitrary neighborhood of $t_1$ in $T(M)$, $W_1 \subset W$ a neighborhood of $t_1$ consisting of ordinary directions, and $l: T(M) \to P_2^*$ the associated map. By Proposition 1, b), $l$ is an immersion on $W_1$.

Let $\Lambda_S$ denote the set of all complex lines in $P_2$ which pass through points of $S$. We claim that it is impossible to have an open set $W' \subset T(M)$ such that $l(W') \subset \Lambda_S$. For if so there exists an open set $W'' \subset W' \subset T(M)$ such that every line $l(t)$, $t \in W''$, passes through some fixed point $P$. If $p \in M$ is any point such that the fibre $T_p$ meets $W''$, choose $t, t' \in T_p \cap W''$, $t \neq t'$. Then $l(t) \cap l(t') = \{f(p)\}$, so that $f(p) = P$. It follows that $f$ maps the whole open set $\pi(W'')$ into $P$, which is impossible. This establishes the claim. Thus $l^{-1}(\Lambda_S)$ is a closed set with empty interior, and we choose an open subset $W_2 \subset W_1$ such that $W_2 \cap l^{-1}(\Lambda_S) = 0$.

Let $T' = T(M) - l^{-1}(\Lambda_S)$. We claim that $l|T'$ is proper onto its image. For let $A \subset l(T')$ be compact. Then

$$A' = \bigcup_{\lambda \in A} \lambda \subset P_2$$

is compact, and $A' \cap S = 0$. Consequently $A' \cap V \subset f(M)$ is compact. Since $f$ is proper onto its image $f^{-1}(A')$ is compact, and since the fibres of $T(M)$ are compact $\pi^{-1}f^{-1}(A')$ is compact. But $l^{-1}(A)$ is closed and contained in $\pi^{-1}f^{-1}(A')$. Hence $l^{-1}(A)$ is compact, which establishes the claim. By Lemma 7, then, there exists an open subset $W_0 \subset W_2$ consisting of good points for $l|T'$. Since $l(W_2) \cap \Lambda_S = 0$ and $\Lambda_S$ is closed, $W_0$ consists of good points for the whole mapping $l: T(M) \to P_2^*$.

Suppose $t_0 \in W_0$ and $t_1 \in T(M)$ are such that $l(t_0) = l(t_1)$, $t_0 \neq t_1$. We claim that then $f\pi(t_0) = f\pi(t_1)$, which is to say that "$f(M)$ has no 3-parameter family of bitangent complex lines". To show this we choose as line at infinity in $P_2$ a line which meets neither $f\pi(t_0)$ nor $f\pi(t_1)$. Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ be the complex coordinates in $R^4 = C^2 \subset P_2$. Apply a complex affine transformation to bring $f\pi(t_0)$ to the origin of $R^4$ so that the tangent plane of $f$ at $\pi(t_0)$ is defined by $y_1 = y_2 = 0$, with $t_0$ in the $x_1$ direction. To each $t$ in a neighborhood of $t_0$ in $T(M)$ associate an orthonormal frame $Xe_1e_2e_3e_4$ in $R^4$ such that $X(t) = f\pi(t)$, $e_1(t)$ is directed along $t$, $e_2$ is in the tangent plane of $f$ at $\pi(t)$, and $e_3$ and $e_4$ are normal, so that $e_3(t_0)$ and $e_4(t_0)$ are along the increasing $y_1$ and $y_2$ directions, and $e_1(t_0)$ and $e_2(t_0)$ along the increasing $x_1$ and $x_2$ directions. If $I$ denotes multiplication by $i = \sqrt{-1}$, we have

$$I(e_1(t_0)) = e_3(t_0) , \qquad I(e_2(t_0)) = e_4(t_0) ,$$
$$I(e_3(t_0)) = -e_1(t_0) , \qquad I(e_4(t_0)) = -e_2(t_0) ,$$

which relations, we must emphasize, hold only at $t_0$.

Let $\omega_i = dX \cdot e_i$, $\omega_{ij} = de_i \cdot e_j$. Since $e_1$ and $e_2$ span the tangent plane of $f$ at each point, we must have $\omega_1, \omega_2$ linearly independent and $\omega_3 = \omega_4 = 0$. If we restrict these differential forms to a fibre $T_p$, we find that $\omega_1 = \omega_2 = 0$, and that $\omega_{12}$ is the differential of the angle of turning. Hence $\omega_1, \omega_2, \omega_{12}$ are everywhere linearly independent. By the structure equations of Maurer-Cartan we have

$$0 = d\omega_4 = \omega_1 \wedge \omega_{14} + \omega_2 \wedge \omega_{24} ,$$

and by a lemma of Cartan we can write

$$\omega_{14} = e\omega_1 + g\omega_2 .$$

Since $t_0$ is a good point of $l$, there exists a diffeomorphism $\varphi$ of some neighborhood of $t_0$ to some neighborhood of $t_1$ such that $l\varphi = l$. Let $Y(t) = f\pi\varphi(t)$. Since $Y(t)$ lies in the complex line $l(t)$, we can write

$$Y = X + \lambda e_1 + \mu I e_1 ,$$

where $\lambda$ and $\mu$ are real-valued functions on a neighborhood of $t_0$ whose differentiability follows from that of $Y$. Calculating, we obtain

$$dY \cdot e_2 = \omega_2 + \lambda\omega_{12} + d\mu(Ie_1)\cdot e_2 + \mu I(\omega_{12}e_2 + \omega_{13}e_3 + \omega_{14}e_4)\cdot e_2 ,$$
$$dY \cdot e_4 = \lambda\omega_{14} + d\mu(Ie_1)\cdot e_4 + \mu I(\omega_{12}e_2 + \omega_{13}e_3 + \omega_{14}e_4)\cdot e_4 .$$

Evaluating at $t_0$, we obtain

$$dY \cdot e_2 = -\mu e\omega_1 + (1 - \mu g)\omega_2 + \lambda\omega_{12} ,$$
$$dY \cdot e_4 = \lambda e\omega_1 + \lambda g\omega_2 + \mu\omega_{12} .$$

