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SOME REMARKS ON STABILITY OF FOLIATIONS

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A foliation $\mathscr{F}$ of a manifold V is said to be C^s-structurally stable (or simply C^s-stable) if there exists a neighborhood U of $\mathscr{F}$, in the C^s-topology, such that for each foliation $\mathscr{F}$ in U, there exists a homeomorphism h of V which sends the leaves of $\mathscr{F}'$ to the leaves of $\mathscr{F}$. We also require that h depend continuously on $\mathscr{F}'$. Stability of vector fields, or one-dimensional foliations, is an extensively studied subject with many applications. For an excellent reference to this subject we recommend [8], [4]. It is very natural to consider vector fields with singularities, and more generally one can study stability properties of foliations with singularities or Haefliger structures. This seems to us an important and difficult subject. We shall make some remarks about stability of Haefliger structures, but for the moment we consider nonsingular foliations.

In this paper we shall study C^1 stability of C^∞ foliations of 3-manifolds. We shall see that many 3-manifolds (S^3 for example) admit no structurally stable foliations whatsoever. We classify the stable foliations of $S^2 \times S^1$; they are relatively simple in structure, and we construct foliations of T^3 which are stable. In § 2, we discuss stability of the intrinsic components of a foliation. We prove there is an open dense set of foliations on any 3-manifold, for which the intrinsic components are stable. This encourages the study of a stratification of the space of foliations. Finally we shall make some remarks on Haefliger structures.

1. Instability on some 3-manifolds

In this section we suppose V is a closed oriented 3-manifold, and $\mathscr{F}$ a transversally oriented foliation of V of codimension one and class C^∞.

Theorem 1.1 (*H. Rosenberg and D. Weil*). *Suppose $(V\mathscr{F})$ satisfies condition*

i) *there exists a closed transversal curve to $\mathscr{F}$ which is null homotopic in V, or*

ii) $\pi_2(V) \neq (0)$ *and* $V \neq S^2 \times S^1$.

Then $\mathscr{F}$ is C^1-instable, i.e., one can approximate $\mathscr{F}$, in the C^1-topology, by foliations nonconjugate to $\mathscr{F}$.

Remark. Conditions i) and ii) are the hypotheses necessary to apply

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Novikov's theorem on the existence of Reeb components. They are always satisfied if, for example, $\pi_1(V)$ is finite or V is a nontrivial connected sum of two 3-manifolds. Thus S^3 admits no stable foliation.

We need some preliminaries before proving Theorem 1.1. Let L be a compact leaf of a foliation with abelian fundamental group. We say that L has k-th order contact with $\mathscr{F}$ if one of the elements of the holonomy group of L can be represented by a local diffeomorphism of R; leaving fixed O, which has k-th order contact with the identity at O. If $k = 1$, we say L is hyperbolic. It is well known that if one of the elements of the holonomy group has k-th order contact with the identity, then the other elements also have k-th order contact or are equal to the identity (here we use $\pi_1(L)$ abelian) [3]. Hence the notion of contact depends only upon L.

Let T be the boundary torus of a Reeb component τ of $\mathscr{F}$, and suppose T has a finite contact k with $\mathscr{F}$. Let α and β be generators of $\pi_1(T)$ such that α is homotopic to zero in τ. Then the holonomy element associated to β is of order k, and the element associated to α is the identity. Hence we can choose a neighborhood U of τ, diffeomorphic to $S^1 \times D^2$, whose boundary is transverse to $\mathscr{F}$, inducing a foliation by circles, each null homotopic in U. Hence we can change $\mathscr{F}$ by replacing the foliation of $\mathscr{F}$ by disks D^2 whose boundary is the induced foliation on ∂U by $\mathscr{F}$. This is the reverse of modification along a simple closed transversal curve. We shall call this process erasing the Reeb component.

Proof of Theorem 1.1. First we shall prove that a foliated 3-manifold V satisfying conditions i) or ii) of Theorem 1.1 must have a flat torus leaf, i.e., a torus leaf having ∞ contact with the identity. We know that every foliation of V has Reeb components. We can assume all the Reeb components of $\mathscr{F}$ are not flat. Let R_1 be a Reeb component of $\mathscr{F}$, and $\mathscr{F}_1$ the foliation obtained by erasing R_1. If $\mathscr{F}_1$ has one fewer Reeb component than $\mathscr{F}$, then we erase another Reeb component of $\mathscr{F}$. Since there is a finite number of Reeb components in $\mathscr{F}$, we can assume $\mathscr{F}_1$ has the same number of Reeb components as $\mathscr{F}$. This means that by erasing R_1 we create a Reeb component R_2 which contains R_1 in its interior, and the boundary leaf of R_2 is a leaf of $\mathscr{F}$. If R_2 is not a flat leaf, then we can erase it (to erase finite contact suffices). Let $\mathscr{F}_2$ be the foliation so obtained. We can assume $\mathscr{F}_2$ has the same number of Reeb components as $\mathscr{F}_1$, hence we have created a Reeb component R_3 whose boundary leaf is in $\mathscr{F}$. If this process continues indefinitely, then $\mathscr{F}$ has an infinite number of torus leaves $T_i = \partial R_i$. The limit of the T_i is a torus leaf of $\mathscr{F}$ which must be flat.

