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RIEMANNIAN SUBMERSIONS WITH TOTALLY GEODESIC FIBERS

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Let M and B be C^∞ Riemannian manifolds. By a *Riemannian submersion* we mean a C^∞ mapping $\pi : M \to B$ from M onto B such that π is of maximal rank and π_* preserves the lengths of horizontal vectors, i.e., vectors orthogonal to the fiber $\pi^{-1}(x)$ for some $x \in B$.

§ 1 is primarily devoted to a summary of known results which will be used in the remaining portion of the paper. § 2 gives a sufficient condition for an isometry f of M to preserve bundle structure where the bundles in question are determined by Riemannian submersions $\pi_i : M \to B$ where M and B are Riemannian manifolds. In this result (Theorem 2.2) we assume M is connected and complete and the fibers are connected and totally geodesic. Now a large class of Riemannian submersions satisfy precisely these conditions. Thus our theorem should have many applications. Some of them (Lemmas 2.4, 2.5 and 2.6) show that many Riemannian submersions from spheres are essentially equivalent to the standard ones (see O'Neill [15], Gray [8]).

In § 3 we classify those B for which there is a Riemannian submersion $\pi : S^m \to B$ where S^m is a sphere and the fibers are connected and totally geodesic. A similar problem for homogeneous sphere bundles was discussed by Nagano in [14]. Since we make no assumption about homogeneity, our proof depends on the properties of submersion metrics.

In differential geometry there has been extensive study of isometric immersions into space forms. Part of Proposition 3.1 together with Theorem 3.4 may be viewed as providing information on the dual question; namely, given a space form what Riemannian submersions are admissible if the fibers are totally geodesic?

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1. In this section we summarize known results on Riemannian submersions

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which will be used in the sequel. Generally, we will use the terminology of Barrett O'Neill whose article [15] constitutes a basic paper on the subject of submersions.

For a Riemannian submersion $\pi : M \to B$, the implicit function theorem tells us that $\pi^{-1}(x)$ is a closed submanifold of M where $\dim \pi^{-1}(x) = \dim M - \dim B$. Given a Riemannian submersion π from M to B we denote by $\mathscr{V}$ the vector subbundle of TM defined by the foliation of M by the fibers of π. $\mathscr{H}$ will denote the complementary distribution of $\mathscr{V}$ in TM determined by the metric on M.

We make one notational remark. If $q \in M$ where M is any manifold, T_qM denotes the tangent space of M at q. Following O'Neill [15] we define the tensor T for arbitrary vector fields E and F by $T_EF = \mathscr{H}\nabla_{\mathscr{V}E}\mathscr{V}F + \mathscr{V}\nabla_{\mathscr{V}E}\mathscr{H}F$ where $\mathscr{V}E$, $\mathscr{H}E$, etc. denote the vertical and horizontal projections of the vector field E. Recall O'Neill has described the following three properties of the tensor T:

(1) T_E is a skew-symmetric operator on the tangent space of M reversing horizontal and vertical subspaces.

(2) $T_E = T_{\mathscr{V}E}$.

(3) For vertical vector fields V and W, T is symmetric, i.e., $T_VW = T_WV$. In fact, along a fiber T is the second fundamental form of the fiber provided we restrict ourselves to vertical vector fields.

Next we define the *integrability tensor* A associated with the submersion. For arbitrary vector fields E and F,

$$A_EF = \mathscr{H}\nabla_{\mathscr{H}E}\mathscr{V}F + \mathscr{V}\nabla_{\mathscr{H}E}\mathscr{H}F .$$

(1′) At each point A_E is a skew-symmetric operator on TM reversing the horizontal and vertical subspaces.

(2′) $A_E = A_{\mathscr{H}E}$.

(3′) For X, Y horizontal A is alternating, i.e., $A_XY = A_YX$.

Proofs of (1′), (2′) and (3′) are found in [15].

Definition. *A basic vector* field is a horizontal vector field X which is π related to a vector field X_* on B, i.e., $\pi_* X_u = X_{*\pi(u)}$ for all $u \in M$.

Lemma 1.1. *Suppose X and Y are basic vector fields on M which are related to X_* and Y_* on B. Then*

(a) *$g(X, Y) = g^*(X_*, Y_*) \circ \pi$ where g^* is the metric on M, and g the metric on B,*

(b) *$\mathscr{H}[X, Y]$ is basic and is π related to $[X_*, Y_*]$,*

(c) *$\mathscr{H}\nabla_XY$ is basic and is π related to $\nabla^*{}_{X_*}Y_*$ where ∇^* is the Riemannian connection B.*

The proofs of these results are found in O'Neill [15].

As before, we assume g is the metric on M and g^* the metric on B.

Lemma 1.2. *Let Z_i be a basic vector field on M corresponding to Z_{i_*} on B.*

Suppose for a horizontal vector field X, $g_p(X, Z_i) = g_{p'}(X, Z_i)$ for all such Z_i and for any $p, p' \in \pi^{-1}(b)$ where $b \in B$. Then $\pi_* X$ is a well defined vector field on B. In particular, X is basic.

Proof. Since π is a Riemannian submersion, we have $g^*_{\pi(p)}(\pi_* X_p, Z_{i*}) = g_p(X, Z_i) = g_{p'}(X, Z_i) = g_{\pi*(p)}(\pi_* X_{p'}, Z_{i*})$, using the assumptions that $\pi(p') = \pi(p)$ and the fact that $\pi_* Z_{ip} = Z_{i*} = \pi_* Z_{ip'}$. Choosing Z_i to be a basis of $\mathscr{H}$ we have from the above equalities $\pi_* X_{p'} = \pi_* X_p$, so $(\pi_* X)_{\pi(p)}$ is well defined.

Lemma 1·3. *Let X and Y be horizontal vector fields, V and W be vertical vector fields. Then each of the following holds:*

(1) $A_X Y = \frac{1}{2}\mathscr{V}[X, Y]$.

(2) $\nabla_V W = T_V W + \hat{\nabla}_V W$ *where $\hat{\nabla}$ denotes the Riemannian connection along a fiber with respect to the induced metric.*

(3) (a) $\nabla_V X = \mathscr{H}\nabla_V X + T_V X$.

(b) *If X is basic* $\mathscr{H}\nabla_V X = A_X V$.

(c) *Suppose the fibers are connected and totally geodesic* (*i.e.*, $T \equiv O$). *Then X is basic if $\nabla_V X = A_X V$ for any vertical V.*

(4) $\nabla_X V = A_X V + \mathscr{V}\nabla_X V$.

(5) $\nabla_X Y = \mathscr{H}\nabla_X Y + A_X Y$.

Proof. The proof of (1) is given in O'Neill [15]. The results of (2), (3) (a), (4) and (5) are direct consequences of the definitions of T and A.

We proceed now to (3) (b). It is known that if X is basic, $[X, V]$ is vertical. Hence

$$0 = \mathscr{H}[X, V] = \mathscr{H}\nabla_X V - \mathscr{H}\nabla_V X = A_X V - \mathscr{H}\nabla_V X .$$

It follows $A_X V = \mathscr{H}\nabla_V X$. Now we proceed to (3) (c).

If X is basic, then $A_X V = \mathscr{H}\nabla_V X$ as follows from (3) (b). Suppose $A_X V = \mathscr{H}\nabla_V X$ and let Y be any basic vector field. It follows

$$\begin{aligned} Vg(X, Y) &= g(\nabla_V X, Y) + g(X, \mathscr{H}\nabla_V Y) = g(A_X V, Y) + g(X, A_Y V) \\ &= g(V, A_X Y) - g(A_Y X, V) = 0 , \end{aligned}$$

since A is skew-symmetric and alternating. Since by assumption the fibers are connected, we may join any two points p and p' of π^{-1} (b) by a path which lies in the fiber. It follows from the fact that $Vg(X, Y) = 0$ that $g_p(X, Y) = g_{p'}(X, Y)$. Applying Lemma 1.2 the result follows.

Lemma 1.4. *Let X be a horizontal vector field and W a vertical vector field. Then*

(a) $(\nabla_X A)_W = -A_{A_X W}$,

(b) $(\nabla_W T)_X = -T_{T_W X}$.

Proof. We will only prove (a) since the proof for (b) is similar. Let E be an arbitrary vector field on M. Then

$$(\nabla_X A)_W E = \nabla_X(A_W E) - A_{\nabla_X W} E - A_W(\nabla_X E) = -A_{\nabla_X W} E\ ,$$

Since $A_W = 0$ by (2′). But $-A_{\mathscr{H}\nabla_X W}E = -A_{A_X W}E$. The next result gives a geometric characterization of the parallelism of the fundamental tensors T and A. This kind of result seems to have originated with Mutō [21].

Corollary 1.5. (a) *If A is parallel, then A is identically zero, i.e., $(\nabla_E A) = 0$ implies $A \equiv 0$.*

(b) *If T is parallel, then T is identically zero, i.e., $(\nabla_E T) = 0$ implies $T \equiv 0$.*

Thus Riemannian submersions with parallel intergrability tensors A are characterized as those whose horizontal distributions are integrable, and Riemannian submersions with parallel tensors T as those whose fibers are totally geodesic.

Proof. (a) $(\nabla_X A)_W X = -A_{A_X W} X = A_X A_X W$ for horizontal X and vertical W. These equalities are immediate from the preceding lemma, the fact $A_X W$ is horizontal and the fact that $A_X Y = -A_Y X$ for horizontal X and Y. Thus $g((\nabla_X A)_W X, W) = g(A_X A_X W, W) = -g(A_X W, A_X W)$. If $\nabla_X A = 0$, then $g(A_X W, A_X W) = 0$; so A_X annihilates the vertical distribution. Since A_X is a skew-symmetric operator on TM which reverses the horizontal and vertical subspaces, A_X also annihilates the horizontal distribution. The result follows since X was an arbitrary horizontal vector and $A_E = A_{\mathscr{H}E}$.