Since $Y$ has rank 2, there must be a tangent vector $\tau$ of $T(M)$ at $t_0$ for which $dY = 0$, and since the plane $l(t_0)$, which is spanned by $e_1$ and $e_3$ at $t_0$, contains a tangent line of $Y$ at $t_0$, we must have $dY \cdot e_2 = dY \cdot e_4 = 0$ for a tangent vector of $T(M)$ at $t_0$ independent of $\tau$. Hence there exist two linearly independent triples of numbers $\xi_1, \xi_2, \xi_3$ such that

$$-\mu e\xi_1 + (1 - \mu g)\xi_2 + \lambda\xi_3 = 0 , \qquad \lambda e\xi_1 + \lambda g\xi_2 + \mu\xi_3 = 0 ,$$

the functions being evaluated at $t_0$. But this implies the vanishing of a subdeterminant of the system:

$$0 = -\mu^2 e - \lambda^2 e = -(\mu^2 + \lambda^2)e .$$

Take a curve on $M$ passing through $\pi(t_0)$ with tangent line $t_0$ at $\pi(t_0)$, and consider the curve of its tangent lines in $T(M)$. On this curve $\omega_2 = 0$ and $d^2X \cdot e_4$

$= \omega_1\omega_{14} = e\omega_1^2$, which is nonzero at $t_0$ because the curvature vector of the curve does not lie in the linear span of $e_1, e_2, e_3$, since $t_0$ is an ordinary direction. Hence $\lambda^2 + \mu^2 = 0$ at $t_0$, which proves that $\pi(t_0) = \pi(t_1)$, as claimed.

We can obtain something more. Let

$$Z(t, \lambda, \mu) = X(\pi(t)) + \lambda e_1(t) + \mu I e_1(t) .$$

By the above calculations, $dZ \cdot e_2$ and $dZ \cdot e_4$ are linearly independent combinations of $\omega_1, \omega_2$ and $\omega_{12}$, provided $(\lambda, \mu) \neq (0, 0)$ and $t$ is an ordinary direction. From the structure equation

$$0 = d\omega_3 = \omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23}$$

we find by the lemma of Cartan that $\omega_{13}$ is a linear combination of $\omega_1$ and $\omega_2$. A short calculation then gives

$$dZ \cdot e_1 = d\lambda + \cdots , \qquad dZ \cdot e_3 = d\mu + \cdots ,$$

where the dots stand for combinations of $\omega_1, \omega_2$ and $\omega_{12}$. Hence $dZ \cdot e_1$, $dZ \cdot e_2$, $dZ \cdot e_3$, $dZ \cdot e_4$ are linearly independent, provided $(\lambda, \mu) \neq (0, 0)$ and $t$ is an ordinary direction. This proves

**Lemma 8.** *$Z(t, \lambda, \mu)$ has rank 4 at $(t, \lambda, \mu)$, provided $t$ is an ordinary direction and $(\lambda, \mu) \neq (0, 0)$.*

Now since $f$ is proper onto its image, by Lemma 7 there exists a point $p_0$ in the open set $\pi(W_0)$ which is a good point for $f$. Let $f^{-1}(f(p_0)) = \{p_0, \cdots, p_a\}$, and let $A_i$ be an open neighborhood with compact closure of $p_i$ in $M$, such that the $A_i$ are pairwise disjoint, $0 \leq i \leq a$, $A_i$ consists of good points for $f$, and $f$ maps each $A_i$ diffeomorphically onto $f(A_0)$. Let $t_0 \in T_{p_0} \cap W_0$, and $\lambda_0 = l(t_0)$. We claim that $f^{-1}(\lambda_0)$ is finite. We first show the following

**Lemma 9.** *Suppose that $\varphi: A \to P_2$ is an embedding of a surface, and $A' \subset A$ an open submanifold with compact closure. Suppose $\lambda = l(t)$, $t \in T_p \subset T(A')$, and $p$ is a point of type $O$. Suppose $\{x_\nu\}$, $\{y_\nu\}$ are sequences of points of $A$, $x_\nu \neq y_\nu$, such that $L(x_\nu, y_\nu) \to \lambda$, where $L$ is the secant map of § 5. Suppose that either $x_\nu \to p$ and $y_\nu \to p$, or else $\lambda$ meets $\varphi(A)$ only in $p$. Then $\{(x_\nu, y_\nu)\}$, considered as lying in $S(A')$, has a subsequence converging to $t$.*

*Proof.* Since $S(A')$ has compact closure in $S(A)$, $\{(x_\nu, y_\nu)\}$ has in either case a convergent subsequence $\{(x_\mu, y_\mu)\}$. Hence $\varpi(x_\nu, y_\nu)$ converges in $A \times A$, where $\varpi$ is the canonical projection of § 5. By continuity $\varphi(\lim x_\mu)$, $\varphi(\lim y_\mu) \in \lambda$. Hence $\lim x_\mu = \lim y_\mu = p$ is either case. It follows that $\{(x_\mu, y_\mu)\}$ converges to $t' \in T_p$. By continuity $L(t') = l(t') = \lambda = l(t)$; since $l$ maps $T_p$ in a one-to-one fashion, by Proposition 1, we have $t = t'$, which completes the proof.

Suppose that $f^{-1}(\lambda_0)$ is infinite. Then since $\lambda_0$ and $V$ are compact, $\lambda_0 \cap V \subset f(M)$ is compact, and since $f$ is proper onto its image $f^{-1}(\lambda_0) = f^{-1}(\lambda_0 \cap V)$ is compact. It follows that $f^{-1}(\lambda_0)$ contains a convergent sequence of distinct points $\{x_\nu\}$. At the limit point $p$ of $x_\nu, f$ cannot be transversal to $\lambda_0$. Hence $\lambda_0 = l(t)$ for some $t \in T_p$. But then, as we have shown, $f(p) = f(p_0)$, which is to say that $p = p_i$ for some $i \leq a$. Hence $x_\nu$ is eventually inside $A_i$. Now since $A_i$ is identified with $A_0$ under the mapping $f$, we may as well assume that $x_\nu$ lies in $A_0 - \{p_0\}$ and $x_\nu \to p_0$. Note that we have $L(x_\nu, p_0) = \lambda_0$. It follows from Lemma 9 that $\{(x_\nu, p_0)\}$ has a subsequence $\{(x_\mu, p_0)\}$ converging to $t_0$. Of course $L(x_\mu, p_0) = \lambda_0$. But this is impossible because $L$ is a fold in a neighborhood of $t_0$ by Proposition 4, so that $t_0$ has a neighborhood which is mapped by $L$ in a two-to-one or one-to-one fashion. This proves the claim that $f^{-1}(\lambda_0)$ is finite.