We shall call a submanifold A of V a "band" of the foliation $\mathscr{F}$ if A is homeomorphic to $T^2 \times I$, and $\mathscr{F}$ induces a foliation of A equivalent to the foliation $T^2 \times \{t\}$ of $T^2 \times I$. We remark that a stable foliation can not contain any bands since the foliation $T^2 \times \{t\}$ of $T^2 \times I$ can be C^1-approximated, rel $\partial(T^2 \times I)$ by a foliation with a finite number of compact leaves.

To complete the proof of Theorem 1.1 it suffices to show a flat leaf can be thickened. More generally, we will prove

Lemma a. *Let L be a flat compact leaf of a codimension one foliation $\mathscr{F}$ of a manifold V^n. Then L can be thickened, i.e., $\mathscr{F}$ can be C^1-approximated by a foliation $\mathscr{F}'$ such that $\mathscr{F}' = \mathscr{F}$ outside of a tubular neighborhood T of L, and $\mathscr{F}'$ has a band of compact leaves in T.*

Proof. Let $f_1, \cdots, f_k$ be local diffeomorphisms of R at O which generate the holonomy group of L. Each f_i is flat at O; $f_i(\mathrm{O}) = \mathrm{O}$. Let a be a (small) positive real number, and $h: R \to R$ a piecewise linear map such that $h(x) = \mathrm{O}$ for $x \in [-a, a]$, h is linearly increasing on $[a, b]$ for some small $b > a$, and $h(x) = x$ for x large. For $x < -a$ we define $h(x) = -h(-x)$. Let $g_i: R \to R$ be the maps:

$$
\begin{aligned}
g_i(x) &= x && \text{if } x \in [-a, +a] \\
&= h^{-1} f_i h(x) && \text{if } x > a \text{ or } x < -a\,.
\end{aligned}
$$

Then g_i is a C^∞ diffeomorphism of R which is C^1-close to f_i, $g_i = 1$ in $[-a, +a]$ and $g_i(x) = f_i(x)$ for x large. Hence $g_1, \cdots, g_k$ provide a representation of $\pi_1(L)$ which gives the desired foliation $\mathscr{F}'$.

Remark. A leaf L, which is a 2-torus such that the holonomy group is generated by one diffeomorphism f having k-th $(k > 1)$ order contact, can also be thickened.

In contrast to Theorem 1.1, we next give some examples of stable foliations; here also, the stability is related to the topology of the manifold.

Theorem 1.2. *A foliation $\mathscr{F}$ of $S^2 \times S^1$ is C^1-stable if and only if $\mathscr{F}$ is isotopic to a foliation of $S^2 \times S^1$ obtained from the product foliation $\{S^2 \times \{y\} / y \in S^1\}$ by a finite number of hyperbolic modifications.*

Remark. It follows from the Reeb stability theorems that the product foliation is stable. The reader will enjoy proving this fact directly.

Before proving Theorem 1.2 we need some preliminaries concerning Reeb components.

Lemma 1.3. *Let τ be a hyperbolic Reeb component of $(V, \mathscr{F})$. Then there are a neighborhood U of τ in V and a neighborhood N of $\mathscr{F}$, in the C^1-topology, such that for each $\mathscr{F}_1 \in N$ there is a homeomorphism h of U conjugating $\mathscr{F}/U$ with $\mathscr{F}_1/U$. Moreover, h depends continuously on $\mathscr{F}_1$.*

Proof. We define $U = U_1 \cup U_2$ where:

i) U_1 is open in V satisfying that $\bar{U}_1$ is contained in the interior of τ and is diffeomorphic to $S^1 \times D^2$, and $\mathscr{F}/\bar{U}_1$ is conjugate to the foliation of $S^1 \times D^2$ by disks $(y) \times D^2$;

ii) $\bar{U}^2$ is an open neighborhood of $\partial\tau = T$ such that $\partial\bar{U}_2$ is diffeomorphic to $T \times [-1, 1]$ (with $T \times [0, 1] \subset \tau$), U_2 is transverse to $\mathscr{F}$, and for each $x \in T$, $x \times [-1, 1]$ is transverse to $\mathscr{F}$.

