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COMPACT QUOTIENT SPACES OF C^2 BY AFFINE TRANSFORMATION GROUPS

TATSUO SUWA

The purpose of this paper is to classify the compact complex surfaces of the form C^2/G, where G is a properly discontinuous and fixed point free group of affine transformations of the two-dimensional complex vector space C^2. Except for the use of some theorems on numerical characters of a compact complex surface, the method is mostly elementary.

§ 1 contains preliminary considerations on some properties of a fixed point free affine transformation group of C^2. In § 2 we perform the classification. Denoting by b_1 the first Betti number of the quotient space $S = C^2/G$, we prove that if $b_1 = 4$ then S is a complex torus (Theorem 1), if $b_1 = 3$ then S is a fiber bundle of elliptic curves over an elliptic curve (Theorem 2), if $b_1 = 2$ then S is a hyperelliptic surface (Theorem 3), and if $b_1 = 1$ then S is an elliptic surface over the projective line with multiple singular fibers (Theorem 4).

1. A fundamental lemma

Let G denote a group of affine transformations of the two-dimensional complex vector space C^2. Assume the action of G is (A) properly discontinuous, i.e., for any pair (K_1, K_2) of compact subsets in C^2, the set $\{g \in G \mid gK_1 \cap K_2 \neq \emptyset\}$ is finite, and (B) fixed point free, i.e., for all $g \in G$, $g \neq 1$, g has no fixed points. Thus the quotient space C^2/G is a complex manifold of complex dimension 2. Finally we assume (C) C^2/G is compact. The problem is to classify the compact complex surfaces of the form C^2/G. In this section we prove a fundamental lemma for this purpose.

First of all, each element g of G is expressed by a 3×3 matrix:

$$g = \begin{pmatrix} a_{11}(g) & a_{12}(g) & b_1(g) \\ a_{21}(g) & a_{22}(g) & b_2(g) \\ 0 & 0 & 1 \end{pmatrix},$$

which acts on $C^2 = \{z \mid z = (z_1, z_2)\}$ by

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$$\begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} z_1' \\ z_2' \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11}(g) & a_{12}(g) & b_1(g) \\ a_{21}(g) & a_{22}(g) & b_2(g) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix} .$$

We put

$$A(g) = \begin{pmatrix} a_{11}(g) & a_{12}(g) \\ a_{21}(g) & a_{22}(g) \end{pmatrix}, \qquad b(g) = \begin{pmatrix} b_1(g) \\ b_2(g) \end{pmatrix} .$$

Note that $\det A(g) \neq 0$. Moreover, that g has no fixed points means the linear equation

$$(A(g) - I)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -b(g)$$

has no solution for $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, where I denotes the 2×2 unit matrix. In particular,

(1) $$\det (A(g) - I) = 0 ,$$

(2) $$\text{if} \quad b(g) = 0 , \quad \text{then} \quad g = 1 .$$

For elements g and h of G we have

$$A(g^{-1}) = A(g)^{-1} , \qquad b(g^{-1}) = -A(g)^{-1}b(g) ,$$
$$A(gh) = A(g)\cdot A(h) , \qquad b(gh) = A(g)b(h) + b(g) .$$

Next we consider the space $E(2, 1)$ of lines in $\boldsymbol{C}^2$ and the action of G on $E(2, 1)$. A line L is a subvariety of $\boldsymbol{C}^2$ defined by a linear equation $\alpha_0 + \alpha_1 z_1 + \alpha_2 z_2 = 0$, $(\alpha_1, \alpha_2) \neq (0, 0)$. Let $E(2, 1)$ denote the set of lines in $\boldsymbol{C}^2$. Two equations $\alpha_0 + \alpha_1 z_1 + \alpha_2 z_2 = 0$ and $\alpha_0' + \alpha_1' z_1 + \alpha_2' z_2 = 0$ represent the same line if and only if there exists a complex number $\lambda \neq 0$ such that $\alpha_\nu' = \lambda \alpha_\nu$ for $\nu = 0, 1, 2$. Hence we have a bijection

$$E(2, 1) \xrightarrow{\sim} \boldsymbol{P}^2 - \{p\} , \qquad p = (1:0:0) ,$$

given by $L = \{(z_1, z_2) \mid \alpha_0 + \alpha_1 z_1 + \alpha_2 z_2 = 0\} \mapsto (\zeta_0 : \zeta_1 : \zeta_2) = (\alpha_0 : \alpha_1 : \alpha_2)$, where $\boldsymbol{P}^2$ denotes the two-dimensional complex projective space with homogeneous coordinates $(\zeta_0 : \zeta_1 : \zeta_2)$. We identify $E(2, 1)$ with $\boldsymbol{P}^2 - \{p\}$ by this bijection. If we denote by $G(2, 1)$ the set of lines in $\boldsymbol{C}^2$ passing through the origin, then $G(2, 1)$ is the projective line $\boldsymbol{P}^1$ in $E(2, 1)$ defined by $\zeta_0 = 0$. We have a fibering $\pi : E(2, 1) \to G(2, 1)$ defined by $(\zeta_0 : \zeta_1 : \zeta_2) \mapsto (\zeta_1 : \zeta_2)$. Thus $E(2, 1)$ is a complex line bundle over $G(2, 1) = \boldsymbol{P}^1$ of degree 1. Since G is a group of affine transformations, G acts naturally on $E(2, 1)$. Take $L \in E(2, 1)$ which is represented by $\alpha_0 + \alpha_1 z_1 + \alpha_2 z_2 = 0$. Then L is transformed by g to the line $\alpha'_0 + \alpha'_1 z_1 + \alpha'_2 z_2 = 0$, where $\alpha'_0 = \alpha_0 + (\alpha_1, \alpha_2)b(g)$, $(\alpha'_1, \alpha'_2) = (\alpha_1, \alpha_2) \cdot A(g)^{-1}$. Since G acts as a group of bundle automorphisms, G acts on the base space $G(2, 1) = \boldsymbol{P}^1 = \{(\zeta_1 : \zeta_2)\}$ by the formula

$$\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix} \mapsto {}^tA(g)^{-1}\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix} .$$

For a point p of $G(2, 1)$, let $H_p = \{g \in G \mid gp = p\}$ be the isotropy subgroup of G at p.