For later use we summarize in the following lemma a few of the results which we have proved so far.

**Lemma 10.** *Let $V \subset P_2$ be a compact subset, $S \subset V$ a finite set, and $f: M \to P_2$ a $C^4$ immersion of surface which is proper onto its image with $f(M) = V - S$. Let $t \in T(M)$ be an ordinary direction, and $W$ a neighborhood of $t$ in $T(M)$. Then $W$ contains an ordinary direction $t_0$ which is a good point for $l$, such that $p_0 = \pi(t_0)$ is a good point for $f$, and such that $l(t_0)$ does not meet $S$. Such a $t_0$ has the properties that $f^{-1}(l(t_0))$ is finite, and if $t \in l^{-1}(l(t_0))$, then $f\pi(t) = f(p_0)$.*

Let $f^{-1}(\lambda_0) = \{p_0, \cdots, p_a, p_{a+1}, \cdots, p_b\}$ with $f$ transversal to $\lambda_0$ at $p_{a+1}, \cdots, p_b$, and $p_0, \cdots, p_a$ as before. For all $i \leq b$ let $A'_i$ be an open neighborhood of $p_i$ such that the $A'_i$ have pairwise disjoint closures, such that $A'_i \subset A_i$ if $i \leq a$, and such that $f$ maps each $A'_i$ diffeomorphically onto $f(A'_0)$ for $i \leq a$. By Lemma 6 there exists an open neighborhood $B$ of $\{f(p_0), \cdots, f(p_b)\}$ in $P_2$ such that $f^{-1}(B) \subset A'_0 \cup \cdots \cup A'_b$. Since the compact sets $\lambda_0$ and $V - B$ are separated, there exists an open neighborhood $C_1$ of $\lambda_0$ in $P_2^*$ such that if $\lambda \in C_1$ then $f^{-1}(\lambda) \subset A'_0 \cup \cdots \cup A'_b$. Since $f$ is transversal to $\lambda_0$ at $p_{a+1}, \cdots, p_b$, there exists a neighborhood $C_2 \subset C_1$ of $\lambda_0$ such that if $\lambda \in C_2$ then $\lambda$ meets $f(A'_i)$ transversally in a single point, $a + 1 \leq i \leq b$.

Let $W_{00} \subset W_0$ be a neighborhood of $t_0$ in $T(M)$ such that $l(W_{00}) \subset C_2$. By Lemma 8, since $t_0$ is a good point of $l$ (so that $f(p_i) \neq f(p_0)$, $i > a$) as $t$ varies in $W_{00}$, $l(t) \cap f(A'_i)$, $i > a$, will describe some open subset of $f(A'_i)$. Since the good points of $f$ are open and dense in $M$, we can find a $t \in W_{00}$ such that $l(t)$ meets $f(A'_i)$ in the image of a good point of $f$ for all $i > a$. Since $A'_0$ consists of good points for $f$ and $l(t) \cap f(M) \subset f(A'_0) \cup \cdots \cup f(A'_b)$, $l(t)$ meets $f(M)$ only in the images of good points. This new $t$, we now call $t_0$, $l(t_0)$ we call $\lambda_0$, and $\pi(t_0)$ we call $p_0$. Since $t_0 \in W_{00}$, $t_0$ is a good point for $l$ and $\lambda_0 \cap S = 0$. It follows from Lemma 10 that $f^{-1}(l(t_0))$ is finite. We redefine $a,b$, and $p_i$ so that $f^{-1}(\lambda_0) = \{p_0, \cdots, p_a, p_{a+1}, \cdots, p_b\}$, where $\lambda_0$ is transversal to $f$ at $p_{a+1}, \cdots, p_b$, and has a real line in common with the tangent plane of $f$ at $p_0, \cdots, p_a$. By Lemma 10, $f^{-1}(f(p_0)) = \{p_0, \cdots, p_a\}$. Let $A''_0, \cdots, A''_b \subset M$ be pairwise-disjoint open sets with compact closure and smooth boundary such that $A''_i$ is a neighborhood of $p_i$, $0 \leq i \leq b$, such that the $A''_i$ consist of good points for $f$, and such that $f$ maps $A_i''$ diffeomorphically onto $f(A_0'')$, $0 \leq i \leq a$. Let $C_3 \subset C_2$ be a neighborhood of $\lambda_0$ such that if $\lambda \in C_3$ then $\lambda \cap V \subset f(A_0'') \cup \cdots \cup f(A_b'')$ and $\lambda$ meets each $f(A_i'')$ transversally in a single point, $a + 1 \leq i \leq b$. (That such a $C_3$ exists is proved by the argument for the existence of $C_2$.)

Now by Proposition 4, the secant map $L$ is a fold at $t_0$ with center $T(M)$. That is to say, there exist a coordinate neighborhood $C_4 \subset C_3$ of $\lambda_0$ in $P_2^*$ with coordinates $y_1, \cdots, y_4$, and a coordinate neighborhood $X$ of $t_0$ in $S(A_0'')$ with coordinates $x_1, \cdots, x_4$ such that $T(M)$ in $X$ is defined by $x_1 = 0$, and $L$ takes the form

$$y_1 = x_1^2 , \quad y_i = x_i , \quad i = 2, 3, 4 .$$

Let

$$C_4^+ = \{(y_1, \cdots, y_4) \in C_4 | y_1 > 0\} ,$$
$$C_4^T = \{(y_1, \cdots, y_4) \in C_4 | y_1 = 0\} ,$$
$$C_4^- = \{(y_1, \cdots, y_4) \in C_4 | y_1 < 0\} .$$

By shrinking $C_4$ if necessary, we arrange that $L$ maps $X$ *onto* $C_4^+$. For any subset $Q \subset P_2^*$ let $Q^+ = Q \cap C_4^+$, $Q^T = Q \cap C_4^T$, $Q^- = Q \cap C_4^-$.