Part (b) is proven in a similar fashion. $g((\nabla_W T)_X W, X) = -g(T_{T_W X} W, X) = -g(T_W T_W X, X) = g(T_W X, T_W X)$ since T is skew-symmetric and $T_U V = T_V U$ when U and V are vertical. If $\nabla_W T = 0$, it follows $g(T_W X, T_W X) = 0$. Thus T_W annihilates the horizontal distribution. Since T_W is a skew-symmetric operator on TM which reverses the horizontal and vertical subspaces, T_W annihilates the vertical distribution as well. The result follows since W was an arbitrary vertical vector.

Let R denote the curvature tensor of M, and R^* the curvature tensor of B. Since there is no danger of ambiguity, we denote the horizontal lift of R^* by R^* as well. Following O'Neill [15] we set $g(R^*{}_{h_1 h_2} h_3, h_4) = g^*(R^*{}_{h_{1*} h_{2*}} h_{3*}, h_{4*})$ where h_i are horizontal vectors and $\pi_{i*} h = h_{i*}$.

For E and F, linearly independent vectors, we denote the tangent plane spanned by these two vectors by $P_E F$. In general, if X is a horizontal vector, then $\pi_* X$ is denoted by X_*. K, K_* and $\hat{K}$ will denote the sectional curvature of M, B and the fiber, respectively. We state now several results of O'Neill [15] which will be of future use.

Proposition 1.6. *Let $\pi : M \to B$ be a Riemannian submersion.*

(a) *Then for horizontal vectors X, Y, Z and H*

$$g(R^*{}_{XY}Z, H) = g(R_{XY}Z, H) + 2g(A_X Y, A_Z H) - g(A_Y Z, A_X H) - g(A_Z X, A_Y H)\ .$$

(b) *If X and Y are horizontal, and V and W are vertical vectors, then*

$$g(R_{XV}Y, W) = g((\nabla_X T)_V W, Y) + g((\nabla_V A)_X Y, W) \\ - g(T_V X, T_W Y) + g(A_X V, A_Y W) .$$

Proposition 1.7. *Let $\pi : M \to B$ be a Riemannian submersion. Then for orthonormal horizontal vectors X and Y and orthonormal vectical vectors V and W we obtain the following relations:*

(a) $K(P_V W) = \hat{K}(P_V W) - g(T_V V, T_W W) + g(T_V W, T_V W)$,

(b) $K(P_X V) = g((\nabla_X T)_V V, X) + g(A_X V, A_X V) - g(T_V X, T_V X)$,

(c) $K(P_X Y) = K_*(P_{X_*} Y_*) - 3g(A_X Y, A_X Y)$.

If we denote $g(E, E)$ by $\|E\|^2$, we have the following corollary crucial to our future work.

Corollary 1.7.1. *Let $\pi : M \to B$ be a Riemannian submersion with totally geodesic fibers. Then for orthonormal horizontal vectors X and Y and a vertical vector V of unit length*

(a) $K(P_X V) = \|A_X V\|^2$,

(b) $K(P_X Y) = K_*(P_{X_*} Y_*) - 3\|A_X Y\|^2$.

Proof. Immediate from Proposition 1.7.

Assume $\pi : M \to B$ has the structure of a fibered space; as usual assume π is a Riemannian submersion and, in addition, M is complete. Let γ be a smooth curve in B with $\gamma(0) = p$ and $\gamma(t_0) = q$. Then the family of unique horizontal lifts of γ to M denoted by $\{\tilde{\gamma}_x\}$ with $\tilde{\gamma}_x(0) = x \in \pi^{-1}(p)$ gives rise to a mapping $F_\gamma : \pi^{-1}(p) \to \pi^{-1}(q)$ defined as follows. For $x \in \pi^{-1}(p)$, we have $F_\gamma(x) = \tilde{\gamma}_x(t_0)$ and therefore the following result:

Proposition 1.8. *Under the above hypotheses, the mapping F_γ are diffeomorphisms between the fibers. Moreover, a necessary and sufficient condition for the mapping F_γ to be isometries is that the fibers be totally geodesic.*

Corollary 1.9. *Under the hypotheses of* (1.8) *a necessary and sufficient condition for the flow of a basic vector field to give rise to an isometry between fibers is that the fibers be totally geodesic.*

Proof. See Nagano [13] and also Hermann [9].

Proposition 1.10. *Let $\pi : M \to B$ be a Riemannian submersion. Assume M is complete and the fibers are totally geodesic. If X and Y are basic vector fields, then $A_X Y$ when restricted to a fiber is a Killing vector field of that fiber.*

Proof. See Bishop [4].

We now recount several other results on submersions which will be useful in our future work.

Theorem 1.11. *Let $\pi : M \to B$ be a Riemannian submersion with M connected. If M is complete so is B, and π is a locally trivial fiber space. If, in addition, the fibers are totally geodesic, then π is a fiber bundle with structure group the Lie group of isometries of the fiber.*

Proof. See Hermann [9] or Nagano [13].

A partial converse to the above theorem is provided by an elegant result of Vilms.

Theorem 1.12. *Let $\pi : M \to B$ be a fiber bundle with standard fiber F and Lie structure group G. Assume the bundle admits a connection in the sense of Ehresemann. Endow B and F with Riemannian metrics, and assume F is G invariant. Then there exists a natural metric on M such that π is a Riemannian submersion with totally geodesic fibers.*

Proof. See Vilms [19].

The following result is originally due to Hermann [9]. O'Neill [16] has given an elegant infinitesimal proof.

Proposition 1.12. *Let $\pi : M \to B$ be a Riemannian submersion. If γ is a geodesic of M which is horizontal at one point, then it is always horizontal, and hence $\pi \circ \gamma$ is a geodesic of B.*

Proof. See O'Neill [16].

2. In this section we consider Riemannian submersions from a complete M onto B. In addition, we assume throughout this section that the fibers of a submersion are connected and totally geodesic.

Let π and $\bar{\pi}$ be two Riemannian submersions from M to B which satisfy these conditions. π and $\bar{\pi}$ are said to be *equivalent* provided there exists an isometry f of M which induces an isometry $\underline{f}$ of B so that the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \bar{\pi}} \\ B & \xrightarrow{\underline{f}} & B \end{array}$$

In this case, the pair $(f, \underline{f})$ is called a *bundle isometry* between π and $\bar{\pi}$. We should note that the term "bundle isometry" is appropriate by the theorem of Hermann [9]. If $\pi = \bar{\pi}$, the pair $(f, \underline{f})$ is called a *bundle automorphism* of π. Finally, we say that a Riemannian submersion π is homogeneous provided for every $p, q \in M$, there exists a bundle automorphism $(f, \underline{f})$ of π with $f(p) = q$.

For the main theorem of this section we need the following result. As before, A will denote the integrability tensor of π.

Proposition 2.1. *Let $\pi : M \to B$ be a Riemannian submersion from a connected complete Riemannian manifold M onto a Riemannian manifold B. Assume the fibers are connected and totally geodesic, and let $F = \pi^{-1}(\mathrm{b})$ for a fixed $b \in B$. Suppose a tensor $\bar{A}$ is defined on F satisfying the following properties at every $r \in F$:*

(1) *For $E \in T_rM$, $\bar{A}_E$ is a skew-symmetric operator on T_rM reversing the horizontal and vertical subspaces.*

(2) *$\bar{A}_E = \bar{A}_{\mathscr{H}E}$ for $E \in T_rM$.*

(3) *For X, Y horizontal, $\bar{A}$ is alternating, i.e., $\bar{A}_X Y = -\bar{A}_Y X$.*

(4) *For X, Y horizontal and V, W vertical, we have*

$$g(R_{XV}Y, W) = g((\nabla_V \bar{A})_X Y, W) + g(\bar{A}_X V, \bar{A}_Y W) .$$

If $\bar{A} = A$ at one point $p \in F$, then $\bar{A}$ coincides with A on F.

Proof. We will drop the bar notation in the calculations for simplicity. By properties (4) and (1) of the hypotheses,

$$g(R_{XV}Y, W) = g((\nabla_V A)_X Y, W) - g(A_Y A_X V, W) .$$

Since W is an arbitrary vertical vector field we have

$$\text{(a)} \qquad A_Y A_X V + \mathscr{V} R_{XV} Y = \mathscr{V}(\nabla_V A)_X Y ,$$

where $\mathscr{V}$ denotes the vertical projection operator. Now

$$(\nabla_V A)_X Y = \nabla_V(A_X Y) - A_{\nabla_V X} Y - A_X \nabla_V Y .$$

Since $T = 0$ when the fibers are totally geodesic, $\nabla_V A_X Y$ is vertical and $\mathscr{H}\nabla_V Y = \nabla_V Y$. These facts together with properties (1) and (2) imply $A_{\nabla_V X} Y$ and $A_X \nabla_V Y$ are vertical. It follows $\mathscr{V}(\nabla_V A)_X Y = (\nabla_V A)_X Y$. Thus our expression (a) becomes

$$\text{(b)} \qquad A_Y A_X V + A_{\nabla_V X} Y + A_X \nabla_V Y + \mathscr{V} R_{XV} Y = \nabla_V (A_X Y) .$$

Let us take $\{X_i\}$ to be horizontal vector fields defined along the fiber F, and choose them so that at each point they span the horizontal distribution and are orthonormal. For any two points p and q of F, there exists a geodesic γ lying in F with $\gamma(0) = p$, $\gamma(1) = q$ since F is totally geodesic, connected and complete. Let us choose vertical vector fields $\{P_\alpha\}$ which form an orthonormal basis for the vertical distribution and which are parallel along γ. See Milnor [12].

We then have

$$\text{(c)} \qquad \mathscr{V} R_{X_i \dot\gamma} X_j + A_{X_j} A_{X_i} \dot\gamma + A_{\nabla_{\dot\gamma} X_i} X_j + A_{X_i} \nabla_{\dot\gamma} X_j = \nabla_{\dot\gamma}(A_{X_i} X_j) .$$

Now set $\nabla_{\dot\gamma} X_k = \sum_l S_k^l X_l$, $\mathscr{V} R_{X_i \dot\gamma} X_j = \sum_\alpha R_{i\dot\gamma j} P_\alpha$ and $A_{X_i}\dot\gamma = \sum_k T_i^k X_k$. This may be done since γ is vertical and the tensor defined in the proposition reverses horizontal and vertical subspaces.