Remark. Thus we get a representation of G into the group of one-dimensional projective linear transformations $PGL(1, \boldsymbol{C})$. The kernel is the subgroup $\{g \in G \mid A(g) = 1\}$, i.e., the group of translations.

Lemma 1.1. *There exists a point p_0 on $G(2, 1)$ for which $H_{p_0} = G$.*

Proof. Suppose for any point p, $H_p \subsetneq G$. Fix an element g which acts non-trivially on $G(2, 1)$. Note that the number of the fixed points of g on $G(2, 1) = \boldsymbol{P}^1$ is 1 or 2.

Case I: g has only one fixed point p_1. By a suitable coordinate transformation, we may assume that $p_1 = 0 = (1 : 0)$ and $g(\infty) = 1$, $\infty = (0 : 1)$, $1 = (1 : 1)$. In view of (1) we have $A(g^{-1}) = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$. By assumption, there exists an element h such that $h(p_1) \neq p_1$. If we put ${}^tA(h)^{-1} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$, then $c \neq 0$. On the other hand, $0 = \det(A(h^{-1}) - I) = (a - 1)(d - 1) - bc$ by (1). Thus we have $\det(A(h^{-1}g^{-1}) - I) = (a - 1)(d - 1) - bc - c = -c \neq 0$. This means gh has a fixed point on $\boldsymbol{C}^2$, a contradiction.

Case II: g has two fixed points p_1 and p_2 on $G(2, 1)$. By a suitable coordinate transformation, we may assume $p_1 = 0$ and $p_2 = \infty$. This implies that $A(g)^{-1} = \begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}$ with $a \neq d$. On the other hand, $0 = \det(A(g)^{-1} - I) = (a - 1)(d - 1)$. By assumption there exist elements $g_i \notin H_{p_i}$, for $i = 1, 2$. Now we can divide our discussion into the following three cases.

(α) $g_1 \notin H_{p_2}$. Put ${}^tA(g_1)^{-1} = \begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix}$. Then $g_1(0) \neq 0$ and $g_1(\infty) \neq \infty$ imply that $b_1 c_1 \neq 0$. On the other hand, $0 = \det(A(g_1)^{-1} - I) = (a_1 - 1) \cdot (d_1 - 1) - b_1 c_1$. Put $\Delta = \det(A(g_1^{-1}g^{-1}) - I)$. Then we have

$$\Delta = (a - 1)(d_1 - 1) + (d - 1)(a_1 - 1) ,$$

where $(a - 1)(d - 1) = 0$ and $(a_1 - 1)(d_1 - 1) \neq 0$. Hence $\Delta \neq 0$, which means gg_1 has a fixed point on $\boldsymbol{C}^2$.

(β) $g_2 \notin H_{p_1}$. We can get a contradiction by the same argument as in case (α).

(γ) $g_1 \in H_{p_2}$ and $g_2 \in H_{p_1}$. We have $g_1 g_2 \notin H_{p_1}$ and $g_1 g_2 \notin H_{p_2}$, and this case is then reduced to case (α) if we replace g_1 by $g_1 g_2$. q.e.d.

By a suitable coordinate transformation, we may assume $p_0 = \infty, H_\infty = G$. Then for any element g of G, $A(g) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ is a triangular matrix. Hence we get

Corollary. *The group G is solvable.*

From now on, we always assume $a_{21}(g) = 0$ for every $g \in G$.

Remarks. 1. In the proof of the lemma, we only used the fact that the action of G on C^2 is fixed point free. Moreover, from this fact we have either $a_{11}(g) = 1$ for all $g \in G$ or $a_{22}(g) = 1$ for all $g \in G$.

2. Every element g of G is compatible with the projection $(z_1, z_2) \mapsto z_2$ of C^2 onto the second factor U_2. This suggests the fiber structure of C^2/G over U_2/G (see the proofs of Theorem 2 and 4).

2. Classification

We need some formulas for numerical characters of a compact complex surface. Denote by S a compact complex surface, i.e., a compact complex manifold of complex dimension 2, and by $\mathcal{O}$ and Ω^ν, respectively, the sheaves over S of germs of holomorphic functions and holomorphic ν-forms. Define $h^{\nu,\mu} = \dim H^\mu(S, \Omega^\nu)$. The geometric genus p_g and the irregularity q of S are defined, respectively, by $p_g = h^{0,2}$ and $q = h^{0,1}$. By the duality theorem, $p_g = h^{0,2} = h^{2,0}$. Moreover, we denote by b_ν the ν-th Betti number, and by c_ν the ν-th Chern class of S. Among these numerical characters, the Noether formula due to Hirzebruch, Atiyah and Singer holds:

$$12(p_g - q + 1) = c_1^2 + c_2 \ . \tag{3}$$

Moreover a theorem of Kodaira [3, I, Theorem 3] says

$$\begin{aligned} &\text{if } b_1 \text{ is even, then } 2q = b_1 \text{ and } h^{1,0} = q; \\ &\text{if } b_1 \text{ is odd, then } 2q = b_1 + 1 \text{ and } h^{1,0} = q - 1 \ . \end{aligned} \tag{4}$$

Take an affine transformation group G of C^2 satisfying conditions (A), (B), and (C) in § 1. Note that G, being the fundamental group of a compact space, is finitely generated.