We claim that there is a neighborhood $C_5 \subset C_4$ of $\lambda_0$ such that if $\lambda \in C_5^+$ then $\lambda$ meets $f(A_0'')$ in just the unique pair of points $f(p)$ and $f(p')$ such that $L(p, p') = \lambda$. For otherwise there exists a sequence of lines $\lambda_\nu \to \lambda_0$, $\lambda_\nu \in C_4^+$, such that $f^{-1}(\lambda_\nu)$ contains three distinct points $p_\nu, p_\nu', p_\nu'' \in A_0''$ for each $\nu$ with $(p_\nu, p_\nu') \in X$. It follows from Lemma 9 that $\{(p_\nu, p_\nu'')\}$ has a subsequence $\{(p_\pi, p_\pi'')\}$ converging to $t_0$, and then that $\{(p_\pi', p_\pi'')\}$ has a subsequence $\{(p_\rho', p_\rho'')\}$ converging to $t_0$. It follows that $\{(p_\rho, p_\rho'')\}$ and $\{(p_\rho', p_\rho'')\}$ are eventually inside $X$. Hence, for some value of the index $\rho$, $L(p_\rho, p_\rho') = L(p_\rho, p_\rho'') = L(p_\rho', p_\rho'')$, which contradicts the fact that $L$ is two-to-one on $X - T(M)$. This establishes the claim.

Now since $t_0$ is a good point for $l$, there exists a connected neighborhood $C_6 \subset C_5$ of $\lambda_0$ in $P_2^*$ such that $l(T(M)) \cap C_6 \subset C_6^T$. It follows that every line in $C_6^+$ or $C_6^-$ meets $f(M)$ transversally. Since any line $\lambda$ in $C_6^+$ meets $f(A_0'')$ transversally in exactly two points and does not meet $\partial f(A_0'')$, its linking number with $\partial f(A_0'')$ must be even. Since $C_6$ is connected and no $\lambda \in C_6$ meets the $\partial f(A_0'')$, the linking number of any $\lambda \in C_6$ with $\partial f(A_0'')$ is even. Hence no $\lambda \in C_6$ meets $f(A_0'')$ transversally in a single point, and there must be an open neighborhood $C_7 \subset C_6$ of $\lambda_0$ such that if $\lambda \in C_7^-$ then $\lambda$ does not meet $f(A_0'')$ in more than one point. For otherwise there would exist a sequence of lines $\lambda_\nu \to \lambda_0$, $\lambda_\nu \in C_6^-$, such that $\lambda_\nu$ meets $f(A_0'')$ in two distinct points $f(x_\nu)$, $f(y_\nu)$. Lemma 9 gives a convergent subsequence of $\{(x_\nu, y_\nu)\}$ which eventually lies inside $X$, so that for some $\nu$, $\lambda_\nu = L(x_\nu, y_\nu) \in C_6^+$, which is a contradiction.

We have constructed, therefore, two open subsets $C_7^+, C_7^- \subset P_2^*$ such that each line in $C_7^-$ does not meet $f(A_0'')$, and each line in $C_7^+$ meets $f(A_0'')$ transversally in exactly two points. Recall that each $A''_i$, $i \leq a$, is identified with $f(A''_0)$ under the map $f$. Each line in $C_7$ meets $f(A''_i)$ transversally in a single point, $i > a$. Consequently the number of points in $f^{-1}(\lambda) \cap (A''_{a+1} \cup \cdots \cup A''_b)$ does not vary as $\lambda$ varies in $C^7$. Since $A''_{a+1}, \cdots, A''_b$ consist of good points, the number of points in $\lambda \cap (f(A''_{a+1}) \cup \cdots \cup f(A''_b))$ does not vary as $\lambda$ varies in $C^7$. Finally, $C_7$ may be made arbitrarily small. We have proved the following.

**Proposition 11.** *Let $t \in T(M)$ be an ordinary direction, and $C$ a neighborhood of $l(t)$ in $P_2^*$. Then there exist nonempty open subsets $C^+, C^- \subset C$ and positive integers $c, d$, and $e$, such that if $\lambda \in C^+$, then $\lambda \cap V$ consists of $c$ points and $f^{-1}(\lambda)$ consists of $d$ points, and such that if $\lambda \in C^-$, then $\lambda \cap V$ consists of $c - 2$ points and $f^{-1}(\lambda)$ consists of $d - e$ points. Furthermore each line in $C^+ \cup C^-$ is transversal to $f$ and does not meet $S$.*

The conclusion of this proposition is clearly incompatible with Hypothesis 2 of the Theorem. We conclude that if $n = k = 1$ and the hypotheses of the Theorem are fulfilled, then $T(M)$ contains no ordinary directions. The proof of the case $n = k = 1$ of the Theorem, that is to say Corollary 1, now stands complete.

**Remark.** If it is desired to prove Lemma 2 of Thom [6], one cuts $L(X)$ by a complex line in $P_2^*$ transversal to $L(T(M) \cap X)$, which line corresponds to a pencil of lines in $P_2$.

## 8. The higher-dimensional case

We next generalize Proposition 11 to higher dimensions. Let $V \subset P_{n+k}$ be a compact subset, $M$ a differentiable manifold of (real) dimension $2k$, and $f: M \to P_{n+k}$ a $C^4$ immersion which is proper onto its image, with $f(M) \subset V$. Let $S = V - f(M)$, and suppose there exists an everywhere dense subset $T \subset G_{n+1,k-1}$ such that if $v \in T$ then $v \cap S$ consists of finitely many points and $v$ is transversal to $f$. Suppose that $T(M)$ contains an ordinary direction $t_1$. By Lemma 7 there exists a good point for $f$ arbitrarily close to $\pi(t_1)$, and hence there exists an ordinary direction $t_2$ such that $\pi(t_2)$ is a good point of $f$. Finally we can find a $v \in T$ containing an ordinary direction $t_3$ so close to $t_2$ that $p_3 = \pi(t_3)$ is a good point for $f$.