Let

$$A_{X_i} X_j = \sum_\beta a_{ij}^\beta P_\beta \quad \text{and} \quad \dot\gamma = \sum_\beta f^\beta(t) P_\beta .$$

Then

$$\begin{aligned} T_i^k &= g(A_{X_i}\dot\gamma, X_k) = -g(\dot\gamma, A_{X_i} X_k) = -g\Big(\sum_\beta f^\beta(t) P_\beta, \sum_\sigma a_{ik}^\sigma P_\sigma\Big) \\ &= -\sum_\beta f^\beta(t) a_{ik}^\beta . \end{aligned}$$

It follows that

$$A_{X_j}A_{X_i}\dot{\gamma} = A_{X_j}\Big(\sum_k T_i^k X_k\Big) = \sum_k T_i^k A_{X_j}X_k = \sum_k \Big(-\sum_\beta f^\beta(t)a_{ik}^\beta\Big)A_{X_j}X_k$$
$$= \sum_k \Big(-\sum_\beta f^\beta(t)a_{ik}^\beta\Big)\Big(\sum_u a_{jk}^u P_u\Big) = \sum_k \Big(-\sum_{k,\beta} f^\beta(t)a_{ik}^\beta a_{jk}^u\Big)P_u\ .$$

We may write (c) as follows:

$$\text{(d)} \qquad R_{i\dot{\gamma}j}^\alpha P_\alpha - \sum_{k,\beta} f^\beta(t)a_{ik}^\beta a_{jk}^\alpha P_\alpha - \sum_l S_i^l a_{lj}^\alpha P_\alpha - \sum_l S_j^l a_{il}^\alpha P_\alpha = \frac{d}{dt}(a_{ij}^\alpha)P_\alpha$$

since the P_α are parallel along γ. Thus we have a system of ordinary differential equations

$$\text{(e)} \qquad R_{i\dot{\gamma}j}^\alpha - \sum_{k,\beta} f^\beta(t)a_{ik}^\beta a_{jk}^\alpha - \sum_l S_i^l a_{lj}^\alpha - \sum_l S_j^l a_{il}^\alpha = \frac{d}{dt}a_{ij}^\alpha\ .$$

Now we know that $\gamma(0) = p$ and that $a_{ij}(0)$ are given at that point by hypothesis. In fact, one solution exists since the components of the integrability tensor A of π certainly satisfy properties (1) through (3) of the proposition and since we also obtain (4) by Proposition 1.6(b) and the assumption on the fibers.

Since everything is assumed to be C^∞, the Lipschitz condition for (e) is satisfied. It follows from the uniqueness theorem for ordinary differential equations that $A = \bar{A}$ at q if they coincide at p. But q was an arbitrary point of F. Thus $A = \bar{A}$ on all of F.

Theorem 2.2. *Let $\pi_i (i = 1, 2)$ be Riemannian submersions from a connected complete Riemannian manifold onto a Riemannian manifold B. Assume the fibers of these submersions are connected and totally geodesic. Suppose f is an isometry of M which satisfies the following two properties at a given point $p \in M$:*

(1) *$f_{*p}: T_pM \to T_{f(p)}M$ maps $\mathcal{H}_{1_p}$ onto $\mathcal{H}_{2f(p)}$ where $\mathcal{H}_i$ denotes the horizontal distribution of π_i.*

(2) *For $E, F \in T_p(M)$, $f_* A_{1E}F = A_{2f_*E}f_*F$ where A_i are the integrability tensors associated with π_i.*

Then f induces an isometry $\underline{f}$ of B so that the pair $(f, \underline{f})$ is a bundle isometry between π_1 and π_2. In particular, π_1 and π_2 are equivalent.

Proof. We begin by proving a lemma which will be needed in the sequel. (Compare [11, p. 235, Lemma 2].)

Lemma. Let M be a connected and complete Riemannian manifold. Suppose N and N' are two closed, connected, totally geodesic submersions. If $x \in N \cap N'$ and $T_xN = T_xN'$, then $N = N'$.

Proof. Suppose $y \in N$. Then there exists a geodesic τ in N joining x to y. Since N is totally geodesic, τ is a geodesic in M. Now the initial tangent vector in T_xN' lies in N', since N' is totally geodesic and complete. It follows $\tau(t) \in N'$ for all values of t. Thus $y \in N'$ so $N \subset N'$. Similarly $N' \subset N$. Hence $N = N'$.

Let $S_{\pi_1(p)}$ denote the fiber through p with respect to π_1, and $S_{\pi_2 f(p)}$ the fiber through $f(p)$ with respect to π_2. Now $f(S_{\pi_1(p)})$ is totally geodesic in M since $S_{\pi_1(p)}$ is totally geodesic in M and f is an isometry. Indeed, $f(S_{\pi_1(p)})$ is connected and complete, by the assumption on the fibers $f(p) \in f(S_{\pi_1(p)}) \cap S_{\pi_2 f(p)}$. In fact, $T_{f(p)} f(S_{\pi_1(p)}) = T_{f(p)} S_{\pi_2 f(p)}$ since $f_*(\mathscr{H}_{1p}) = \mathscr{H}_{2f(p)}$ and since the above distributions are complementary distributions with respect to the metric on M. It follows $f(S_{\pi_1(p)}) = S_{\pi_2 f(p)}$ by the lemma which was proven above. From this we see that for any $q \in S_{\pi_1(p)}$, $f_* T_q(S_{\pi_1(p)}) = T_{f(q)} S_{\pi_2 f(p)}$. Thus $f_{*q} : \mathscr{H}_{1q} \to \mathscr{H}_{2f(q)}$ is an isometry onto, since these latter are the complementary spaces to the respective fibers and f is an isometry of M.

Let γ be a geodesic in B starting at $\pi_1(p)$ which we lift horizontally to $\tilde{\gamma}$ starting at $q \in S_{\pi_1(p)}$. Then $f \circ \tilde{\gamma}$ is a geodesic starting at $f(q)$ and, moreover, $f_* \dot{\tilde{\gamma}}(0)$ is horizontal. From Proposition 1.12 it follows $f \circ \tilde{\gamma}$ is always horizontal. Let $S_{\gamma(t)}$ denote the fiber over $\gamma(t)$ with respect to π_1. Notice $S_{\tilde{\gamma}(0)} = S_{\pi_1(p)}$. Then the mapping $F_{\gamma(t)} : S_{\pi_1(p)} = S_{\gamma(0)} \to S_{\gamma(t)}$ defined by mapping the point $q \in S_{\pi_1(p)}$ to the endpoint (at time t) of the unique horizontal lift of γ starting at q gives ries to an isometry of the fibers $S_{\gamma(0)}$ and $S_{\gamma(t)}$ as we let q vary over $S_{\gamma(0)}$. See Corollary 1.9.

To complete the proof of the theorem we need the following result which we will demonstrate at the end of the proof of the theorem.

Lemma 2.3. *The family of geodesics $\{f \circ \tilde{\gamma}\}$ obtained from the family of geodesics $\{\tilde{\gamma}\}$ as the initial point of $q = \gamma(0)$ varies over $S_{\pi_1(p)}$ are the unique horizontal lifts with respect to π_2 of some geodesic ν in B starting at $\pi_2 f(p)$.*

If we admit this lemma it follows that we have a map $F_{\nu(t)} : S_{\pi_2(p)} = S_{\nu(0)} \to S_{\nu(t)}$ where $S_{\nu(t)}$ is the fiber with respect to π_2 over $\nu(t)$ and where $F_{\nu(t)}(x) = f \circ \tilde{\gamma}(t)$ for $f \circ \gamma(0) = x$, where $x \in S_{\nu(0)} = S_{\pi_2 f(p)}$. Again, by the result of Hermann and Nagano (Corollary 1.9), $F_{\nu(t)}$ is an isometry from $S_{\nu(0)}$ onto $S_{\nu(t)}$. Since $f(S_{\pi_1(p)}) = S_{\pi_2 f(p)} = S_{\nu(0)}$, it follows

$$\text{(a)} \qquad F_{\nu(t)}(f(S_{\pi_1(p)})) = S_{\nu(t)} .$$

But we also have

$$\text{(b)} \qquad f(F_{\gamma(t)} S_{\pi_1(p)}) = f(F_{\gamma(t)} S_{\gamma(0)}) .$$

Finally

$$\text{(c)} \qquad F_{\nu(t)}(f(S_{\pi_1(p)})) = f(F_{\gamma(t)} S_{\pi_1(p)})$$

by the definition of these maps.

Using (c) followed by (b) and (a) it follows $f(S_{\gamma(t)}) = S_{\nu(t)}$. From this we may conclude that f maps fibers onto fibers since the completeness of M implies the completeness of B by Theorem 1.11 and so for any $r \in B$, there exists a geodesic γ from $\pi_1(p)$ to r in B and hence a family $\{\tilde{\gamma}\}$ of horizontal lifts between the fibers $S_{\pi_1(p)} = S_{\gamma(0)}$ and $S_{\gamma(t)}$.

Since f is fiber-preserving, we have the result that f induces a map $\underline{f}: B \to B$ defined by $\underline{f}(\pi_1(q)) = \pi_2 f(q)$. This definition of $\underline{f}$ is easily seen to be well defined. Since f is an isometry which is fiber-preserving, it follows $f_{*q}: \mathscr{H}_{1q} \to \mathscr{H}_{2f(q)}$ isometrically for all $q \in M$. Since π_i maps $\mathscr{H}_i$ (for $i = 1, 2$) isometrically onto TB and since f is itself an isometry, we see easily that $\underline{f}$ is an isometry.