The following proposition is obvious.

Proposition 1. *If $H_p = G$ for every point p of $G(2, 1)$, i.e., if every element of G is a translation, then $S = C^2/G$ is a complex torus.*

From now on, we assume that there exists an element of G which is not a translation. We classify the cases as follows:

$$\exists\, g_0, a_{12}(g_0) \neq 0 \qquad \begin{cases} \forall\, g, a_{11}(a) = a_{22}(g) = 1 \ . & (\alpha) \\ \exists\, g_1, a_{11}(g_1) \neq 1 \ . & (\gamma 1) \\ \exists\, g_2, a_{22}(g_2) \neq 1 \ . & (\gamma 2) \end{cases}$$

$$\forall\, g, a_{12}(g) = 0\ , \qquad \exists\, g_2, a_{22}(g_2) \neq 1\ . \tag{β}$$

Lemma 2.1. *Case* (γ1) *is reduced to case* (β).

Proof. Take two elements g and h of G. Their commutator is given by

$$ghg^{-1}h^{-1} = \begin{pmatrix} 1, & a_{12}(h)(a_{11}(g) - 1) - a_{12}(g)(a_{11}(h) - 1), & * \\ 0, & 1 & , 0 \\ 0, & 0 & , 1 \end{pmatrix}.$$

Since $ghg^{-1}h^{-1}$ has no fixed points on C^2, we have

$$a_{12}(h)(a_{11}(g) - 1) - a_{12}(g)(a_{11}(h) - 1) = 0\ .$$

By assumption, there exist g_0 and g_1 with $a_{12}(g_0) \neq 0$ and $a_{11}(g_1) \neq 1$. Thus there exists a nonzero complex number λ such that $a_{12}(g) - \lambda(a_{11}(g) - 1) = 0$ for any g. If we introduce new coordinates (z_1', z_2') of C^2 by $\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, we see that case ($\gamma$1) is reduced to case ($\beta$). q.e.d.

In view of this lemma, we may assume $a_{11}(g) = 1$ for any $g \in G$ in any case (cf. Remark 1 at the end of § 1).

Lemma 2.2. *Case* (γ2) *is reduced to case* (β) *if there exists a complex number* λ *such that, for any* g,

$$a_{12}(g) + \lambda(a_{22}(g) - 1) = 0\ . \tag{5}$$

Proof. This can be done by applying the coordinate transformation $\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$.

Thus in case (γ2), we assume that

($*$) for any complex number λ, there exists an element g such that (5) does not hold.

Lemma 2.3. *In cases* (β) *and* (γ2), *the center* C *of* G *is given by*

$$C = \{g \in G \,|\, A(g) = I,\ b_2(g) = 0\}\ .$$

Proof. It is clear that an element g with $A(g) = I$ and $b_2(g) = 0$ is in C. Take an element g in C. For any element h of G, we have

$$\begin{aligned} &a_{12}(g)(a_{22}(h) - 1) - a_{12}(h)(a_{22}(g) - 1) = 0\ , \\ &(a_{22}(g) - 1)b_2(h) - (a_{22}(h) - 1)b_2(g) = 0\ , \\ &a_{12}(g)b_2(h) - a_{12}(h)b_2(g) = 0\ . \end{aligned} \tag{6}$$

We claim that $a_{22}(g) = 1$. In case (γ2), this is trivial in view of the assumption ($*$). In case (β), this is proved as follows. Assume $a_{22}(g) \neq 1$ and put $\lambda =$

$b_2(g)/(a_{22}(g) - 1)$. Introducing new coordinates of C^2 by $(z'_1, z'_2) = (z_1, z_2 + \lambda)$, we see that we can assume $b_2(h) = 0$ for any $h \in G$. Then G acts on the line $z'_2 = 0$ effectively, and the action is properly discontinuous. Hence we have $G \subset Z \oplus Z$, where Z denotes the ring of integers. Thus C^2/G cannot be compact (see the following proposition).

Finally, the existence of an element g_2 with $a_{22}(g_2) \neq 1$ implies $a_{12}(g) = b_2(g) = 0$.

Proposition 2. *Let F be a free abelian group acting on C^2 freely and properly discontinuously. If the rank of F is less than or equal to* 3, *then the quotient space C^2/F cannot be compact.*

Proof. As C^2 is an acyclic space, we have an isomorphism

$$H^n(C^2/F, Z) \xrightarrow{\sim} H^n(F, Z) , \qquad n = 0, 1, \cdots ,$$

where $H^n(F, Z)$ denotes the n-th cohomology group of F with coefficients in the trivial F-module Z. Let r be the rank of F. Then the cohomology groups $H^n(F, Z)$ are isomorphic to the cohomology groups of the real r-torus T^r. If C^2/F were compact, we would have $H^4(C^2/F, Z) = Z$. On the other hand, $H^4(F, Z) = H^4(T^r, Z) = 0$ since $r \leq 3$, which is a contradiction.