Let $M' = f^{-1}(v)$. Then $M'$ is an embedded submanifold of $M$ of dimension 2; let $f'$ denote the restriction of $f$ to $M'$. We claim that $t_3 \in T(M')$ is an ordinary direction for $f'$. For consider a curve $p(s)$ on $M'$ tangent to $t_3$; since its curvature vector at $p_3$ does not lie in the tangent space to $f$ at $p_3$, *a fortiori* it does not lie in the tangent space of $f'$ at $p_3$. Since $l(t_3)$ does not lie in the span of this curvature vector and the tangent space to $f$ at $p_3$, *a fortiori* it does not lie in the span of the curvature vector and the tangent space of $f'$ at $p_3$, which establishes the claim.

Let $A(v)$ denote the locus swept out in $v$ by all the complex lines joining points of the finite set $S \cap v$ to points of $f'(M')$; it depends on 4 real parameters. Let $B(v)$ denote the locus swept out in $v$ by the complex lines each of which meets a tangent plane of $f'$ in a real line at least; it depends on 5 real parameters. Let $C(v, t_3)$ denote the complex 2-plane spanned by the tangent space of $f'$ at $p_3$. Now choose as hyperplane at infinity in $P_{n+k}$ a hyperplane which does not pass through $f(p_3)$, and identify its complement with $C^{n+k} = R^{2(n+k)}$ in the canonical way. Let $L$ be the real line in $R^{2(n+k)}$ through $f(p_3)$ in the direction of the curvature vector of a curve $p(s)$ on $M'$ tangent to $t_3$. Choose a point $P$ on $L$ distinct from $f(p_3)$, and let $D(v, t_3)$ denote the closure of the locus of complex lines joining $P$ to the points of the real 3-plane spanned by the tangent plane of $f'$ at $p_3$ and $l(t_3)$; $D(v, t_3)$ depends on 5 real parameters. Let $E(v, t_3)$ denote the closure of the locus of points swept out by the complex lines joining the points of $f(M')$ to $f'(p_3)$; $E(v, t_3)$ depends on 4 real parameters. It follows that we can find a complex $(n - 2)$-plane $H$ in $v$ which does not meet $A(v), B(v), C(v, t_3), D(v, t_3)$, nor $E(v, t_3)$. Let $P_2 \subset v$ be so situated that $P_2 \cap H = 0$, and let $\varpi$ denote projection of $v$ into $P_2$ with $H$ as center. Let $S'$ denote the finite set $\varpi(v \cap S)$.

Since $H$ does not meet $B(v)$, $\varpi f' : M' \to P_2$ is an immersion. Since $H$ does not meet $A(v)$, $\varpi f'(M') \cap S' = 0$. Since $v \cap V$ is compact, so is $V' = \varpi(v \cap V)$. Since $H$ does not meet $C(v, t_3)$, the tangent space of $\varpi f'$ at $p_3$ is not a complex line, so that this tangent plane does not contain the complex line spanned by $(\varpi f')_*(t_3)$. Since $H$ does not meet $D(v, t_3)$, the curvature vector of $\varpi f' p(s)$ is in general position at $p_3$ with this complex line and tangent space, so that $t_3$ is an ordinary direction for $\varpi f'$. Let $K \subset \varpi f'(M')$ be a compact set. Then $\varpi^{-1}(K) \cup H$ is compact, and $(\varpi^{-1}(K) \cup H) \cap V \subset f'(M')$. Since $f$ is proper onto its image, $f^{-1}((\varpi^{-1}(K) \cup H) \cap V) = (\varpi f')^{-1}(K)$ is compact, which proves that $\varpi f'$ is proper onto its image. By Lemma 10 we conclude that there is a $t_4 \in T(M')$ such that the complex line $\lambda$ in $P_2$ generated by $(\varpi f')_*(t_4)$ meets $V'$ in only finitely many points, does not meet $S'$, and such that if $\lambda$ is non-transversal to $\varpi f'$ at $p$ then $\varpi f'(p) = \varpi f' \pi(t_4)$. Since $t_4$ can be taken arbitrarily close to $t_3$, we can arrange that in addition $t_4$ is an ordinary direction for $f$ and $p_4 = \pi(t_4)$ is a good point for $f$. Since $H$ does not meet the compact set $E(v, t_3)$, no complex line joining $f(p_3)$ to any other point of $V$ meets $H$, so we can take $t_4$ so close to $t_3$ that no complex line joining $f(p_4)$ to any other point of $V$ meets $H$.

It follows that the complex $n$-plane $h = \varpi^{-1}(\lambda) \cup H$ meets $V$ in finitely many points, does not meet $S$, contains $t_4$, and has the property that if it is not transversal to $f$ at $p$, then $f(p) = f(p_4)$. It may happen that the points in which $h$ meets $V$, other than $f(p_4)$, are not good points of $f$. We must therefore vary $t_4$ and $h$ containing $t_4$ so that $f^{-1}(h \cap V)$ will consist only of good points of $f$ and $h \cap S = 0$. To show that this is possible we observe first that if $t_4$ and $h$ are varied by sufficiently small amounts then $t_4$ will remain an ordinary direction of $f$, $p_4$ will remain a good point of $f$, $h$ will remain transversal to $f(M)$ at all points other than an open neighborhood of good points of $f(p_4)$, and $h \cap S = 0$. Since there is a good point for $f$ arbitrarily close to any point of $M$, and since $h \cap V$ consists of finitely many points, it suffices to show that any point $q \in h - f(p_4)$ can be varied in any direction transversal to $h$ by an appropriate variation of $t_4$ and $h$, which may be shown by showing that $q$ can be varied in any one of a maximal linearly independent set of directions transversal to $h$ by varying $t_4$ and $h$. Now if $q \in h$, $q \notin l(t_4)$, $q$ can be varied in any direction perpendicular to $h$ by a kind of rotation of $h$ about $l(t_4)$. Suppose $q \in l(t_4) - f(p_4)$, and $f(p_4) \in R^{2(n+k)} = C^{n+k} \subset P_{n+k}$. Since $v$ is transversal to the tangent space of $f$ at $p_4$, we can choose tangent vectors $e_1, \cdots, e_{2k-2}$ which, together with $v$, span $R^{2(n+k)}$. Then the rotations of $t_4$ about $p_4$ in the tangent space of $f$ at $p_4$ in the directions $e_i$ give rise to linearly independent movements of $q \in l(t_4)$ transversal to $v$, as can be shown by an elementary computation. Finally, by Lemma 8, $\varpi(q)$ may be varied in any direction in $P_2$ by varying $t_4$ and leaving $H$ fixed. Hence $q$ may be varied in any direction transversal to $h$ in $v$ by varying $t_4$.