Now by Reeb stability, there exists a neighborhood N_1 of $\mathscr{F}$ (even in the

C^0-topology) such that if $\mathscr{F}_1 \in N_1$, then $\mathscr{F}_1/\bar{U}_1$ is conjugate to $\mathscr{F}/\bar{U}_1$. By the Hirsch stability theorem [2], there is a neighborhood N_2 of $\mathscr{F}$ such that if $\mathscr{F}_2 \in N_2$, then $\mathscr{F}_2/U_2$ has exactly one compact leaf T_2 in U_2, diffeomorphic to T. This can be proved directly without too much difficulty, but it is easier to refer to Hirsch's theorem. We choose N_2 small enough so that $\mathscr{F}_2 \in N_2$ implies that $\mathscr{F}_2$ is transverse to $\partial\bar{U}_2$ and the arcs $x \times [-1, 1], x \in T$.

Now let $N = N_1 \cap N_2$. If $\mathscr{F}' \in N$, then $\mathscr{F}'/U_2$ has a unique compact leaf T' in U_2. Since T' is close to T, T' bounds a solid torus τ' in U. It is clear that the leaves of $\mathscr{F}'$ in U_2 with the exception of T' are planes or cylinders (this can be seen by observing that the leaves are transverse to the fibres $x \times [-1, 1]$, $x \in T$, and to ∂U_2, hence the projection along these fibres realises the leaves as covering spaces of T'). Since these leaves intersect $\partial\bar{U}_1$ in circles, they must be cylinders in $\tau' \cap U_2$. Also T' is hyperbolic (in the proof of Hirsch's theorem, one obtains that the holonomy maps of T' are close to those of T), and hence the leaves of $\mathscr{F}'$ in $U_2 - \tau'$ are also cylinders. Now it is easy to construct a homeomorphism of U conjugating $\mathscr{F}'/U$ with $\mathscr{F}/U$. We leave this to the reader.

Lemma 1.4. *Suppose a foliation $\mathscr{G}$ is obtained from a foliation $\mathscr{F}$ by a finite number of hyperbolic modifications. Then $\mathscr{G}$ is C^1-stable if and only if $\mathscr{F}$ is C^1-stable.*

Proof. We can suppose $\mathscr{G}$ is obtained from $\mathscr{F}$ by one hyperbolic modification. Let us prove the stability of $\mathscr{F}$ implies that $\mathscr{G}$ is stable. First we remark that the operation of erasing hyperbolic Reeb components can be done to be locally continuous in the C^∞-topology, i.e., if $\mathscr{F}'$ is the foliation obtained by erasing a hyperbolic component from $\mathscr{G}'$, then the map $\mathscr{G}' \to \mathscr{F}'$ is continuous in a neighborhood of $\mathscr{G}$.

Now suppose $\mathscr{G}$ is obtained by hyperbolic modification along a simple closed transversal curve γ in a foliated tubular neighborhood U of γ. If $\mathscr{G}'$ is close enough to $\mathscr{G}$, then $\mathscr{G}'$ comes from modification along γ, in U, of a foliation $\mathscr{F}'$ close to $\mathscr{F}$. $\mathscr{F}$ is stable, so if $\mathscr{F}'$ is close enough to $\mathscr{F}$ (which happens when $\mathscr{G}'$ is close enough to $\mathscr{G}$) then there is a homeomorphism h_1 conjugating $\mathscr{F}'$ with $\mathscr{F}$. Since h_1 is close to the identity, we can suppose $h_1(U) = U$, and hence $h_1(V - U) = V - U$.

By Lemma 1.3 we know there exists a homeomorphism $h_2 \colon U \to U$ conjugating $\mathscr{G}'/U$ with $\mathscr{G}/U$. Now $h_2^{-1}h_1$ is a homeomorphism of ∂U sending the circles of $G/\partial U$ onto themselves. It is clear then that $h_2^{-1} \circ h_1$ extends to a homeomorphism of U to itself, leaving $\mathscr{G}/U$ invariant. Denote by λ such an extension. Define a conjugation h of $\mathscr{G}$ to $\mathscr{G}'$ by

$$h/V - U = h_1/V - U\,, \qquad h/U = h_2 \circ \lambda\,.$$

To prove the converse of Lemma 1.4, one uses the same methods; observing that hyperbolic modification can be done to be locally continuous, i.e., if

$\mathscr{F}$ is a foliation, γ a simple closed transversal curve to $\mathscr{F}$, and U a foliated tubular neighborhood of γ, then one can choose the same γ and U to perform modification for all foliations $\mathscr{F}'$ sufficiently close to $\mathscr{F}$ such that the map $\mathscr{F}' \to \mathscr{G}'$ is continuous in a neighborhood of $\mathscr{F}$.