Lemma 2.4. *For any* $g \in G$, $a_{22}(g)$ *is a root of unity.*

Proof. First we prove that $|a_{22}(g)| = 1$ for every $g \in G$. Assume there exists an element g with $|a_{22}(g)| \neq 1$. By taking its inverse, if necessary, we may assume $|a_{22}(g)| < 1$. The n-th power of g is given by

$$g^n = \begin{pmatrix} 1 & \alpha_n & \beta_n \\ 0 & \gamma_n & \delta_n \\ 0 & 0 & 1 \end{pmatrix} ,$$

where

$$\alpha_n = \frac{a_{22}(g)^n - 1}{a_{22}(g) - 1} \cdot a_{12}(g) ,$$

$$\beta_n = nb_1(g) + \left(\frac{a_{22}(g)^n - 1}{(a_{22}(g) - 1)^2} - \frac{n}{a_{22}(g) - 1} \right) \cdot a_{12}(g) b_2(g) ,$$

$$\gamma_n = a_{22}(g)^n , \qquad \delta_n = \frac{a_{22}(g)^n - 1}{a_{22}(g) - 1} \cdot b_2(g) .$$

Put $\alpha = -a_{12}(g)/(a_{22}(g) - 1)$ and $\delta = -b_2(g)/(a_{22}(g) - 1)$. Then $\alpha_n \to \alpha$, $\gamma_n \to 0$, and $\delta_n \to \delta$ as $n \to +\infty$.

For any element h, we have

$$A(g^n h g^{-n}) = \begin{pmatrix} 1, & \gamma_n^{-1}(\alpha_n(a_{22}(h) - 1) + a_{12}(h)) \\ 0, & a_{22}(h) \end{pmatrix} ,$$

$$b(g^n h g^{-n}) = \begin{pmatrix} -\gamma_n^{-1}\delta_n(\alpha_n(a_{22}(h) - 1) + a_{12}(h)) + \alpha_n b_2(h) + b_1(h)) \\ \gamma_n b_2(h) - \delta_n(a_{22}(h) - 1) \end{pmatrix} .$$

Thus

$$g^n h g^{-n}\begin{pmatrix} z_1 \\ \delta \end{pmatrix} \to \begin{pmatrix} z_1 + \varepsilon(h) \\ \delta \end{pmatrix} , \qquad \text{as } n \to +\infty ,$$

where $\varepsilon(h) = \delta(\alpha(a_{22}(h) - 1) + a_{12}(h)) + \alpha b_2(h) + b_1(h)$.

Choose positive numbers c_1 and c_2 so that $|\varepsilon(h)| < c_1$. Consider the compact set K in C^2 defined by

$$K = \{(z_1, z_2) \,|\, |z_1| \le c_1 \quad \text{and} \quad |z_2 - \delta| \le c_2\} .$$

Since $g^n h g^{-n}(0, \delta)$ converges to the point $(\varepsilon(h), \delta)$ as $n \to +\infty$, $g^n h g^{-n}(0, \delta) \in K$ for any large n. Since the action of G on C^2 is properly discontinuous, some positive power of g should commute with h. Moreover, since G is finitely generated, some power g^N of g should be contained in the center C. Hence we have $a_{22}(g)^N = 1$ by Lemma 2.3, which is a contradiction. Thus we have proved $|a_{22}(g)| = 1$ for any $g \in G$.

Since each entry of the matrix $g^n h g^{-n}$ remains bounded as n tends to infinity, by a similar argument as above we can prove $a_{22}(g)^n = 1$ for a positive integer n. q.e.d.

Let G^* be the normal subgroup of G defined by $G^* = \{g \in G \,|\, a_{22}(g) = 1\}$. Since G is finitely generated, Lemma 2.4 implies G/G^* is finite. Moreover, G^* is a nilpotent group. Thus we have

Corollary. *The group G^* is a nilpotent subgroup of G of finite index.*

Lemma 2.5. *The first Betti number b_1 of the quotient space $S = C^2/G$ is given by*

$$b_1 = \begin{cases} 4 \text{ or } 3 , & \text{in case } (\alpha) , \\ 2, & \text{in case } (\beta) , \\ 2 \text{ or } 1 , & \text{in case } (\gamma 2) . \end{cases}$$

Proof. First we note that $\partial/\partial z_1$ is a nonvanishing G-invariant holomorphic vector field on C^2. Hence by a theorem of Bott [1], we have $c_1^2 = c_2 = 0$ in each case. Next we find the number of lineraly independent G-invariant holomorphic forms on C^2. The pullbacks g^*dz_i of dz_i, $i = 1, 2$, by an element g of G are given by $g^*dz_1 = dz_1 + a_{12}(g)dz_2$ and $g^*dz_2 = a_{22}(g)dz_2$. Thus we have $g^*(dz_1 \wedge dz_2) = a_{22}(g)dz_1 \wedge dz_2$.

Case (α). Since $a_{22}(g) = 1$ for every g in G, a holomorphic 2-form $f(z)dz_1 \wedge dz_2$ on C^2 is G-invariant if and only if f is G-invariant. If f is G-invariant, f is considered to be a holomorphic function on the quotient space C^2/G, which is compact. Thus f is a constant, so that the geometric genus p_g

of $S = \boldsymbol{C}^2/G$, which is equal to the number of linearly independent holomorphic 2-forms on S, is equal to 1. Since the Noether formula (3) implies $q = 2$, by (4) we have $b_1 = 4$ or 3.