Proof of the lemma. If X is a basic vector field, it is known that for any vertical V, $\nabla_V X = A_X V$. In our case we set $X = \dot{\tilde{\gamma}}$ where $\tilde{\gamma}$ is the unique horizontal lift of γ. Now X is defined along $S_{\gamma(0)}$ and, in fact, is projectible, i.e., $\pi_{*q}X = \pi_{*q'}X$ for all $q, q' \in S_{\gamma(0)}$. Now we may extend $\dot{\gamma}$ in B to a vector field which we denote by X_*. Take the horizontal lift of X_* and call it X'. Then X' coincides with X along $S_{\gamma(0)}$. It follows that $\nabla_V X' = \nabla_V X$ on $S_{\gamma(0)}$. But $\nabla_V X' = A_{X'}V = A_X V$. Thus $\nabla_V X = A_X V$ on $S_{\gamma(0)}$.

Now f preserves the horizontal and vertical subspaces along the fiber $S_{\pi_1(p)}$ since we showed $f(S_{\pi_1(p)}) = S_{\pi_2 f(p)}$. Since f is an isometry, it also preserves ∇.

Consider $\{f_* X_i\}$ for the horizontal distribution along $S_{\pi_2(p)}$ where the X_i are horizontal on $S_{\pi_1(p)}$. Similarly, consider a local basis $\{f_* V_i\}$ for the vertical distribution on $S_{\pi_2 f(p)}$ where the $\{V_i\}$ form a local basis for the vertical distribution on $S_{\pi_1(p)}$.

Define the tensor $\bar{A}$ on $S_{\pi_2 f(p)}$ as follows:

$$\bar{A}_{f_* X_i} f_* V_j = f_* A_{1X_i} V_j \,, \qquad \bar{A}_{f_* X_i} f_* X_j = f_* A_{1X_i} X_j \,.$$

Then $\bar{A}$ is skew symmetric and reverses horizontal and vertical subspaces on $S_{\pi_2 f(p)}$ as may be checked easily.

Recall A satisfies property 4 of Proposition 2.1 on $S_{\pi_1(p)}$.

Since f is an isometry which along $S_{\pi_1(p)}$ preserves the horizontal and vertical distributions, it follows that $\bar{A}$ as defined above satisfies property 4 of Proposition 2.1 on $S_{\pi_2 f(p)}$. By assumption, $\bar{A}_{f(p)} = A_{2f(p)}$. It follows $\bar{A}$ coincides with A_2 on $S_{\pi_2 f(p)}$.

Now let X be a basic vector field on $S_{\pi_1(p)}$, and V a vertical vector field. On the one hand we have $f_* \nabla_V X = \nabla_{f_* V} f_* X$. But since $\bar{A} = A_2$ on $S_{\pi_2 f(p)}$, we have $f_* A_{1X} V = \bar{A}_{f_* X} f_* V = A_{2 f_* X} f_* V$. Now, since X is basic on $S_{\pi_1(p)}$, we have $\nabla_V X = A_{1X} V$ so $\nabla_{f_* X} f_* V = A_{2 f_* X} f_* V$ by what we showed in the beginning of the proof of the lemma. Recall f is an isometry, so any $W \in TS_{\pi_2 f(p)}$ may be written $f_* V$ for some appropriate choice of $V \in TS_{\pi_1(p)}$. Thus, for all π_2 basic vector fields Z,

$$\begin{aligned} Wg(f_* X, Z) &= g(\nabla_W f_* X, Z) + g(f_* X, \nabla_W X) \\ &= g(\nabla_{f_* V} f_* X, Z) + g(f_* X, \nabla_{f_* V} Z) \\ &= g(A_{2 f_* X} f_* V, Z) + g(f_* X, A_{2Z} f_* V) \end{aligned}$$

by the preceding argument and the fact that Z is π_2 basic. Using the last expression we see

$$Wg(f_*X, Z) = -g(f_*V, A_{2f_*X}Z) - g(A_{2Z}f_*X, f_*V) = 0 ,$$

since $A_{2Y_1}Y_2 = -A_{2Y_2}Y_1$ for horizontal Y_1, Y_2. Thus $Wg(f_*X, Z) = 0$, and so f_*X projects to a well defined vector field at $\pi_2 f(p)$ since $S_{\pi_2 f(p)}$ is connected and Z is basic (see Lemma 1.2).

The geodesic ν starting at $\pi_2 f(q)$ with $\pi_{2*}f_*X$ as initial tangent vector is uniquely determined by the well known theorem for ordinary differential equations.

The π_2 horizontal geodesic lift of ν is likewise uniquely determined. Since the horizontal lift of ν has f_*X as its initial tangent vector, it follows this lift must coincide with $f \circ \tilde{\gamma}$. Thus the proof is complete.

The next few results provide information on the rigidity of Riemannian submersions π from the unit sphere S^n onto a Riemannian manifold B under the assumption that the fibers are totally geodesic.

Consider the sphere S^{2n+1} as a subspace of $R^{2n+2} = C^{n+1}$. Let N denote the outward unit normal to S^{2n+1}, and J the natural almost complex structure on C^{n+1}. Then JN_p is tangent to S^{2n+1} at p. In fact, JN gives rise to a foliation of S^{2n+1} with standard fiber S^1. If we identify the leaves of this foliation as points, we obtain $CP(n)$. Indeed, this procedure gives rise to a mapping $\pi_1 : S^{2n+1} \to CP(n)$ which can in an obvious way be made into a Riemannian submersion with totally geodisic fibers. We call π_1 the standard or natural submersion of S^{2n+1} onto $CP(n)$, and denote the integrability tensor of this submersion by θ. (See O'Neill [16], Gray [8].)

Lemma 2.4. *Let $\pi_i : S^{2n+1} \to CP(n)$ be Riemannian submersions with totally geodesic fibers, and assume $n = 2$. π_i will denote the standard submersion. Then there exists an isometry f of S^{2n+1} which induces an isometry f of $CP(n)$ so that*

$$\begin{array}{ccc} S^{2n+1} & \xrightarrow{f} & S^{2n+1} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ CP(n) & \xrightarrow{f} & CP(n) \end{array}$$

is commutative. In fact $f = \mathrm{id}_{CP(n)}$.

Proof. Let $p \in CP(n)$, $q \in \pi_1^{-1}(p)$, $q' \in \pi_2^{-1}(p)$. Denote the horizontal distributions of π_1 and π_2 by $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively. Let X be a unit horizontal vector with $X \in \mathscr{H}_{1q}$. Then the π_2 horizontal lift to q' of $\pi_{1*}X$ will be denoted by $\bar{X}$. This procedure gives rise to a mapping $L : \mathscr{H}_{1q} \to \mathscr{H}_{2q'}$ which is a linear isometry. This fact follows from the fact that the π_i are submersions and preserve the metrics of the respective horizontal distributions and the fact that the horizontal lifts of vectors are unique.

By means of L we may define an almost complex structure $\bar{J}$ on $\mathscr{H}_{2q'}$ as follows: for $X \in \mathscr{H}_{1q}$ set $\bar{J}L(X) = L(JX)$. Then $\bar{J}\bar{J}L(X) = \bar{J}L(JX) = L(J^2X) = -L(X)$ so $\bar{J}^2L(X) = -L(X)$. Since L is onto $\mathscr{H}_{2q'}$, this gives a well defined almost complex structure on the vector space $\mathscr{H}_{2q'}$.

Recall from O'Neill [16] that the integrability tensor of π_1, which we denote by θ, has the following form. For orthonormal vectors X, Y in $\mathscr{H}_{1q}$, $\theta_X Y = -g(JX, Y)JN$. Actually this formula is valid for any horizontal X and Y but in the steps which follow X and Y are assumed to be orthonormal. A will denote the integrability tensor of π_2.

Step A. As before, R^* denotes the curvature tensor of the base or its horizontal lift. By Corollary 1.71 (a) we have

$$g(R^*_{XY}X, Y) = g(R_{XY}X, Y) + 3\,\|\theta_X Y\|^2 .$$

But since the total space is S^{2n+1}, we have

$$g(R^*_{XY}X, Y) = 1 + 3\,\|\theta_X Y\|^2 .$$

In particular, if $Y = JX$, then

$$\text{(a)} \qquad g(R^*_{XJX}X, JX) = 1 + 3\,\|\theta_X JX\|^2 = 4$$

by our formula for θ. Consider now the situation for $\bar{X} = L(X)$, $\overline{JX} = L(JX)$. By the way L was defined we know

$$g(R^*_{XJX}, X, JX) = g(R^*_{\bar{X}\,\overline{JX}}\bar{X}, \overline{JX}) ,$$

and so by Corollary 1.71 (a) and (a) above, we have

$$\text{(b)} \quad 4 = g(R^*_{\bar{X}\,\overline{JX}}\bar{X}, \overline{JX}) = g(R^*_{\bar{X}\,\overline{JX}}\bar{X}, \overline{JX}) + 3\,\|A_{\bar{X}}\overline{JX}\|^2 = 1 + 3\,\|A_{\bar{X}}\overline{JX}\|^2 ,$$

it follows $\|A_{\bar{X}}\overline{JX}\| = 1$.