Thus we find $t_5 \in T(M)$ and a complex $n$-plane $h' \subset P_{n+k}$ such that $t_5$ is an ordinary direction for $f$, $p_5 = \pi(t_5)$ is an ordinary point of $f$, $h'$ contains $t_5$, $h' \cap S = 0$, and $f^{-1}(h')$ consists of good points for $f$. Since all these properties remain unchanged under small changes of $t_5$ and $h'$ containing $t_5$, we may assume that $h' \subset v' \in T \subset G_{n+1,k-1}$. We now repeat the construction of this section beginning with the second paragraph. Let $M'' = f^{-1}(v')$, and $f''$ be the restriction of $f$ to $M''$. Then $t_5$ is an ordinary direction for $f''$. Let $H'$ be a complex $(n-2)$-plane in $v'$ which does not meet $A(v')$, $B(v')$, $C(v', t_5)$, $D(v', t_5)$, nor $E(v', t_5)$, and which is sufficiently close to $h'$ that the projective span of $H'$ and $l(t_5)$ meets $f(M)$ only in good points for $f$. Let $P_2 \subset v'$ be chosen so that $P_2 \cap H' = 0$, and let $\varpi'$ denote the projection into $P_2$ with $H'$ as center. Let $V'' = \varpi'(v' \cap V)$, $S'' = \varpi'(v' \cap S)$. Then $S''$ consists of finitely many points, $V''$ is compact, $\varpi' f''$ maps $M''$ onto $V'' - S''$, $\varpi' f''$ is an immersion proper onto its image, and $t_5$ is an ordinary direction for $f''$. It follows by Proposition 11 that there exist complex lines $\lambda_1$ and $\lambda_2$ in $P_2$ which meet $V''$ in different numbers of points, which do not meet $S''$, which are transversal to $\varpi' f''$ and which may be taken so close to $\varpi' l(t_5)$ that the complex $n$-planes $h_1 = (\varpi')^{-1}(\lambda_1)$ and $h_2 = (\varpi')^{-1}(\lambda_2)$ are so close to the projective span of $H'$ and $l(t_5)$ that $h_1$ and $h_2$ meet $V$ only in good points of $f$. Moreover $h_1$ and $h_2$ are transversal to $f$. It follows that $h_1$ and $h_2$ have open neighborhoods $U_1, U_2$ in $G_{n,k}$ such that for each $i = 1, 2$, each $h \in U_i$ meets $V$ in the same number of points and each $f^{-1}(h)$ consists of the same number of points, and these numbers are distinct for $h \in U_1$ and $h \in U_2$. But this situation is incompatible with the hypotheses of the Theorem. Hence, if the hypotheses of the Theorem are satisfied, $M$ can have no ordinary directions. To complete the proof of the Theorem, we then need only prove the following.

**Proposition 12.** *Let $f: M \to P_{n+k}$ be an immersion of class $C^3$ of a connected differentiable manifold of dimension $2k$. Then $f(M)$ lies in a complex* *projective transform of a semi-real flat of dimension* $2k$ *and type strictly less than* $2k$ *if and only if every point of* $M$ *is of type* $F$.

*Proof.* The proof of the forward implication goes as in the proof of Proposition 2. To prove the converse, we first note that by the argument of that proof if every point of $M$ is of type $F$, then through every point $x$ and tangent to every direction $t \in T_x^G$ there passes a unique curve on $M$ whose image under $f$ lies in a complex line. These curves are called $s$-curves, and by an argument essentially that found in the proof of Proposition 2, if $x, p \in M$ are joined by an $s$-curve, $f(x) \neq f(p)$, then the set of points of $M$ which may be joined to $x$ by $s$-curves contains a neighborhood of $p$ in $M$. Assume that every point of $M$ is of type $F$.

We prove the local form of the converse first, so we assume for the time being that $f$ is an embedding. Let $t \in T_y^G$, let $C$ be the $s$-curve on $M$ tangent to $t$, and let $Q \subset P_{n+k}$ be a complex $(n+1)$-plane which meets $f$ transversally at $y$ and contains $t$ (explicit construction below). Then $f^{-1}(Q)$ contains a surface $R$ passing through $y$, and $Q$ contains all $s$-curves of $M$ tangent to tangent directions of $R$. Since $t \in T_y^G$, the tangent space of $R$ at $y$ is not a complex line. We restrict $R$ to a connected neighborhood of $y$ small enough that the tangent space at every point of $R$ is not a complex line. It follows that tangent to every direction in $T(R)$ there lies on $R$ a portion of an $s$-curve of $M$. It follows that every point of $R$ is of type $F$, from which it follows by Proposition 2 that $R$ is a surface of type $F$. Hence all the portions of $s$-curves of $M$ lying on $R$ are portions of "circles". It follows that $C$, in particular, contains a portion of a "circle" passing through $y$. Since $y$ and $t \in T_y^G$ were arbitrary, we conclude that all the $s$-curves of $M$ are portions of circles.

Let $p \in M$ be an arbitrary point, and let $x \in M$, $x \neq p$, be a point which is joined to $p$ by an $s$-curve in $M$; $p, x$ will remain fixed for the remainder of the discussion. We will show that all the $s$-curves on $M$ through $x$ can be turned into straight lines simultaneously by a complex projective transformation of $P_{n+k}$. We first apply a complex projective transformation to bring $f(x)$ to the origin of $C^{n+k} \subset P_{n+k}$.