Case (β). Since $a_{12}(g) = 0$ for every g in G, the subgroup $G^* = \{g \in G \mid a_{22}(g) = 1\}$ of G consists of translations. Moreover, by the corollary to Lemma 2.4, the quotient space $T = \boldsymbol{C}^2/G^*$ is a finite unramified covering of S, which is compact. Thus T is a complex torus. Any G^*-invariant holomorphic 2-form on $\boldsymbol{C}^2$ is of the form $c dz_1 \wedge dz_2$ with c a constant. Since we have an element g_2 in G with $a_{22}(g_2) \neq 1$, no holomorphic 2-form on $\boldsymbol{C}^2$ is G-invariant, so that $p_g = 0$. Moreover, any G^*-invariant holomorpnic 1-form on $\boldsymbol{C}^2$ is of the form $a dz_1 + b dz_2$ with a and b constants. Since $g_2^*(a dz_1 + b dz_2) = a dz_1 + b a_{22}(g_2) dz_2$, the scalar multiples of dz_1 are the only G-invariant holomorphic 1-forms on $\boldsymbol{C}^2$, which means $h^{1,0} = 1$. Therefore (3) and (4) imply $b_1 = 2$.

Case ($\gamma 2$). Consider $G^* = \{g \in G \mid a_{22}(g) = 1\}$. The quotient space $S^* = \boldsymbol{C}^2/G^*$ is a finite unramified covering of $S = \boldsymbol{C}^2/G$ and is a surface of case (α). As is seen in case (α), any G^*-invariant holomorphic 2-form on $\boldsymbol{C}^2$ is of the form $c dz_1 \wedge dz_2$ with c a constant. Since there is an element g_2 in G with $a_{22}(g_2) \neq 1$, we have $p_g = 0$, and therefore $q = 1$ by (3). Hence (4) implies $b_1 = 2$ or 1.

Theorem 1. *If $b_1 = 4$, then $S = \boldsymbol{C}^2/G$ is a complex torus.*

Proof. If G consists of only translations, the theorem is obvious. Thus we consider case (α) with $b_1 = 4$. From the assumption, we have $h^{1,0} = 2$. Let φ and ψ denote linearly independent G-invariant holomorphic 1-forms on $\boldsymbol{C}^2$, and write $\varphi = \varphi_1(z) dz_1 + \varphi_2(z) dz_2$ and $\psi = \psi_1(z) dz_1 + \psi_2(z) dz_2$. Conditions for φ and ψ to be G-invariant are given by

$$(8) \qquad \varphi_1(gz) = \varphi_1(z)\,, \qquad \psi_1(gz) = \psi_1(z)\,,$$

$$(9) \qquad \varphi_2(gz) = \varphi_2(z) - \varphi_1(gz) a_{12}(g)\,, \qquad \psi_2(gz) = \psi_2(z) - \psi_1(gz) a_{12}(g)\,,$$

for any $g \in G$. From (8), we have $\varphi_1(z) = \varphi_1$ and $\psi_1(z) = \psi_1$ are constants, so that (9) reduces to

$$(10) \qquad \varphi_2(gz) = \varphi_2(z) - \varphi_1 a_{12}(g)\,, \qquad \psi_2(gz) = \psi_2(z) - \psi_1 a_{12}(g)\,.$$

Since $\varphi \wedge \psi = (\varphi_1 \psi_2(z) - \psi_1 \varphi_2(z)) dz_1 \wedge dz_2$ is a G-invariant holomorphic 2-form on $\boldsymbol{C}^2$, $\varphi_1 \psi_2(z) - \psi_1 \varphi_2(z) = c$ is a constant. We have $\psi_1 \varphi - \varphi_1 \psi = (\psi_1 \varphi_2(z) - \varphi_1 \psi_2(z)) dz_2 = c dz_2$. If $c = 0$, we would have $\varphi_1 = \psi_1 = 0$, and then $\varphi_2(z)$ and $\psi_2(z)$ would be constant by (10), which is a contradiction. Hence $c \neq 0$. Consider the Albanese variety A of $S = \boldsymbol{C}^2/G$. Since A is a complex torus whose lattice Γ is generated by the periods of φ and ψ on four free generators for $H_1(S, \boldsymbol{Z})$, we have a canonical mapping $\Phi: S \rightarrow A$ dfined by $\Phi(z) = \left(\int^z \varphi, \int^z \psi\right) \pmod{\Gamma}$ for $z \in S$. The Jacobian of Φ is given by $\varphi_1 \psi_2(z) -$

$\psi_1\varphi_2(z) = c$, so that Φ is an unramified covering mapping. Hence $S = C^2/G$ is a complex torus.

Example. Consider the group G generated by four elements:

$$g_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$g_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_4 = \begin{pmatrix} 1 & i & 0 \\ 0 & 1 & i \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, by a suitable coordinate transformation φ, say $\varphi(z_1, z_2) = (z_1 - \frac{1}{2}z_2^2, z_2)$, G is transformed into a group of translations. Moreover, $\varphi g_i \varphi^{-1}, i = 1, \cdots, 4$, are linearly independent over R. Thus C^2/G is a complex torus.

Theorem 2. *If $b_1 = 3$, then $S = C^2/G$ is a fiber bundle of elliptic curves over an elliptic curve.*

Proof. Take two elements g and h of G. Their commutator is given by

$$ghg^{-1}h^{-1} = \begin{pmatrix} 1 & 0 & a_{12}(g)b_2(h) - a_{12}(h)b_2(g) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $G^{(1)} = [G, G]$ be the commutator group of G. Then we have the following exact sequence:

$$(11) \qquad 1 \longrightarrow G^{(1)} \longrightarrow G \xrightarrow{\varphi} H_1(S, Z) \longrightarrow 0,$$

where $S = C^2/G$. Note that for any element g of $G^{(1)}$, $A(g) = I$ and $b_2(g) = 0$ and that $G^{(1)}$ is commutative.