Step B. Suppose $Y \in \{X, JX\}^\perp$ where $\{X, JX\}$ denotes the horizontal subspace of $\mathscr{H}_{1q}$ spanned by X and JX, and $\{X, JX\}^\perp$ denotes the complement of that subspace in $\mathscr{H}_{1q}$. We have on the one hand

$$\text{(c)} \qquad g(R^*_{XY}X, Y) = 1 + 3\,\|\theta_X Y\|^2 = 1 ,$$

since $\theta_X Y = 0$ under this assumption on Y as may be easily verified. But we also have a corresponding formula for X, Y keeping in mind that $g(R^*_{XY}X, Y) = g(R^*_{\bar{X}\bar{Y}}\bar{X}, \bar{Y})$ by the definition of L. Thus

$$\text{(d)} \qquad 1 = g(R^*_{\bar{X}\bar{Y}}\bar{X}, \bar{Y}) = 1 + 3\,\|A_{\bar{X}}\bar{Y}\|^2 ,$$

and so

$$\|A_{\bar{X}}\bar{Y}\| = 0 \quad \text{for} \quad \bar{Y} \in \{\bar{X}, \overline{JX}\}^{\perp}, \quad \text{i.e.,} \quad Y \in \{X, JX\}^{\perp} .$$

Step C. If we appeal to Proposition 1.6 (a) and replace X, Y, Z and H in that proposition by X, JX, Y and JY for $Y \in \{X, JX\}$, we get the following formula at q:

$$\text{(e)} \qquad \begin{aligned} g(R^*_{XJX}Y, JY) &= g(R_{XJX}Y, JY) + 2g(\theta_X JX, \theta_Y JY) \\ &\quad - g(\theta_{JX}Y, \theta_X JY) - g(\theta_Y X, \theta_{JX}JY) . \end{aligned}$$

Now the first term on the right is zero since this is the curvature tensor of the sphere. By direct calculation we see easily that $\theta_X JX = \theta_Y JY$ and $\theta_{JX}Y = \theta_X JY = \theta_Y X = \theta_{JX}JY = 0$, so $g(R^*_{XJX}Y, JY) = 2$. If we consider the analogue of (e) at q' for $L(X) = \bar{X}, L(JX) = \overline{JX}, L(Y) = \bar{Y}$ and $L(JY) = \overline{JY}$ we get, observing that $g(R^*_{XJX}Y, JY) = g(R^*_{\bar{X}\overline{JX}}\bar{Y}, \overline{JY})$, the following formula:

$$\text{(f)} \qquad \begin{aligned} 2 = g(R^*_{\bar{X}\overline{JX}}\bar{Y}, \overline{JY}) &= g(R_{\bar{X}\overline{JX}}\bar{Y}, \overline{JY}) + 2g(A_{\bar{X}}\overline{JX}, A_{\bar{Y}}\overline{JY}) \\ &\quad - g(A_{\overline{JX}}\bar{Y}, A_{\bar{X}}\overline{JY}) - g(A_{\bar{Y}}\bar{X}, A_{\overline{JX}}\overline{JY}) . \end{aligned}$$

Since the first term on the right hand side is the curvature tensor of the sphere and $A_{\bar{Y}}\bar{X} = A_{\overline{JX}}\overline{JY} = A_{\overline{XJ}}\bar{Y} = A_{\bar{X}}\overline{JY} = 0$, by step B formula (f) becomes

$$2 = 2g(A_{\bar{X}}\overline{JX}, A_{\bar{Y}}\overline{JY}) .$$

By step (a) $\|A_{\bar{X}}\overline{JX}\| = \|A_{\bar{Y}}\overline{JY}\| = 1$, so we may conclude $A_{\bar{X}}\overline{JX} = A_{Y}\overline{JY}$. Now for any unit $X' \in \{\bar{X}, \overline{JX}\}$ we have $A_{X'}JX' = A_{\bar{Y}}\overline{JY}$. We also have $A_{\bar{Y}}\overline{JY} = A_{\bar{X}}\overline{JX}$. It follows for any unit $\bar{Z}$ in $\mathscr{H}_{2q'}$, that $A_{\bar{Z}}\overline{JZ} = A_{\bar{X}}\overline{JX}$.

Step D. For $q \in \pi_1^{-1}(p)$ and $q \in \pi_2^{-1}(p)$ we will construct bases for $T_q S^{2n+1}$ and $T_{q'} S^{2n+1}$ as follows: the basis for $T_q S^{2n+1}$ is given by

$$\{X_1, J_1X_1, X_2, JX_2, \cdots, X_n, JX_n, \theta_{X_1}JX_1\} ,$$

where the X_i are horizontal of unit length and for $i \neq j$, $X_i \notin \{X_j, JX_j\}$.

If $\bar{X}_i = L(X_i)$ and $\overline{JX}_i = L(JX_i)$, then we have the following basis for $T_{q'} S^{2n+1}$

$$\{\bar{X}_1, \overline{JX}_1, \bar{X}_2, \overline{JX}_2, \cdots, \bar{X}_n, \overline{JX}_n, A_{\bar{X}_1}\overline{JX}_1\} .$$

We will define a linear isometry from $T_q S^{2n+1} \to T_{q'} S^{2n+1}$ which extends $L\colon \mathscr{H}_{1q} \to \mathscr{H}_{2q'}$. We will denote this new map also by L. On the basis elements, L behaves as follows:

$$L\colon X_i \to \bar{X}_i ,$$

$$L\colon JX_i \to \overline{JX}_i ,$$

$$L\colon \theta_{X_1}JX_1 \to A_{\bar{X}_1}\overline{JX}_1 \ .$$

Since $L\colon T_qS^{2n+1} \to T_{q'}S^{2n+1}$ is the extension of the linear isometry from $\mathscr{H}_{1q}$ onto $\mathscr{H}_{2q'}$ and since both $\theta_{X_1}JX_1$ and $A_{\bar{X}_1}\overline{JX}_1$ have unit length, this new L is a linear isometry.

Now L preserves the tensors θ and A. To see this, note

$$\theta_{X_i}JX_k = A_{\bar{X}_i}\overline{JX}_k = 0 \ , \qquad \theta_{X_i}X_k = A_{\bar{X}_i}\bar{X}_k = 0 \ , \qquad \text{for } i \neq k$$

by step B. Thus

$$\begin{aligned} 0 &= (\theta_{X_i}JX_k) = A_{\bar{X}_i}\overline{JX}_k = A_{L(X_i)}L(JX_k) \ , \\ 0 &= L(\theta_{X_i}X_k) = A_{\bar{X}_i}\bar{X}_k = A_{L(X_i)}L(X_k) \qquad \text{for } i \neq k \ . \end{aligned}$$

If $i = k$, then $\theta_{X_i}JX_i = \theta_{X_1}JX_1$ and $A_{\bar{X}_i}\overline{JX}_i = A_{\bar{X}_1}\overline{JX}_1$, so

$$A(\theta_{X_i}JX_i) = L(\theta_{X_1}JX_1) = A_{\bar{X}_1}\overline{JX}_1 = A_{\bar{X}_i}\overline{JX}_i = A_{L(X_i)}L(JX_i) \ .$$

Using the skew symmetry of θ and A we see L preserves θ and A since $\{X_i, JX_i\}_{1\leq i\leq n}$ are a basis for the respective horizontal distributions.

Now we know from a well known Theorem [20, 2.3.12] that there exists an isometry f of S^{2n+1} with $f(q) = q'$ and $f_{*q} = L$. Since the fibers of π_1 and π_2 are closed, they are complete. A simple homotopy argument shows the fiber of π_2 is connected. Now it is known that the only complete, connected, totally geodesic submanifolds of spheres are spheres, so it follows the fibers of π_1 and π_2 are isometric to S^1. Our theorem applies since S is connected, and we have already shown that L and hence f_{*q} preserves the integrability tensors θ and A. Thus f is fiber-preserving and induces an isometry $\underline{f}$ of $CP(n)$ so that

$$\begin{array}{ccc} S^{2n+1} & \xrightarrow{f} & S^{2n+1} \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\pi_2} \\ CP(n) & \xrightarrow{\underline{f}} & CP(n) \end{array}$$

commutes.

This diagram implies $\underline{f}(p) = p$. Now $\underline{f}_{*p}\pi_{1*}X_i = \pi_{2*}f_{*q}X_i = \pi_{2*}\bar{X}_i = \pi_{1*}X_i$ by the way L was defined in the beginning. Also $\underline{f}_{*p}\pi_{1*}JX_i = \pi_{1*}JX_i$. Thus $\underline{f}_{*q} =$ identity. It follows $\underline{f} = \mathrm{id}_{CP(n)}$.

Our procedure in the next result is akin to that of Lemma 2.4. Consider S^{4n+3} as a unit sphere in R^{4n+4}, and let N denote the outward unit normal to S^{4n+3}. Let I, J and K be the natural almost complex structures on R^{4n+4} with $IJ = K$, $JK = L$, $KI = J$. Then IN, JN, KN give rise to a distribution on S^{4n+3} which is integrable. To check this, let ∇ denote the covariant derivative on R^{4n+4}.

Now $[IN, JN] = \nabla_{IN}JN - \nabla_{JN}IN = J\nabla_{IN}N - I\nabla_{JN}N = JIN - IJN = -2KN$, since the structures I, J, K are parallel and the sphere is umbilical. The other identities for integrability are similarly vcrified. Identifying the leaves of the distribution on S^{4n+3} as points we obtain $QP(n)$. This procedure gives rise to a mapping $\pi_1 : S^{4n+3} \to QP(n)$ which can be made into a Riemannian submersion by taking as the horizontal distribution the distribution complementary to IN, JN, KN in TS^{4n+3}. Direct calculation shows that the tensor T of π_1 is zero. As before, we call π_1 the standard or natural fibration of S^{4n+3} over $QP(n)$.

Lemma 2.5. *For $i = 1, 2$ let $\pi_i : S^{4n+3} \to QP(n)$ be Riemannian submersions with totally geodesic fibers. Assume $n = 2$, and π_1 is the natural fibration. Then there exist an isometry f of S^{4n+3} and an isometry $\underline{f}$ of $QP(n)$ so that*

$$\begin{array}{ccc} S^{4n+3} & \xrightarrow{f} & S^{4n+3} \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ QP(n) & \xrightarrow{\underline{f}} & QP(n) \end{array}$$

is commutative. In fact, $\underline{f} = \mathrm{id}_{QP(n)}$.

Proof. The proof follows the same pattern as that of Lemma 2.4 but is more complicated.

Let $p \in QP(n)$, $q \in \pi_1^{-1}(p)$, $q' \in \pi_2^{-1}(p)$. Denote the horizontal distributions of π_1 and π_2 by $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively. If X is a unit horizontal vector with $X \in \mathscr{H}_{1q}$, then the unique π_2 horizontal lift to q' of $\pi_{1*}X$ will be denoted by $\bar{X}$. This procedure gives use to a linear isometry $L : \mathscr{H}_{1q} \to \mathscr{H}_{2q'}$, since the submersions π_i preserve the metrics of the respective horizontal distributions, and the horizontal lifts are unique as we just mentioned.