Let $\tau_x^C$ denote the complex part of the tangent space of $f$ at $x$, i.e., $\tau_x^C = \tau_x \cap I\tau_x$, where $I$ denotes multiplication in $C^{n+k}$ by $\sqrt{-1}$. Let $s_1, \cdots, s_r$ form a complex basis of $\tau_x^C$. We say that $s_{r+1}, s_{r+2} \in \tau_x - \tau_x^C = \tau_x^G$ form a *generic pair* if $s_1, \cdots, s_r, is_1, \cdots, is_r, s_{r+1}, s_{r+2}$ are linearly independent over the reals. Let $s_{r+1}, s_{r+2}$ be an arbitrary generic pair, and extend to a real basis $s_1, \cdots, s_r$, $is_1, \cdots, is_r, s_{r+1}, s_{r+2}, \cdots, s_{2k-r}$ of $\tau_x$ and to a complex basis $s_1, \cdots, s_r$, $s_{r+1}, \cdots, s_{2k-r}, s_{2k-r+1}, \cdots, s_{k+n}$ of $C^{n+k}$. The complex $(n+1)$-plane $Q$ through the origin spanned by $s_{r+1}, s_{r+2}, s_{r+3} + is_{r+4}, \cdots, s_{2k-r-1} + is_{2k-r}$, $s_{2k-r+1}, \cdots, s_{k+n}$ is transversal to $f$ at $x$. Note that any tangent vector in $\tau_x^G$ can be made a member of a generic pair, since $M$ is even-dimensional, so that the construction of the previous $Q$ is now explicit. Since our new $Q$ contains $s_{r+1}$ and $s_{r+2}$, $Q \cap f(M)$ contains an $F$-surface $R$ tangent to $s_{r+1}$ and $s_{r+2}$, according to our previous remarks.

Suppose $r \geq 1$, and let $s \in \tau_x^C$, $s \neq 0$. We change the complex basis $s_1, \cdots, s_r$ of $\tau_x^C$ so that $s = s_1$. Now apply a complex linear transformation to $C^{n+k}$ to bring $s_j$ to $\partial/\partial x_j$, $1 \leq j \leq k+n$, where $z_j = x_j + iy_j$ are the complex coordinates of $C^{n+k}$. Consider the four vectors

$$s_{r+1}, s_{r+2}, s_1 + 2s_{r+1} + s_{r+2}, is_1 + s_{r+1} + s_{r+2} .$$

As one checks, each pair of these is a generic pair. Since each lies in $\tau_x^C$, there lies on $M$ an $s$-curve tangent to each.

On each of the $s$-curves tangent to $s_{r+1}, s_{r+2}$ and $s_1 + 2s_{r+1} + s_{r+2}$, choose a point distinct from $x$. Through these three points, pass a hyperplane in $P_{n+k}$ which does not contain $x$. (Such a hyperplane exists, since otherwise $s_1, s_{r+1}$ and $s_{r+2}$ would lie in a 2-complex-dimensional complex plane.) Take this hyperplane as hyperplane at infinity in $P_{n+k}$ without disturbing the tangent space at the origin of $C^{n+k}$. The $s$-curves tangent to $s_{r+1}, s_{r+2}$ and $s_1 + 2s_{r+1} + s_{r+2}$ are now straight lines in $C^{n+k}$. Choose a point $q$ distinct from $x$ on the $s$-curve on $M$ tangent to $is_1 + s_{r+1} + s_{r+2}$. Its coordinates in $C^{n+k}$ have the form $z_1 = -b + ia$, $z_{r+1} = a + ib$, $z_{r+2} = a + ib$, $z_j = 0$, $j \neq 1, r+1, r+2$, with $a, b$ real, $(a, b) \neq (0, 0)$, since the $s$-curve lies in the complex line through the origin spanned by $is_1 + s_{r+1} + s_{r+2}$. Consider the complex hyperplane

$$H\colon -bz_1 + az_{r+1} = a^2 + b^2 .$$

One verifies that $H$ meets the real lines through the origin tangent to $s_{r+1}$, $s_{r+2}$, and $s_1 + 2s_{r+1} + s_{r+2}$ (the points of intersection are at infinity possibly), that it passes through $q$, and that it does not pass through the origin. Take $H$ as hyperplane at infinity in $P_{n+k}$ without disturbing the tangent space at the origin of $C^{n+k}$. The $s$-curves tangent to $s_{r+1}, s_{r+2}, s_1 + 2s_{r+1} + s_{r+2}$ and $is_1 + s_{r+1} + s_{r+2}$ are now all straight lines. Since each pair of these vectors is a generic pair, there passes tangent to each pair an $F$-surface lying in $M$, with each such $F$-surface containing a portion, containing the origin, of the straight line in $C^{n+k}$ tangent to each vector of the pair. From Lemma 3, b) we conclude that each of these $F$-surfaces is now contained in a semi-real flat of dimension 2.

Let $S$ be the complex $(n+2)$-plane through the origin of $C^{n+k}$ spanned by $s_1, s_{r+1}, s_{r+2}, s_{r+3} + is_{r+4}, \cdots, s_{2k-r-1} + is_{2k-r}, s_{2k-r+1}, \cdots, s_{k+n}$. It meets $f(M)$ transversally at $x$, so that its intersection with $f(M)$ contains a 4-real-dimensional manifold $N$ containing the origin of $C^{n+k}$. The vectors $s_{r+1}, s_{r+2}, s_1 + 2s_{r+1} + s_{r+2}$ and $is_1 + s_{r+1} + s_{r+2}$ form a (real) basis for the tangent space of $N$ at the origin. Since $S$ is a linear space, it contains the semi-real flats tangent to these vectors in pairs. Hence $N$ contains an open set, containing the origin, of each of these semi-real flats. Choose local coordinates $x_1, \cdots, x_4$ on $N$, for example geodesic normal coordinates, such that these semi-real flats are defined in $N$ by $x_j = x_k = 0$, for pairs $j, k$, $1 \leq j < k \leq 4$, and represent $f|N$ by a vector function $X(x_1, \cdots, x_4)$ in $C^{n+k} = R^{2(n+k)}$. We must have that the component of $X_{jk}(0, \cdots, 0)$ normal to $N$ vanishes for all $j, k$, $1 \leq j, k \leq 4$, because $N$ contains these semi-real flats. Consequently the second fundamental form of $N$ in $C^{n+k}$ vanishes identically at $x$, which implies that all the $s$-curves on $N$ through $x$ are straight lines lying in the tangent space to $N$ at $x$.