Proof of Theorem 1.2. By the Reeb stability theorems and Lemma 1.4, it suffices to prove that if $\mathscr{F}$ is a stable foliation of $S^2 \times S^1$, then $\mathscr{F}$ is obtained from the product foliation by a finite number of hyperbolic modifications. If $\mathscr{F}$ has no Reeb components, then $\mathscr{F}$ is isotopic to the product foliation. So we can assume $\mathscr{F}$ has Reeb components which are finite in number. They must all be hyperbolic, otherwise we could introduce a band. We erase the Reeb components as in the proof of Theorem 1.1. A finite number of erasures suffice to obtain a foliation with no Reeb components since $\mathscr{F}$ has no flat leaves. Then the foliation thus obtained is the product foliation.

Stable foliations of T^3. The study of stability on T^3 relies very heavily on the thesis of Nancy Kopell [3]. Before discussing this we need a sharpening of Lemma a which will permit us to dispose of compact torus leaves which have finite $R \neq 1$.

Lemma b. *Let L be a compact leaf of foliation $\mathscr{F}$ of codimension one. Suppose $\pi_1(L)$ is abelian and not* (1), *and L has a finite contact $R \neq 1$. Then $\mathscr{F}$ can be C^1-approximated by a foliation $\mathscr{F}'$ such that $\mathscr{F}' = \mathscr{F}$ outside of a tubular neighborhood T of L and inside T, $\mathscr{F}'$ has two or three compact leaves.*

Corollary 1.5. *Let $\mathscr{F}$ be a C^1-stable foliation of codimension one, and L a compact leaf of $\mathscr{F}$ with $\pi_1(L)$ abelian, and $\pi_1(L) \neq (1)$. Then L is hyperbolic.*

Proof of Lemma b. Let $f_1, \cdots, f_m$ be local diffeomorphisms of R at O which generate the holonomy group of L, and let $h: R \to R$ be a map as in the proof of Lemma a. We know there is a $k > 1$ such that $f_1^{(k)}(\mathrm{O}) \neq \mathrm{O}$, and $f_1^{(s)}(\mathrm{O}) = \mathrm{O}$ for $1 < s < k$. Define maps g_i by

$$g_i(y) = h^{-1} f_i h(t) \qquad \text{if } |t| > a \ .$$

Let $\bar{g}_i(\pm a)$ be the formal power series of g_i at $\pm a$. By a theorem of N. Kopell [3], we know there exists a formal vector field $\bar{X}_1$ at $+a$, with a one-parameter formal group $\bar{\varphi}_1(\tau)$ such that $\bar{\varphi}_1(\tau_i) = \bar{g}_i(a)$ for certain values $\tau_1, \cdots, \tau_m$.

Similarly, there exists a formal vector field $\bar{X}_2$ at $-a$ with analogous properties for the same values of τ_i (they are determined by the k-th derivatives of f_i at O).

Let $X(t)$ be a C^∞ vector field on $[-a, a]$ such that $X(a) = \bar{X}_1$, $X(-a) = \bar{X}_2$ and X is C^1-close to the zero vector field. Clearly such a vector field can be chosen to have two or three zeros (depending on the parity of k), at the points $\pm a$ and eventually at an interior point.

Let $\varphi(\tau, t)$ be the one-parameter group of X, and define g_i on $[-a, a]$ by

$$g_i(t) = \varphi(\tau_i, t) \ .$$

Then the representation of $\pi_1(L)$ given by $g_1, \cdots, g_m$ defines the desired foliation $\mathscr{F}'$.

Now we are ready to analyse C^1-stable foliations of T^3. Suppose $\mathscr{F}$ is a C^1-stable foliation of T^3. We know $\mathscr{F}$ is obtained from a C^1-stable foliation G by a finite number of hyperbolic modifications, and G has no Reeb components. Moreover, all the compact leaves of G are tori and hence hyperbolic.