Let U_1 and U_2 denote the first and second factors of the product C^2. Then $G^{(1)}$ acts on U_1 effectively as a group of translations. Moreover, since the action of $G^{(1)}$ on C^2 is "parallel" to the z_1-axis, we see that $G^{(1)}$ acts on U_1 properly discontinuously. Hence $G^{(1)}$ is a subgroup of $Z \oplus Z$.

(i) First we assume $G^{(1)} = 0$. Then we have $G = H_1(S, Z)$. The free part F of G is a free abelian group of rank 3. The quotient space C^2/F, being a finite covering of C^2/G, is compact, which is a contradiction (see Proposition 2).

(ii) Secondly we assume $G^{(1)} = Z$. Let h_0 be a generator of the infinite cyclic group $G^{(1)}$, γ_1, γ_2, and γ_3 generators of the free part of $H_1(S, Z)$, and $\tau_1, \cdots, \tau_t$ generators of the torsion part of $H_1(S, Z)$. Choose elements $h_i, i = 1, 2, 3$, and $k_j = 1, \cdots, t$, of G so that $\varphi(h_i) = \gamma_i$ and $\varphi(k_j) = \tau_j$. Then G is generated by $h_0, h_1, h_2, h_3, k_1, \cdots, k_t$.

Lemma 2.6. *Let g be an element of G. If $\varphi(g)$ is a torsion element, then $b_2(g) = 0$.*

Proof. The condition implies that some positive power g^n of g is contained in $G^{(1)}$. Hence we have $0 = b_2(g^n) = nb_2(g)$.

Lemma 2.7. *For any element g of G there exist integers n_i, $i = 1, 2, 3$, such that*

$$b_2(g) = \sum_{i=1}^{3} n_i b_2(h_i) .$$

Proof. Since $b_2(gh) = b_2(g) + b_2(h)$ for any two elements g and h of G, the lemma follows from Lemma 2.6. q.e.d.

Consider the natural action of G on the second factor U_2 of C^2, which is given by $g: z_2 \mapsto z_2 + b_2(g)$ for $g \in G$, and let G_1 denote the kernel of the action. Since G is free on C^2, if $b_2(g) = 0$ then $a_{12}(g) = 0$. Thus an element g of G is contained in G_1 if and only if $b_2(g) = a_{12}(g) = 0$.

Lemma 2.8. *G/G_1 acts properly discontinuously on U_2.*

Proof. Since the commutator group $G^{(1)}$ is generated by h_0, there exists an integer n_{ij} for each pair (h_i, h_j), $i, j = 1, 2, 3$, such that

$$a_{12}(h_i)b_2(h_j) - a_{12}(h_j)b_2(h_i) = n_{ij}b_1(g) . \tag{12}$$

From (12), we get

$$n_{12}b_2(h_3) + n_{23}b_2(h_1) + n_{31}b_2(h_2) = 0 . \tag{13}$$

Assume $n_{12} = n_{23} = n_{31} = 0$. Then G should be commutative, which is a contradiction. Therefore at least one of of n_{12}, n_{23} or n_{31} is nonzero, and we get a nontrivial linear relation (13) among $b_2(h_i)$ with integer coefficients. This fact, together with Lemma 2.7, implies the lemma. q.e.d.

Now we have $C^2/G = (C^2/G_1)/(G/G_1)$, where $C^2/G_1 = (U_1/G_1) \times U_2$. Since G/G_1 acts properly discontinuously on U_2, C^2/G is a fiber bundle over the one-dimensional complex manifold $U_2/(G/G_1)$ with fiber U_1/G_1. Hence U_1/G_1 and $U_2/(G/G_1)$ are compact. Moreover, since G_1 and G/G_1 act on U_1 and U_2 respectively as groups of translations, U_1/G_1 and $U_2/(G/G_1)$ are elliptic curves.

(iii) Finally, we assume $G^{(1)} = Z \oplus Z$. We have $C^2/G = (C^2/G^{(1)})/(G/G^{(1)})$. Since $G^{(1)} = Z \oplus Z$ acts trivially on U_2, $C^2/G^{(1)} = (U_1/G^{(1)}) \times U_2$ is the product of the elliptic curve $U_1/G^{(1)}$ and U_2. Let Γ denote the kernel of the natural action of $G/G^{(1)}$ on U_2. Since $G/G^{(1)}$ acts properly discontinuously on $(U_1/G^{(1)}) \times U_2$, whose first factor is compact, $(G/G^{(1)})/\Gamma$ acts properly discontinuously on U_2. Now as in case (ii), take elements h_1, h_2, and h_3 of G such that $\varphi(h_1)$, $\varphi(h_2)$, and $\varphi(h_3)$ generate the free part of $H_1(S, Z)$. Then $G^{(1)}$ is generated by $h_i h_j h_i^{-1} h_j^{-1} = \begin{pmatrix} 1 & 0 & \omega_{ij} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $i, j = 1, 2, 3$, where $\omega_{ij} = a_{12}(h_i)$