By means of L we may define three almost complex structures $\bar{I}, \bar{J}$ and $\bar{K}$ on $\mathscr{H}_{2q'}$ as follows: For $X \in \mathscr{H}_{1q}$, set

$$\bar{I}L(X) = L(IX)\ , \qquad \bar{J}L(X) = L(JX)\ , \qquad \bar{K}L(X) = L(KX)\ .$$

Then $\bar{I}^2 = \bar{J}^2 = \bar{K}^2 = -1$ as was shown in the proof of Lemma 2.4. In fact, we have the usual relations $\overline{IJ} = \bar{K}, \overline{JK} = \bar{I}, \overline{KI} = \bar{J}$. For example, $\overline{IJ}L(X) = \bar{I}L(JX) = L(IJX) = L(KX) = \bar{K}L(X)$. But L is onto $\mathscr{H}_{2q'}$ so it follows $\overline{IJ} = \bar{K}$.

Recall from Gray [8] that the integrability tensor θ of π_1 has the following form for any horizontal X and Y:

$$\theta_X Y = -g(IX, Y)IN - g(JX, Y)JN - g(KX, Y)KN\ .$$

In what follows X and Y will be assumed to be orthonormal. A will denote the integrability tensor of π_2.

Step A. By Corollary 1.71 (a) we have at q

$$g(R^*_{XY}X, Y) = g(R_{XY}X, Y) + 3\,\|\theta_X Y\|^2 ,$$

where R^* is the lift of the curvature tensor on $QP(n)$, and R is the curvature tensor of the sphere. It follows $g(R^*_{XY}X, Y) = 1 + 3\,\|\theta_X Y\|^2$. In particular, if we set $Y = IX$ we get

$$\text{(a)} \qquad g(R^*_{XIX}X, IX) = 1 + 3\,\|\theta_X IX\|^2 = 4 ,$$

since $\|\theta_X IX\| = 1$ as may be verified from the formula for θ.

Consider now the situation for $L(X) = \overline{X}$ and $L(IX) = \overline{IX}$ at $q' \in \pi_2^{-1}(p)$. First observe $g(R^*_{XIX}X, IX) = g(R^*_{\overline{X}\,\overline{IX}}\overline{X}, \overline{IX})$, since $\pi_{1*}X = \pi_{2*}\overline{X}$ and $\pi_{2*}\overline{IX} = \pi_{1*}IX$. Then again appealing to Corollary 1.71 (a) and (a) above, we have

$$\text{(b)} \qquad 4 = g(R^*_{\overline{X}\,\overline{IX}}\overline{X}, \overline{IX}) = 1 + 3\,\|A_{\overline{X}}\overline{IX}\|^2 ,$$

and so $\|A_{\overline{X}}\overline{IX}\| = 1$. In a similar way we see $\|A_{\overline{X}}\overline{JX}\| = \|A_{\overline{X}}\overline{KX}\| = 1$. In fact, if $S, S' \in \{\overline{I}, \overline{J}, \overline{K}\}$ with $S \neq S'$ we have $\|A_{S\overline{X}}S'\overline{X}\| = 1$. For example, $A_{\overline{JX}}\overline{KX} = A_{\overline{JX}}\overline{I}(\overline{JX})$ but $\|A_{\overline{JX}}\overline{I}(\overline{JX})\| = 1$ by the above.

Step B. Suppose $Y \in \{X, IX, JX, KX\}^{\perp}$, where $\{X, IX, JX, KX\}$ is the space spanned by these vectors and $\{X, IX, JX, KX\}^{\perp}$ denotes its complement in $\mathscr{H}_{1q}$. Then

$$\text{(c)} \qquad g(R^*_{XY}X, Y) = 1 + 3\,\|\theta_X Y\|^2 = 1 ,$$

since $\theta_X Y = 0$ as may be checked directly. For $\overline{X} = L(X), \overline{Y} = L(Y)$ we have the following formula

$$\text{(d)} \qquad 1 = g(R^*_{\overline{X}\,\overline{Y}}\overline{X}, \overline{Y}) = 1 + 3\,\|A_{\overline{X}}\overline{Y}\|^2 .$$

This follows from (c) and the fact that $g(R^*_{\overline{X}\,\overline{Y}}\overline{X}, \overline{Y}) = g(R^*_{XY}X, Y)$. It follows $A_{\overline{X}}\overline{Y} = 0$ for $\overline{Y} \in \{\overline{X}, \overline{IX}, \overline{JX}, \overline{KX}\}^{\perp}$. Indeed, for any $X' \in \{\overline{X}, \overline{IX}, \overline{JX}, \overline{KX}\}$ $\cdot A_{X'}\overline{Y} = 0$.

Step C. Assume $Y \in \{X, IX, JX, KX\}^{\perp}$. By Proposition 1.6 (a) we have

$$\text{(e)} \qquad \begin{aligned} &g(R^*_{XJX}Y, JY) \\ &\quad = 2g(\theta_X JX, \theta_Y JY) - g(\theta_{JX}Y, \theta_X JY) - g(\theta_Y X, \theta_{JX}JY) = 2 , \end{aligned}$$

since $\theta_Y X = \theta_{JX}JY = \theta_{JX}Y = \theta_X JY = \theta_X JY = 0$ and $\theta_X JX = \theta_Y JY$ as may be verified from the formula. Thus, by (e) and observations similar to those made before, it follows

$$\text{(f)} \qquad \begin{aligned} 2 &= g(R^*_{\overline{X}\,\overline{JX}}\overline{Y}, \overline{JY}) \\ &= 2g(A_{\overline{X}}\overline{JX}, A_{\overline{Y}}\overline{JY}) - g(A_{\overline{JX}}\overline{Y}, A_{\overline{X}}\overline{JY}) - g(A_{\overline{Y}}\overline{X}, A_{\overline{JX}}\overline{JY}) . \end{aligned}$$

By step B, $A_{\overline{JX}}\overline{Y} = A_{\overline{Y}}\overline{X} = A_{\overline{X}}\overline{JY} = A_{\overline{JX}}\overline{JY} = 0$ and so $2 = 2g(A_{\overline{X}}\overline{JX}, A_{\overline{Y}}\overline{JY})$. By step A, $\|A_{\overline{X}}\overline{JX}\| = \|A_{\overline{Y}}\overline{JY}\| = 1$ and so $A_{\overline{X}}\overline{JX} = A_{\overline{Y}}\overline{JY}$. Now if $X' \in \{\overline{X}, \overline{IX}, \overline{JX}, \overline{KX}\}$ and X' has unit length, we have $A_{X'}\bar{J}X' = A_{\overline{Y}}\overline{JY}$. But $A_{\overline{Y}}\overline{JY} = A_{\overline{X}}\overline{JX}$. It follows for any unit $\overline{Z} \in \mathscr{H}_{2q'}$, $A_{\overline{Z}}\overline{JZ} = A_{\overline{X}}\overline{JX}$. In a similar way we may show $A_{\overline{X}}\overline{IX} = A_{\overline{Y}}\overline{IY}$, $A_{\overline{X}}\overline{KX} = A_{\overline{Y}}\overline{KY}$ for all unit $\overline{X}, \overline{Y} \in \mathscr{H}_{2q'}$. More generally, $A_{S\overline{X}}S'\overline{X} = A_{S\overline{Y}}S'\overline{Y}$ for such $\overline{X}$ and $\overline{Y}$ if $S, S' \in \{\bar{I}, \bar{J}, \bar{K}\}$ and $S \neq S'$. For example

$$A_{\overline{JX}}\overline{KX} = A_{\overline{JX}}\bar{I}(\bar{J}(\overline{X})) = A_{\overline{JY}}\bar{I}(\overline{JY}) = A_{\overline{JY}}\overline{KY}$$

by the above

Step D. Let $\overline{X}$ be a unit horizontal vector at q', and set $V_1 = A_{\overline{X}}\overline{IX}$, $V_2 = A_{\overline{X}}\overline{JX}$, $V_3 = A_{\overline{X}}\overline{KX}$. Then $\{V_1, V_2, V_3\}$ are orthonormal vertical vectors. That V_i have unit length is immediate from step A.

$$g(V_1, V_2) = g(A_{\overline{X}}\overline{IX}, V_2) = -g(\overline{IX}, A_{\overline{X}}V_2) .$$

But

$$1 = g(A_{\overline{X}}\overline{JX}, V_2) = -g(\overline{JX}, A_{\overline{X}}V_2) ,$$

so $A_{\overline{X}}V_2 = -\overline{JX}$ as follows from Corollary 1.71 (a), since $1 = g(A_{\overline{X}}V_2, A_{\overline{X}}V_2)$ by that corollary. Thus

$$g(V_1, V_2) = -g(\overline{IX}, A_{\overline{X}}V_2) = g(\overline{IX}, \overline{JX}) = 0 .$$

Similarly,

$$g(V_2, V_3) = g(V_1, V_3) = 0 .$$

Step E. Consider the following orthonormal basis for T_qS^{4n+3}. Let X_i be unit horizontal vectors with $X_j \in \{X_i, JX_i, KX_i\}^\perp$ for $i \neq j$. The basis for T_qS^{4n+3} is given by

$$\{X_1, IX_1, JX_1, KX_1, \cdots, X_n, IX_n, JX_n, KX_n, \theta_{X_1}IX_1, \theta_{X_1}JX_1, \theta_{X_1}KX_1\} .$$

Suppose $L(X_i) = \overline{X}_i, L(IX_i) = \overline{IX}_i, L(JX_i) = \overline{JX}_i, L(KX_i) = \overline{KX}_i$. We construct a basis for $T_{q'}S^{4n+3}$ as follows:

$$\{\overline{X}_1, \overline{IX}_1, \overline{JX}_1, \overline{KX}_1, \cdots, \overline{X}_n, \overline{IX}_n, \overline{JX}_n, \overline{KX}_n, A_{\overline{X}_1}\overline{IX}_1, A_{\overline{X}_1}\overline{JX}_1, A_{\overline{X}_1}\overline{KX}_1\} .$$

These two bases are both orthonormal.