Now one can cut G by a transverse 2-torus (see [5] and [7] for details of this method), and G becomes the suspension of a foliation g of T^2 by a diffeomorphism α of T^2, isotopic to the identity and leaving g invariant. More precisely, G is equivalent to the foliation $g \times I$ of $T^2 \times I$, quotient by the identification $(x, 0) = (\alpha(x), 1)$ for $x \in T^2$. We shall write this as

$$G = g \times I/\alpha \ .$$

There are two possibilities:

Case 1: *G has no compact leaves.* In this case, the foliation g is the suspension of a diffeomorphism of S^1, and it can be shown (by the process of cutting along transverse tori) that G is equivalent to a foliation of $T^2 \times S^1$ transverse to the factor S^1. Such a foliation is completely determined by the holonomy representation:

$$\rho\colon \pi_1(T^2) \to \mathrm{Diff}\,(S^1) \ .$$

If the image of ρ were generated by one diffeomorphism β, then G could not be stable. For β would have no periodic points (G has no compact leaves), yet we could C^1-approximate β by a diffeomorphism with periodic points, and the associated representation ρ' would give a foliation C^1-close to G with compact leaves. Hence the image of ρ is isomorphic to $Z \oplus Z$; it is generated by two diffeomorphisms of S^1, without periodic points and whose rotation numbers are irrationally independent. Thus G is a foliation of T^3 by planes R^2, and we must admit (with much consternation) that we do not know if a stable foliation of T^3 by planes does exist. This is an important problem. We have previously proved that a foliation by planes of T^3 is topologically equivalent to a linear foliation. If this equivalence could be chosen differentiable, then the foliation would not be stable. We have some more success when G has compact leaves.

Case 2: *G has at least one compact leaf.* One can choose a transverse torus to G so that G is the suspension of a hyperbolic foliation g of T^2. If g is the suspension of a diffeomorphism of S^1 (topologically, this means g has no Reeb components of dimension two), then G is given by a representation

$$\rho\colon \pi_1(T^2) \to \mathrm{Diff}\,(S^1) \ ,$$

and the image of ρ consists of hyperbolic diffeomorphisms of S^1 (or the identity). N. Kopell has analysed such subgroups.

Lemma 1.6 [3]. *Let H be an abelian hyperbolic subgroup of* Diff (S^1). *Then $H = \{\gamma^n / n \in Z\}$ for some diffeomorphism γ of S^1 or H is a one-parameter group $\{\gamma^t / t \in R\}$.*

Clearly, the case when $(\pi_1 T^2)$ embeds in a one-parameter group gives rise to an unstable foliation. Now in the general case, when g can have Reeb components, we proceed as follows.

Let $C(g)$ be the group of diffeomorphisms of T^2 which leave g invariant and are isotopic to the identity. Let $C_0(g)$ be the normal subgroup of $C(g)$ of diffeomorphisms isotopic to the identity by an isotopy which at each stage leaves g invariant. Let $\bar{C}(g) = C(g)/C_0(g)$. The lemma of N. Kopell yields

Lemma 1.7. *Let g be a hyperbolic foliation of T^2. Then either*

i) *$C_0(g)$ is open in $C(g)$, or*

ii) *$\bar{C}(g)$ is nontrivial and generated by a one-parameter group $\{\beta^t\}$ of $C(g)$, modulo $C_0(g)$.*

Remark. It is clear that two elements of $C(g)$, which differ by an element of $C_0(g)$, define the same foliation by suspension.

The topological study of foliations of T^3 done in [7] permits us to state Lemma 1.7 precisely as follows.

If a foliation G of T^3 is obtained by suspension of a hyperbolic foliation g of T^2, then all the noncompact leaves of G are of the same topological type, i.e., they are all planes or all cylinders. In case (i) of Lemma 1.7, the noncompact leaves are all cylinders. In case (ii), we can put in correspondence the noncompact leaves of $G_t = g \times I/\beta^t$ with the number t as follows: the noncompact leaves of two such foliations G_{t_1} and G_{t_2} are homeomorphic if and only if t_1 and t_2 are rationaly dependent. From this it follows easily that a foliation of type (ii) of Lemma 1.7 is unstable. Thus, if G is stable, then G is of type (i), and all the noncompact leaves are cylinders. Now one can prove that there is a transverse torus to G such that G is the suspension of another foliation g of T^2 by *the identity*: $G = g \times I/1$. Hence the leaves of G are the product of the leaves of g by S^1.

If $g \times S^1$ is a stable foliation, then condition (i) is satisfied for all foliations near g; more precisely we have

Definition. Let Σ be the foliation g of T^2 satisfying: g is hyperbolic, and there are a neighborhood $V(g)$ of 1 in Diff (S^1), in the C^1-topology, and a neighborhood $U(g)$ of g in the C^1-topology such that for $g' \in U(g)$,

$$C(g') \cap V(g) \subset C_0(g') .$$

Then one can generalise to the following Lemma 1.8 the N. Kopell's result that the set of diffeomorphisms β of S^1 whose commutator subgroup is $\{\beta^n / n \in Z\}$ contains an open dense set of Diff (S^1).

Lemma 1.8. *$\sum$ is open and dense in the space of foliations of T^2 with the C^1-topology.*

As one should expect, the foliations in Σ yield stable foliation of T^3.