$\cdot b_2(h_j) - a_{12}(h_j)b_2(h_i)$. On the other hand, since $(G/G^{(1)})/\Gamma$ acts on U_2 properly discontinuously, we have a nontrivial relation:

$$(14) \qquad \sum_{i=1}^{3} n_i b_2(h_i) = 0 ,$$

where $n_i, i = 1, 2, 3$, are integers with $(n_1, n_2, n_3) \neq (0, 0, 0)$. Note that (14) implies $\sum_{i=1}^{3} n_i a_{12}(h_i) = 0$. Thus we have the following equalities:

$$(15) \qquad n_1\omega_{12} - n_3\omega_{23} = 0 , \quad n_2\omega_{23} - n_1\omega_{31} = 0 , \quad n_3\omega_{31} - n_2\omega_{12} = 0 .$$

Since $(n_1, n_2, n_3) \neq (0, 0, 0)$, (15) implies that rank $G^{(1)} \leq 1$, which is a contradiction. This completes the proof of Theorem 2.

A comact complex surface S is said to be an *elliptic surface* if there exists a holomorphic mapping Ψ of S onto a nonsingular curve Δ such that the inverse image $\Psi^{-1}(u)$ of any general point $u \in \Delta$ is an elliptic curve. For the theory of elliptic surfaces we refer to Kodaira [2]. Let $\Psi : S \to \Delta$ be a (holomorphic) fiber bundle of elliptic curves over an elliptic curve Δ, and assume that the first Betti number b_1 of S is equal to 3. Then the functional invariant of S is constant and the homological invariant of S is trivial [2, II, § 7], [4, p. 470]. Thus the basic member B is trivial; $B = C \times \Delta$, where C denotes the typical fiber of $S \to \Delta$. Hence the canonical bundle K of S is simply given by $K = \Psi^*(\kappa)$, where κ denotes the canonical bundle of Δ, [3, I, Theorem 12]. Since κ is trivial, so is K. Therefore, by Theorem 19 in [3, I], S is biholomorphic to a quotient space of C^2 by an affine transformation group G, which is generated by four elements g_1, g_2, g_3 and g_4 with a fundamental relation $g_3g_4 = g_2^m g_4 g_3$, where m is a positive integer.

The fiber bundles over an elliptic curve Δ with fiber an elliptic curve C whose homological invariants are trivial are described as follows. First we express C as a quotient group: $C = \mathbf{C}/\Gamma$, where Γ denotes a discrete subgroup of $\mathbf{C}$ generated by 1 and ω, Im $\omega > 0$, and for any $\zeta \in \mathbf{C}$ we denote by $[\zeta]$ the corresponding element of $C = \mathbf{C}/\Gamma$. We have the following sheaf exact sequence over Δ

$$0 \to \Gamma \to \Omega \to \Omega(C) \to 0 ,$$

where Ω and $\Omega(C)$ denote the sheaves of germs of holomorphic functions and holomorphic mappings into C respectively. We have the corresponding cohomology exact sequence

$$\cdots \longrightarrow H^1(\Delta, \Omega) \xrightarrow{h} H^1(\Delta, \Omega(C)) \xrightarrow{c} H^2(\Delta, \Gamma) \longrightarrow 0 .$$

Any fiber bundle S over Δ with fiber C whose homological invariant is trivial is written in the form $(C \times \Delta)^\eta$, for some $\eta \in H^1(\Delta, \Omega(C))$, [2, II, Theorem 10.1], [4, p. 470]. Moreover, $S = (C \times \Delta)^{\eta}$ is a deformation of $S' = (C \times \Delta)^{\eta'}$ if the characteristic classes are the same; $c(\eta) = c(\eta')$, [2, III, Theorem 11.4]. The first Betti number b_1 of $S = (C \times \Delta)^{\eta}$ is 4 or 3 according as $c(\eta) = 0$ or $c(\eta) \neq 0$, [2, III, Theorem 11.9]. For each element $\gamma \in H^2(\Delta, \Gamma) = \Gamma \xrightarrow{\sim} \boldsymbol{Z} \oplus \boldsymbol{Z}$, we can construct a bundle S_γ with characteristic class γ as follows (cf. [3, II, p. 684]). Take a point p on Δ, and let z be a local coordinate with center p and $U = \{z \mid |z| < \varepsilon\}$ a small disk around p. S_γ is defined by $S_\gamma = U \times C \cup (\Delta - p) \times C$, where $(z, [\zeta]) \in U \times C$ and $(z, [\zeta']) \in (\Delta - p) \times C$ are identified if and only if $[\zeta'] = [\zeta + (\gamma/2\pi i) \log z]$. Thus any fiber bundle S over Δ with fiber C with $b_1 = 3$ is a deformation of S_γ for some $\gamma \in \Gamma, \gamma \neq 0$. If $\gamma = h + k\omega$, h and $k \in \boldsymbol{Z}$, we have $H_1(S, \boldsymbol{Z}) = \boldsymbol{Z} \oplus \boldsymbol{Z} \oplus \boldsymbol{Z} \oplus \boldsymbol{Z}_m$, where $m = (h, k)$.

A *hyperelliptic surface* is a fiber bundle of elliptic curves over an elliptic curve with $b_1 = 2$. For the classification of hyperelliptic surfaces we refer to [4].

Theorem 3. *If $b_1 = 2$, then $S = \boldsymbol{C}^2/G$ is a hyperelliptic surface.*

Proof. By the characterization (D) in [4, p. 476] of hyperelliptic surfaces.

Remark. S is algebraic as $p_g = 0$ and b_1 is even [3, I. Theorem 10]. We can also prove (A), (B) or (C) in [4, p. 476] directly.