We define an isometry from T_qS^{4n+3} onto $T_{q'}S^{4n+3}$ which extends $L\colon \mathscr{H}_{1q} \to \mathscr{H}_{2q'}$. We denote this extension also by L which is given as follows:

$$
\begin{aligned}
&L\colon T_qS^{4n+3} \to T_{q'}S^{4n+3}\,, \\
&L\colon X_i \to X_i\,, \\
&L\colon SX_i \to \overline{SX}_i\,, && \text{where } S \in \{I, J, K\} \text{ and } \bar{S} \in \{\bar{I}, \bar{J}, \bar{K}\}\,, \\
&L\colon \theta_{X_1}SX_1 \to A_{\bar{X}_1}\overline{SX}_1\,, && S = I, J, K \text{ if } \bar{S} = \bar{I}, \bar{J}, \bar{K}, \text{ respectively.}
\end{aligned}
$$

Then L is an extension of a linear isometry and is an isometry of the vertical subspaces. Hence it preserves the respective horizontal and vertical subspaces of π_1 and π_2. In fact L preserves the integrability tensors θ and A. We check this as follows: If $X \in \{X_i, IX_i, JX_i, KX_i\}$ and $Y \in \{X_j, IX_j, JX_j, KX_j\}$ for $i \neq j$, then $\theta_X Y = A_{\bar{X}}\bar{Y} = 0$ by step B. Thus $\theta = L(\theta_X Y) = A_{\bar{X}}\bar{Y} = A_{\bar{X}}\bar{Y} = A_{L(X)}L(Y)$. Next we have $L\theta_{X_i}SX_i = L\theta_{X_1}SX_1 = A_{\bar{X}_1}\overline{SX_1} = A_{\bar{X}_1}\overline{SX_i} = A_{L(X_i)}L(SX_i)$. In fact, $L\theta_{SX_i}S'X_i = A_{L(SX_i)}L(S'X_i)$ for $S, S' \in \{I, J, K\}$. For example,

$$
\begin{aligned}
L\theta_{IX_i}KX_i &= L\theta_{IX_i}(-JIX_i) = -L\theta_{IX_i}J(IX_i) = -L\theta_{IX_i}J(IX_i) \\
&= -L\theta_{X_1}JX_1 = -A_{\bar{X}}\overline{JX}_1 = -A_{\overline{IX_1}}\overline{JIX}_1 = A_{IX_i} - \overline{JIX}_i \\
&= A_{\overline{IX_i}}\overline{KX}_i = A_{L(IX_i)}L(KX_i)
\end{aligned}
$$

by the conclusion to step C.

The other relations follow in a similar manner. By a well known theorem there exists an isometry f on S^{4n+3} with $f(q) = q'$ and $f_{*_q} = L$. In the proof of Lemma 2.4 the fibers of π_i are connected. It follows the fibers of π_1 and π_2 are isometric with S^3. Then, by our theorem, f is fiber-preserving and induces $\underline{f}$ of $QP(n)$ so that

$$
\begin{array}{ccc}
S^{4n+3} & \xrightarrow{f} & S^{4n+3} \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\
QP(n) & \xrightarrow{\underline{f}} & QP(n)
\end{array}
$$

is commutative. From this we have $\underline{f}(p) = p$ and, as in Lemma 2.4, $\underline{f}_{*_p} =$ identity.

Lemma 2.6. *Let $\pi_i\colon S^3 \to S^2$ be Riemannian submersions with totally geodesic fibers for $i = 1, 2$, and assume π_1 is the natural fibration. Then there exists an isometry f of S^3 which induces $\underline{f}$ of S^2 so that*

$$
\begin{array}{ccc}
S^3 & \xrightarrow{f} & S^3 \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\
S^2 & \xrightarrow{\underline{f}} & S^2
\end{array}
$$

is commutative. In fact $\underline{f} = \mathrm{id}_{S^2}$.

Proof. Omitted. Similar to that of Lemma 2.4 but simpler.

Remarks. Unfortunately we did not find analogues of Lemma 2.6 for the cases $S^7 \to S^4$ and $S^{15} \to S^8$. The question is unsettled.

Lemma 2.7. *The natural submersions*

$$\text{(a)} \quad \pi_1 : S^{2n+1} \to CP(n) , \qquad \text{(b)} \quad \pi_1 : S^{4n+3} \to QP(n)$$

$n \geq 2$ *are homogeneous.*

Proof. We omit this proof which uses techniques similar to those of Lemmas 2.5 and 2.6. It is much simpler because we only have to deal with the integrability tensor θ of π_1, and we have a formula for θ in these two cases.

3. In this section we wish to consider Riemannian submersions with totally geodesic fibers. We refrain from calling them "totally geodesic Riemannian submersions" since this term has been used by Vilms [19] in his classification of Riemannian submersions for which geodesics project to geodesics. His result implies that the horizontal distribution is integrable whereas in the above situation no such condition is in general required.

Instead of viewing these submersions from an exclusively bundle-theoretic viewpoint, we propose to look at them by imposing certain restrictions on the curvatures of the manifolds in question. It turns out that such restrictions on the curvature lead to very strong conditions on the mappings themselves.

The case where M is a manifold of nonpositive sectional curvature is the easiest, so we consider it first.

Proposition 3.1. *Let M be a Riemannian manifold with nonpositive sectional curvature, and π a Riemannian submersion from M onto a Riemannian manifold B. Assume the fibers are totally geodesic.*

(1) *Then π is integrable in the sense that the horizontal distribution is integrable, and*

(2) *B has nonpositive sectional curvature.*

(3) *If M has strictly negative curvature, so must B and moreover $\pi^{-1}(q)$ is discrete for every $q \in B$. If M is complete and connected, π is characterized as covering projection.*

Proof. (1) Suppose there exist orthonormal horizontal vectors X and Y on M with $A_X Y \neq 0$. Then there is a unit vector V on M such that $g(A_X Y, V) \neq 0$. It follows $0 \neq g(A_X Y, V) = -g(Y, A_X V)$ since A is skew-symmetric. But, by Corollary 1.71, $K(P_X V) = g(A_X V, A_X V) > 0$. This contradicts the curvature assumption on M. Hence $A \equiv 0$, and the horizontal distribution of π on M is integrable.

(2) From (1), $A \equiv 0$. Hence using Corollary 1.71 (b) we have $0 \geq K(P_X Y) = K_*(P_{X*} Y_*)$ so B has nonpositive sectional curvature.

(3) We know from (1) that $A \equiv 0$, and from Corollary 1.7 (b) that B has negative sectional curvature if M does. The implicit function theorem tells us $\pi^{-1}(q)$ is a submanifold of M for any $q \in B$. If $\dim \pi^{-1}(q) \geq 1$, then there exists

a nonzero vertical vector V tangent to $\pi^{-1}(q)$. Taking a unit horizontal vector X, we have again from Corollary 1.7 (a) that $K(P_X V) = \|A_X V\|^2 = 0$ which contradicts the assumption in (3) that M has strictly negative curvature. It follows the fiber is discrete. Suppose M is connected and complete. Then so is B by Theorem 1.11. Thus π is an isometric immersion from M onto B. It follows from a well known theorem [10, p. 176] that π is a covering projectien.

Proposition 3.2. *Let $\pi: M \to B$ be a Riemannian submersion with totally geodesic fibers where $1 \leq \dim$ fiber $< \dim M$, and assume M has strictly positive sectional curvature.*

(1) *If $0 \neq X \in \mathscr{H}$ and $0 \neq V \in \mathscr{V}$, then $A_X V \neq 0$. In fact, $A_X: \mathscr{V} \to \mathscr{H}$ is an injective but not a subjective mapping from the vertical distribution $\mathscr{V}$ into the horizontal distribution $\mathscr{H}$.*

(2) $\dim B > \frac{1}{2} \dim M$ *and, in particular,* $\dim B > 1$.

(3) *B has strictly positive sectional curvature.*

Proof. We may assume without loss of generality that X has unit length. Let V be a unit vector of V. Then by Corollary 1.71, $0 < K(P_X V) = \|A_X V\|^2$ and hence A_X is an injective mapping. A_X maps $\mathscr{V}$ into $\mathscr{H}$ since A_X reverses horizontal and vertical subspaces. Using properties (1′) and (3′) of § 1, we find $0 = g(A_X X, V) = -g(X, A_X V)$ for every $V \in \mathscr{V}$. Hence $A_X \mathscr{V}$ is a proper subspace of $\mathscr{H}$. This proves (1). It follows $\dim \mathscr{V} < \dim \mathscr{H}$ and hence the dimension of a fiber $F < \dim B$. Thus $\dim B > \frac{1}{2} \dim M > 1$ and (2) follows. (3) is immediate from Corollary 1.71.

It should be remarked that Ferus [6] has obtained related results for the case of foliations.

Consider the case when $M = S^n(1)$ where $S^n(1)$ denotes the limit n-dimensional sphere. It is natural to ask for what B are there Riemannian submersions from $S^n = S^n(1)$ to B with totally geodesic fibers? We begin with a preliminary result.

Proposition 3.3. *Let $\pi: M \to B$ be a Riemannian submersion with* $\dim B \neq 0$. *If M is δ-pinched, then B is $\frac{1}{4}$-piched.*

Proof. We will show if $K(P_E F) \leq 1$, then $\|A_X Y\| \leq 1$ for X, Y orthonormal and horizontal. This is obvious if $A_X Y = 0$. If not, let $V = A_X Y / \|A_X Y\|$, and we have

$$\begin{aligned} \|A_X Y\| &= g(A_X Y, A_X Y / \|A_X Y\|) = g(A_X Y, V) = g(Y, A_X V) \\ &\leq g(A_X V / \|A_X V\|, A_X V) = \|A_X V\| \leq 1 , \end{aligned}$$

since by Corollary 1.71 and the curvature hypothesis $\|A_X V\|^2 = K(P_X V) \leq 1$. Thus $\|A_X Y\| \leq 1$, so $\|A_X Y\|^2 \leq 1$ for any orthonormal horizontal X, Y. But again by Corollary 1.71, $K_*(P_X Y) = K(P_X Y) + 3\|A_X Y\|^2$ for any orthonormal horizontal unit vectors. We may then conclude for any orthonormal horizontal unit vectors X, Y that $\delta \leq K_*(P_X Y) \leq 4$ since $K(P_X Y) \leq 1$ and $\|A_X Y\|^2 \leq 1$. Normalizing the metric on B we have $\frac{1}{4}\delta \leq K_*(P_X Y) \leq 1$.