Theorem 1.9. *If $g \in \Sigma$, then the product foliation $g \times S^1$ is C^1-stable.*

Proof. Suppose G' is a foliation of T^3, and C^1 close to $G = g \times S^1$. G' can be written as $g' \times I/\alpha'$ where g' and α' depend continuously on G'. Thus, if G' is sufficiently close to G, then $g' \in U(g)$ and $\alpha' \in V(g)$ which implies $\alpha' \in C_0(g')$. Hence one can find an isotopy sending $g' \times I/\alpha'$ to $g' \times S^1$, which depends continuously on α' and thus on G'.

Now g is hyperbolic, so if g' is sufficiently C^1-close to g then g' is conjugate to g by a homeomorphism $h(g')$ depending continuously on g'. Thus G is C^1-stable.

We summarise our results on T^3 as follows.

Theorem 1.10. *The C^1-stable foliations of T^3 are, up to hyperbolic modification,*

(i) *foliations by planes (perhaps), or*

(ii) *foliations of the form $g \times S^1$ with $g \in \Sigma$ where Σ is the open dense set of foliations of T^2 defined above.*

2. Intrinsic components

We now assume $\mathscr{F}$ is a transversally orientable codimension one foliation of a closed n-manifold V. Novikov has defined the components of $\mathscr{F}$ as follows: two points x and y of V are in the same component if there exists a closed transversal curve to $\mathscr{F}$ passing through x and y. It is easy to see that a foliation has three types of components:

1. the entire manifold V,
2. an open submanifold of V, whose boundary consists of the union of compact leaves of $\mathscr{F}$.
3. a compact leaf.

Rigorously speaking, it is the points of type 3 which are the components. Among the components of type 2, we distinguish the intrinsic components: A is intrinsic if the normal vector field to $\mathscr{F}$ points in the same direction along each connected component of ∂A, i.e., into A or towards $V - A$. We also agree to call components of type 1, intrinsic components. It is easy to see that $\mathscr{F}$ always has at least one intrinsic component, and the intrinsic components are characterised by the property of remaining components whenever they are embedded in some foliated manifold.

We now prove some lemmas which show that intrinsic components can only get bigger after perturbation.

Lemma 2.1. *Let $\mathscr{F}$ be a foliation of V such that V is an intrinsic component (hence of type 1). If $\mathscr{F}'$ is a foliation of V and C^0-close to $\mathscr{F}$, then V is also an intrinsic component of $\mathscr{F}'$.*

Proof. Let C be a closed transversal curve to $\mathscr{F}$ which intersects each leaf of $\mathscr{F}$. Cover V by a finite number of distinguished neighborhoods $U_1, \cdots, U_k$.

Let $W_1, \cdots, W_k$ be distinguished neighborhoods such that $\overline{W}_i \subset U_i$ for each i, and $W_1, \cdots, W_k$ cover V. For each i, $1 \leq i \leq k$, and $x \in W_i$, there exist $\varepsilon_x > O$ and a neighborhood N_x of x such that if $\mathscr{F}'$ is a foliation of V with $d(\mathscr{F}', \mathscr{F}) < \varepsilon_x$ (in the C^0-topology), then each leaf of $\mathscr{F}'$ which intersects N_x also intersects C. This can be seen as follows: let Γ be a path in the leaf of $\mathscr{F}$ containing x, which joins x to a point of C. Cover Γ by a finite number of distinguished neighborhoods and carry out the reasoning for each distinguished neighborhood, and the intersection of successive neighborhoods. Now by compactness one can choose a smallest ε_x such that if $d(\mathscr{F}', \mathscr{F}) < \varepsilon_x$, then each leaf of $\mathscr{F}'$ intersects C. Hence V is an intrinsic component of $\mathscr{F}'$.

Lemma 2.2. *Let L be a compact leaf of $\mathscr{F}$, and T a tubular neighborhood of L with $\mathscr{F}$ transverse to the fibers of T. Suppose $\mathscr{F}'$ is a perturbation of $\mathscr{F}$, and $\mathscr{F}'$ has no compact leaves in T. Then some leaf of $\mathscr{F}'$ goes from one component of ∂T to the other.*

Proof. Identify T with $L \times [-1, +1]$, and L with $L \times (0)$. Assume $\mathscr{F}'$ close enough to $\mathscr{F}$ so that $\mathscr{F}'$ is transverse to the fibres $x \times [-1, 1]$ for each $x \in L$. Observe that $\mathscr{F}'$ has no closed invariant nonempty sets A in the interior of T; for otherwise, $\{y_x / y_x = (x, \sup \{t/(x, t) \in A\})\}_{x \in L}$ would be a compact leaf of $\mathscr{F}'$ inside T. Hence for each $z \in T$, the closure of the $\mathscr{F}'$ leaf of z must intersect ∂T.