Theorem 4. *If $b_1 = 1$, then $S = \boldsymbol{C}^2/G$ has the following structure:*

(**)
(1) *S is an elliptic surface over the projective line $\boldsymbol{P}^1$,*
(2) *S has no singular fibers over the base curve $\boldsymbol{P}^1$ other than multiple fibers of the form $m\Theta$, where Θ is a nonsingular elliptic curve and m the multiplicity (type ${}_mI_0$ in [2]),*
(3) *the multiplicities m_i of the multiple fibers $m_i\Theta_i$, $i = 1, \cdots, r$, of S satisfy the equality $\sum_{i=1}^{r} (1 - 1/m_i) = 2$.*

Proof. Consider the normal subgroup $G^* = \{g \in G \mid a_{22}(g) = 1\}$ of G. By the corollary to Lemma 2, 4, G/G^* is finite. We have $\boldsymbol{C}^2/G = (\boldsymbol{C}^2/G^*)/(G/G^*)$. The surface $S^* = \boldsymbol{C}^2/G^*$ is compact and is a surface of case (α). Thus the first Betti number b_1^* of S^* is either 3 or 4. If b_1^* were equal to 4, then by Theorem 1, S^* would be a complex torus, which is a Kähler manifold. Thus the finite quotient space $S = S^*/(G/G^*)$ is also a Kähler manifold, which is a contradiction since the first Betti number of S is odd. Hence $b_1^* = 3$. By Theorem 2, S^* is a fiber bundle of elliptic curves over an elliptic curve Δ^*. Let G_1^* be the kernel of the natural action of G^* on the second factor U_2 of $\boldsymbol{C}^2$. Then as is seen in the proof of Theorem 2, the base curve Δ^* is the quotient space $U_2/(G^*/G_1^*)$, and the typical fiber of the fiber bundle $S^* \to \Delta^*$ is the quotient space U_1/G_1, where U_1 denotes the first factor of $\boldsymbol{C}^2$. For $z = (z_1, z_2) \in \boldsymbol{C}^2$ and $g \in G$, the second component of gz is given by $a_{22}(g)z_2 + b_2(g)$ and depends only on z_2. Hence G acts naturally on U_2, which means that the action of G/G^* on the fiber bundle $S^* \to \Delta^*$ is fiber preserving. We have the following commutative diagram:

$$\begin{array}{ccc} S^* & \xrightarrow{\Pi} & S = S^*/(G/G^*) \\ {\scriptstyle \Psi^*}\downarrow & & \downarrow{\scriptstyle \Psi} \\ \Delta^* & \xrightarrow{\pi} & \Delta = \Delta^*/(G/G^*) \,. \end{array}$$

Since each element (different from the identity) of the group G/G^* is represented by an element g of G with $a_{22}(g) \neq 1$, the action of G/G^* on Δ^* is effective. Moreover, the action is properly discontinuous since the projection map Ψ^* is proper. Thus G/G^* is a finite cyclic group acting on the elliptic curve Δ^* with fixed points, and the quotient space $\Delta^*/(G/G^*)$ is biholomorphic to the projective line. For $z_1 \in U_1$, $z_2 \in U_2$, and $g \in G$, we denote by $[z_1],[z_2]$ and $[g]$ the corresponding points in U_1/G_1^*, $U_2/(G/G_1^*)$ and G/G^*, respectively. If a point p on Δ^* is not a fixed point of G/G^*, the fiber $\Psi^{-1}(\pi(p))$ is biholomorphic to the elliptic curve U_1/G_1^*. Consider a fixed point $p = [z_2^o]$ of G/G^* on Δ^*, and let $[g^o]$ be a generator of the isotropy subgroup $(G/G^*)_p$ of G/G^* at p and m the order of $[g^o]$. The group $(G/G^*)_p$ acts on the fiber $\Psi^{*-1}(p) = U_1/G_1^*$ by $[z_1] \mapsto [z_1 + a_{12}(g^o)z_2^o + b_1(g^o)]$. This action is effective since otherwise some power of $[g^o]$ would have fixed points on S^*. Thus we get a multiple fiber $m\Theta$, $\Theta \xrightarrow{\sim} (U_1/G_1^*)/(G/G^*)_p$, of type ${}_mI_0$ in the elliptic surface $\Psi: S \to \Delta$ over the point $\pi(p)$. Moreover, the mapping π is a ramified covering map with ramification exponent m at p. Hence the Hurwitz formula impiles the equality in (3).

Remarks. 1. For the structure of a neighborhood of a multiple fiber of type ${}_mI_0$, see [3, II, p. 685].

2. As is seen in the proof of Theorem 4, G/G^* is a finite cyclic group acting effectively on an elliptic curve with fixed points. Thus the order of G/G^* is 2, 3, 4 or 6.

3. Let S be a complex surface with the property (**). Then the first Betti number b_1 of S is either 2 or 1, [3, II, p. 686]. Moreover, S admits a fiber bundle S^* of elliptic curves over an elliptic curve as an unramified covering [3, II, p. 690], [4, p. 476]. If $b_1 = 2$, then S is a hyperelliptic surface [4, p. 476(C)], and S^* is a complex torus. If $b_1 = 1$, then the first Betti number b_1^* of S^* is 3, and S^* is a quotient space of $\boldsymbol{C}^2$ by an affine transformation group (see p. 239). The canonical bundle of S^* is trivial. Thus in both cases, S is a quotient space of $\boldsymbol{C}^2$ by an affine transformation group [3, II, § 11, especially Theorem 39].

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