Let $y \in N$, $y \neq x$, be a point which is joined to $x$ by an $s$-curve lying in $N$. The set of all points in $N$ distinct from $x$ which may be joined to $x$ by $s$-curves lying in $N$ contains an open subset of $y$ in $N$, by the argument of the proof of Proposition 2. This open set lies in the tangent space of $N$ at $x$ in $R^{2(n+k)} = C^{n+k}$, as we have just shown, which is to say that a neighborhood of $y$ in $N$ is contained in a semi-real flat of dimension 4. Now, by the argument of the proof of Proposition 2, the set of points of $N$ which can be joined to $y$ by $s$-curves contains an open neighborhood of $x$. Since all these $s$-curves lie in the tangent space of $N$ at $y$, we have shown that a neighborhood of $x$ in $N$ lies in a semi-real flat of dimension 4.

From this we draw two conclusions, which we formulate in a form invariant under complex projective transformations of $P_{n+k}$, taking cognizance of the fact that $s_1$ was an arbitrary nonzero vector in $\tau_x^C$ and $s_{r+1}, s_{r+2} \in \tau_x^G$ an arbitrary generic pair. First, given any $s \in \tau_x^C$, $s \neq 0$, an open neighborhood of $x$ in the complex line in $C^{n+k}$ spanned by $s$ is contained in $f(M)$. Secondly, given any $s \in \tau_x^C$, $s \neq 0$, and any generic pair $s', s'' \in \tau_x^G$, there exist unique $F$-surfaces $S_1, S_2 \subset f(M)$ tangent to $s, s'$, and $s, s''$ respectively. The intersection of either with the complex line spanned by $s$ is an open subset of a single "circle" tangent to $s$, which circle also contains $S_1 \cap S_2$. We now drop the assumption $r > 0$.

Consider again the real basis $s_1, \cdots, is_r, \cdots, s_{2k-r}$ of $\tau_x$. We make the following index convention: $1 \leq j \leq r$; $r + 1 \leq \alpha, \beta \leq 2k - r$. Let $L_j$ denote the complex line through the origin of $C^{n+k}$ spanned by $s_j$, and let $C_j$ denote the "circle" tangent to $s_j$ which contains the intersection of $L_j$ with the unique $F$-surface contained in $M$ tangent to $s_j$ and $s_{r+1}$. For each $\alpha \geq r + 2$, $s_{r+1}$ and $s_\alpha$ are a generic pair, from which it follows that the unique $F$-surface lying on $M$ tangent to $s_j$ and $s_\alpha$ passes through $C_j$. Similarly, there lies in $L_j$ a "circle" $C'_j$ tangent to $is_j$ which contains the intersection of $L_j$ with the unique $F$-surface on $M$ tangent to $is_j$ and $s_\alpha$ for all $\alpha$. Since the two "circles" $C_j$ and $C'_j$ lie in a complex line $L_j$ and intersect at the origin with distinct tangents, they must intersect in one further point $q_j$. On the unique $s$-curve of $M$ tangent to $s_\alpha$, choose a point $p_\alpha$ distinct from $x$. Then there must exist a complex hyperplane $H$ in $P_{n+k}$ passing through the $2k - r$ points $q_j, p_\alpha$ and not passing through the origin, since otherwise $\tau_x$ lies in a complex projective subspace of complex dimension $2k - r - 1$, which is clearly impossible. Take $H$ as hyperplane at infinity in $P_{n+k}$ without disturbing the tangent space at the origin. The "circles"

$C_j, C'_j$, as well as the $s$-curves tangent to $s_\alpha$ are now euclidean straight lines, and by Lemma 3, b) the $F$-surfaces tangent to pairs $s_j, s_\alpha$; $is_j, s_\alpha$; and $s_\alpha, s_\beta$ and lying in $f(M)$ are now contained in semi-real flats of dimension 2.

Take local coordinates $u_1, \cdots, u_r, u'_1, \cdots, u'_r, u_{r+1}, \cdots, u_{2k-r}$ on $M$ in a neighborhood of $x$, for example geodesic normal coordinates, such that these $F$ surfaces are defined by conditions $u'_A = 0, u_B = 0, B \neq j, \alpha$; $u'_A = 0, A \neq j, u_B = 0, B \neq \alpha$; $u'_A = 0, u_B = 0, B \neq \alpha, \beta$; and such that $L_j \cap f(M)$ is defined in a neighborhood of the origin by $u'_A = 0, u_B = 0, A, B \neq j$. Represent $f$ by a position vector function

$$X(u_1, \cdots, u_r, u'_1, \cdots, u'_r, u_{r+1}, \cdots, u_{2k-r}) \qquad \text{in } C^{n+k} .$$

Then the components of

$$\frac{\partial X}{\partial u_j \partial u_\alpha}, \frac{\partial X}{\partial u'_j \partial u_\alpha}, \frac{\partial X}{\partial u_\alpha \partial u_\beta}$$

at the origin normal to $f(M)$ vanish for all $j, \alpha, \beta$. Since for every $s \in \tau_x^C$, $s \neq 0$, $M$ contains a portion of a straight line tangent to $s$, the second fundamental form of $M$ must vanish identically on $\tau_x^C$ and therefore vanishes identically on all of $\tau_x$. Thus all the $s$-curves of $M$ through $x$ are now straight lines, and hence lie in the tangent space to $M$ in $R^{2(n+k)}$, which is a semi-real flat of dimension $2k$. It follows that a neighborhood in $M$ of the arbitrary point $p \in M$ lies in this semi-real flat. By analytic continuation ($f$ no longer assumed to be an embedding) we conclude that all of $M$ lies in this semi-real flat. This completes the proof of Proposition 12 and therewith the proof of the Theorem.

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