Let $U_1 = \{z \in T/$ the $\mathscr{F}'$ leaf of z meets $L \times (3/4, 1]\}$, and $U_{-1} = \{z \in T/$ the $\mathscr{F}'$ leaf of z meets $L \times [-1, -3/4)\}$. Clearly U_1 and U_{-1} are open in $L \times (-3/4, 3/4)$, and by our previous remark, each point of $L \times (-3/4, 3/4)$ is in U_1 or U_{-1}. Hence $U_1 \cap U_{-1} \neq \emptyset$, and some leaf of $\mathscr{F}'$ intersects $L \times (-3/4)$ and $L \times (3/4)$, and consequently goes from one side of T to the other.

Lemma 2.3. *Let A be an intrinsic component of type 2 of $\mathscr{F}$. Suppose the compact leaves in ∂A are stable under C^k-perturbation. Then A is stable under C^k perturbation of $\mathscr{F}$.*

Proof. Let $T_1, \cdots, T_j$ be the leaves of $\mathscr{F}$ in ∂A, and let $\varepsilon > O$ such that if $\mathscr{F}'$ is C^k ε-close to $\mathscr{F}$ then $\mathscr{F}'$ has compact leaves $T_1', \cdots, T_j'$ close to $T_1, \cdots, T_j$ respectively. Let C be a closed transversal curve to $\mathscr{F}$ which meets all the leaves of the interior of A. Let $\delta > O$ and $B = \{x \in A / d(x, A) \geq \delta\}$; for δ sufficiently small, B is homeomorphic to A. By reasoning as in the proof of Lemma 2.1, we see there exists $\varepsilon_1 > O$ such that if $\mathscr{F}'$ is a foliation of V, C^0 ε_1-close to $\mathscr{F}$, then each leaf of $\mathscr{F}'$ which intersects B also intersects C. Now let $\varepsilon_2 = \min \{\varepsilon, \varepsilon_1\}$. If $\mathscr{F}'$ is C^k ε_2-close to $\mathscr{F}$, then $\mathscr{F}'$ has an intrinsic component A' which is close to A. To see this we remark that all points of B are in the same component of $\mathscr{F}'$ (all the leaves meet C), and $\mathscr{F}'$ has compact leaves $T_1', \cdots, T_j'$ close to $T_1, \cdots T_j$; hence the submanifold bounded by $T_1', \cdots, T_j'$ and containing C can not be properly contained in a component since no closed transversal curve to $\mathscr{F}'$ can intersect $T_1' \cup \cdots \cup T_j'$.

Lemma 2.4 (*M. Hirsch* [2]). *Let L be a compact leaf of a foliation of codimension k. Suppose that for some $\alpha \in \pi_1(L)$, in the center of $\pi_1(L)$ the* *holonomy of α is hyperbolic. Then L is stable under perturbation.*

Corollary 2.5. *A hyperbolic torus leaf is stable.*

One can prove this corallary directly for 2-torus leaves in 3-manifolds (cf. [6]).

Theorem 2.6. *Let V be a closed orientable 3-manifold, and Ω the space of codimension one foliations of V, with the C^1-topology. Then there is an open dense subset S of Ω such that if $\mathscr{F} \in S$, then the intrinsic components of $\mathscr{F}$ are stable.*

Proof. Let S be the set of foliations in Ω whose intrinsic components are stable under C^1-perturbation. Clearly S is open in Ω. It remains to prove S is dense. Let $\mathscr{F} \in \Omega$, and A be an intrinsic component of $\mathscr{F}$. For reasons of Euler characteristic, the boundary components of A are 2-tori. Hence by Lemma 2.3 and Corollary 2.5, it suffices to prove that a torus leaf can be C^1-approximated by a hyperbolic torus leaf by an approximation whose support is in an arbitrarily small tubular neighborhood of the torus leaf. Let L be a flat torus leaf, and T a tubular neighborhood of L homeomorphic to $L \times [-1, 1]$ with $L = L \times (\mathrm{O})$. But we can C^1-approximate $\mathscr{F}$ by a foliation $\mathscr{F}'$, equal to $\mathscr{F}$ on $V - (L \times [-3/4, 3/4])$ and with a band of compact leaves: $L \times (t)$, $-a \leq t \leq a$, with a as small as we wish. Now we C^1-approximate $\mathscr{F}'$ to make $L \times (\mathrm{O})$ hyperbolic and without changing $\mathscr{F}'$ outside of $L \times [-a, a]$. This can be done simply by approximating the identify map $i: R \to R$ by a diffeomorphism $f: R \to R$ such that O is a hyperbolic fixed point of f and $f = i$ on the complement of $[-a/2, a/2]$. This completes the proof of Theorem 2.6.