Corollary 3.4. *Let $\pi: S^m \to B$ be a Riemannian submersion with totally geodesic fibers* $1 \leq \dim$ *fiber* $\leq m - 1$. *Then B is $\frac{1}{4}$-pinched.*

Proof. Immediate from Proposition 3.3.

With these preliminaries we state the main theorem of this section. We exclude the case where $\dim B = 0$.

Theorem 3.5. *Let $\pi: S^m \to B$ be a Riemannian submersion with connected totally geodesic fibers, and assume* $1 \leq \dim$ *fiber* $\leq m - 1$. *Then, as a fiber bundle, π is one of the following types:*

(a) $$\begin{array}{ccc} S^1 & \longrightarrow & S^{2n+1} \\ & & \downarrow \pi \\ & & CP(n) \end{array} \quad \text{for } n \geq 2$$

(b) $$\begin{array}{ccc} S^3 & \longrightarrow & S^{4n+3} \\ & & \downarrow \pi \\ & & QP(n) \end{array} \quad \text{for } n \geq 2$$

(c) $$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \pi \\ & & S^2(\frac{1}{2}) \end{array}$$

(d) $$\begin{array}{ccc} S^3 & \longrightarrow & S^7 \\ & & \downarrow \pi \\ & & S^4(\frac{1}{2}) \end{array}$$

(e) $$\begin{array}{ccc} S^7 & \longrightarrow & S^{15} \\ & & \downarrow \pi \\ & & S^8(\frac{1}{2}) \end{array}$$

In cases (a) *and* (b), *B is isometric to complex and quaternionic projective space with sectional curvature K_* with $1 \leq K_* \leq 4$. In cases* (c), (d) *and* (e), *B is isometric to a sphere of curvature* 4.

Moreover if π and $\bar{\pi}$ are any two submersions both in class (a), (b) *or* (c) *and satisfying the above hypothesis, then π is equivalent to $\bar{\pi}$. In fact any submersion of type* (a), (b) *or* (c) *is homogeneous.*

Proof. Since $M = S^m(1)$, B is $\frac{1}{4}$-pinched by the above corollary. Now the fibers are totally geodesic and complete, and hence they are spheres. Now by a result of Adem [2], m and j (the fiber dimension) are both odd. It follows $\dim B$ is even. Since $M \xrightarrow{\pi} B$ is a fiber bundle by Hermann's theorem [9], we have the homotopy sequence

$$\pi_1(S^j) \to \pi_1(S^m) \to \pi_1(B) \to 0 \ ,$$

and it follows B is simply connected. Thus, by Berger's theorem [3], it follows B is isometric to one of the projective spaces or is homeomorphic to a sphere. First we deal with the projective spaces. By Adem's result [2], the only fiberings of spheres by spheres in dimensions other than the standard ones, i.e.,

$$\begin{array}{cccccc} S^n & S^{2n+1} & S^{4n+3} & S^3 & S^7 & S^{15} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ RP^n & CP(n) & QP(n) & S^2 & S^4 & S^8 \end{array}$$

are those with total space of dimension $2^{k+1} - 1$ and fiber dimension $2^k - 1$ with $k > 3$. Now if B where $CP(r)$ for some such r, then the homotopy sequence $\pi_2(S^{2^{k+1}-1}) \rightarrow \pi_2(CP(r)) \rightarrow \pi_1(S^{2^k-1})$ implies $\pi_2(CP(r)) = 0$ since $2^k - 1 \geq 15$. But $\pi_2(CP(r)) = 2$, and so this alternative is inadmissible. Similarly, if B were $QP(r)$, then $\pi_4(S^{2^{k+1}-1}) \rightarrow \pi_4(QP(r)) \rightarrow \pi_3(S^{2^k-1})$ yields $\pi_4(QR(r)) = 0$ which is also a contradiction.

We must dispose of the case where $B = C_a^2(P)$, the Cayley projective two-plane. Now $\dim C_a^2(P) = 16$, and so by Adem [2] $k = 4$, $2^{k+1} - 1 = 31$ and $2^k - 1 = 15$. Thus our fibration is

$$\begin{array}{ccc} S^{15} & \longrightarrow & S^{31} \\ & & \downarrow{\scriptstyle \pi} \\ & & C_a^2(P) \ . \end{array}$$

But then using the fiber homotopy sequence $\pi_i(S^{15}) \rightarrow \pi_i(S^{31}) \rightarrow \pi_i(C_a^2(P)) \rightarrow \pi_{i-1}(S^{15})$ we get $\pi_i(C_a^2(P)) = 0$, $1 \leq i \leq 15$. The theorem of Hurewicz implies $C_a^2(P)$ is a homology 16-sphere which it is not. Thus the only fibrations with one of the projective spaces as base are:

$$\begin{array}{ccc} S^1 & \longrightarrow & S^{2n+1} \\ & & \downarrow \\ & & CP^n(n) \end{array} \qquad \begin{array}{ccc} S^3 & \longrightarrow & S^{4n+3} \\ & & \downarrow \\ & & QP(n) \end{array}$$

We now must deal with the case where B is homeomorphic to a sphere. Adem has shown that the only fiber bundles with spheres as total space fiber and base are:

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \\ & & S^2 \end{array} \qquad \begin{array}{ccc} S^3 & \longrightarrow & S^7 \\ & & \downarrow \\ & & S^4 \end{array} \qquad \begin{array}{ccc} S^7 & \longrightarrow & S^{15} \\ & & \downarrow \\ & & S^8 \end{array}$$

Our procedure will allow us to treat the three cases simultaneously. We will show that under the hypotheses of the theorem, S^2, S^4 and S^8 are in fact spheres of constant curvature 4. Let $\mathscr{V}$ denote the vertical distribution, and $\mathscr{H}$ the horizontal distribution. Then in the above three cases $\dim \mathscr{V} + 1 = \dim \mathscr{H}$. Let X and Y be any pair of orthonormal horizontal vectors. We wish to show

$K_*(P_XY) = 4$.

To see this establish the following result:

Lemma 3.6. *$A_X : \mathscr{V} \to (X)^\perp$ is a linear isometry onto $(X)^\perp$ where $(X)^\perp$ denotes the orthogonal complement of X in $\mathscr{H}$. Then*

(a) *$A_X(\mathscr{V}) \perp X$ by the skew-symmetry of A_X and the fact $A_XX = 0$,*

(b) *$A_X : \mathscr{V} \to \mathscr{H}$ is a liear isometry onto its image since $g(A_XV, A_XV) = K(P_XV) = 1$ for all unit $V \in \mathscr{V}$,*

(c) *$\dim (X)^\perp = \dim \mathscr{V} = k$ for $k = 1, 3$ or 7.*

We proceed now to show $K_*(P_XY) = 4$ as follows. Since $Y \perp X$ and $\|Y\| = 1$, there is some unit vertical V such that $A_XV = Y$ by the lemma. We wish to obtain some information about $\|A_XY\|$. From the proof of our pinching proposition, $\|A_XY\| \leq 1$. On the other hand, by the Schwartz inequality $\|A_XY\| \geq g(A_XY, V) = g(Y, A_XV) = \|Y\|^2 = 1$. Thus $\|A_XY\| = 1$. Now from O'Neill's equation (see Corollary 1.71 (b)), $K_*(P_XY) = K(P_XY) + 3\|A_XY\|^2 = 4$. Since X and Y are arbitrary orthonormal vectors, it follows $K_*(P_XY) = 4$ for all 2 planes. Hence S^2, S^4 and S^8 are spheres of constant curvature 4 or radius $\frac{1}{2}$.

We now outline a proof of the last part of Theorem 3.5. Let π and $\bar{\pi} : S^m \to B$ be submersions both of type (a), (b) or (c). π_1 will denote the natural submersion discussed in § 2. From Lemma 2.7 we know that the natural submersions are homogeneous. (Compare Steenrod [18, §§ 20.2, 20.3].) Consider the following diagram:

$$\begin{tikzcd} S^m \arrow[r, "f_1"] \arrow[d, "\pi"'] & S^m \arrow[r, "f_2"] \arrow[d, "\pi_1"] & S^m \arrow[r, "f_3"] \arrow[d, "\pi_1"] & S^m \arrow[d, "\pi'"] \\ B \arrow[r, "\underline{f}_1"] & B \arrow[r, "\underline{f}_2"] & B \arrow[r, "\underline{f}_3"] & B \end{tikzcd}$$

Now $(f_1, \underline{f}_1)$ and $(f_3, \underline{f}_3)$ are bundle isometries whose existence is guaranteed by Lemmas 2.3, 2.4, 2.5. Suppose $r, s \in S^m$. Choose f_2 so that $f_2(f(r)) = f_3^{-1}(s)$. This may be done since $\pi_1 : S^m \to B$ is homogeneous for classes (a), (b) and (c). Letting $\pi' = \bar{\pi}$ we see π and $\bar{\pi}$ are equivalent. If we set $\pi' = \pi$ we see any π in class (a), (b) or (c) is homogeneous. Of course the desired bundle isometry (automorphism) is $(f, \underline{f})$ when $f = f_3 \circ f_2 \circ f_1$ and $\underline{f} = \underline{f}_3 \circ \underline{f}_2 \circ \underline{f}_1$. This completes the proof of Theorem 3.5.

Added in proof. The author has obtained some results similar to Theorem 3.5 for submersions from complex projective space. These will be discussed in a future paper.

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