3. Remarks on stability of Haefliger structures

A Haefliger structure on M is a foliation with a certain type of singularities; we refer the reader to [4] and [1] for the definition and discussion of Haefliger structures. We recall that to a C^p-Haefliger structure H on M, are associated a C^p vector bundle E over M, a section $i: M \to E$ and a C^p foliation $\mathscr{F}$ defined in a neighborhood of $i(M)$ and transverse to the fibres. The dimension of the fibre is the codimension of H. The triple (E, H, i) is called the graph of H; it is defined up to C^p-equivalence and determines H.

One can define stability of a Haefliger structure by its graph.

Definition. H is C^1-stable if for each couple $(i', \mathscr{F}')$ where $i': M \to E$ is a section C^1-close to i, and $\mathscr{F}'$ is a foliation defined in neighborhood of $i'(M)$, transverse to the fibres, and C^1-close to $\mathscr{F}$ in some neighborhood, there exists a couple (h, ϕ) of homeomorphisms, ϕ being defined in a neighborhood of $i(M)$, such that the following diagramm in commutative:

$$\begin{array}{ccc} M & \xrightarrow{i} & E \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \phi} \\ M & \xrightarrow{i'} & E \end{array}$$

Clearly, when i is transverse to $\mathscr{F}$, this is the usual definition of stability. A good exercise to further the understanding of this definition is the Haefliger structure on S^2 defined as follows: Let $E = S^2 \times R = R^3 - \{0\}$, and $\mathscr{F}$ be the foliation whose leaves are $S^2 \times \{t\}$. Let $i: S^2 \to E$ be a section such that $z \circ i$ has exactly two nondegenerate critical points where $z: E \to R$ is the "z-coordinate". Then this Haefliger structure is not stable, even though $\mathscr{F}$ is stable.

One can also put a topology on the space of Haefliger structures using the fact that they can be defined by differential forms. Then defining stability in terms of this topology is equivalent to the previous definition.

Now consider Haefliger structures of codimension one. Generically, a section $i: M \to E$ is in general position with respect to $\mathscr{F}$; so the contact points are singularities of a Morse function. Consequently, if H is C^1-stable, and M is compact, then H has a finite number of singularities, each of Morse type, and H is a foliation of codimension one elsewhere. The singularities are conic or centers.

By convention, a leaf of H will be one of three types:

i) a leaf of a foliation (usual sense),

ii) a leaf which has a conic singularity,

iii) a center.

A difficult and interesting problem is the study of stable Haefliger codimension one structures of S^3. One of the crucial points in the proof of Theorem 1.1 is the existence of a compact leaf for foliations of S^3. This fails for Haefliger structures.

Proposition 3.1. *There is a generic Haefliger structure of S^3 with no compact leaf.*

In order to construct such a structure we need some preliminaries.

The connected sum of foliations. Let M and M' be manifolds of the same dimension, and $\mathscr{F}, \mathscr{F}'$ codimension one foliations of M, M', tangent to ∂M, $\partial M'$, with $\partial M \neq \emptyset \neq \partial M'$. Define $(M, \mathscr{F}) \# (M', \mathscr{F}')$ to be $M \# M'$ (connected sum along the boundary) together with a generic Haefliger structure $\mathscr{F} \# \mathscr{F}'$ defined as follows.

Let $m \in \partial M$, $m' \in \partial M'$, and U, U' be distinguished neighborhoods of m, m', diffeomorphic to $R^{n-1} \times R^+$, where the foliations are defined by projection on R^+.

Let W, W' be the complements in U, U' of the (half) open disc of radius one. Clearly, one can reparametrize W and W' by coordinates $(x_1, \cdots, x_n)$ and $(x_1', \cdots, x_n')$ such that:

$$W = \left\{x_i \,|\, 0 \leq x_1 \leq 1,\ \sum_2^n x_i^2 \leq 1\right\}, \qquad W' = \left\{x_i' \,|\, 0 \leq x_1' \leq 1,\ \sum_2^n x_i'^2 \leq 1\right\},$$

and the foliations are given by the functions:

$$f = x_1 - \sum_2^n x_i^2 \quad \text{for} \quad \sum_2^n x_i^2 \leq \tfrac{1}{4}, \qquad f = -\sum_2^n x_i^2 \quad \text{for} \quad \sum_2^n x_i^2 \geq \tfrac{3}{4},$$

and a similar definition for f'. Notice that $\{x_1 = 0\}$ corresponds to the boundary C of the half disc which we have removed, and $\{\sum_2^n x_i^2 = 1\}$ is in ∂M.

Define $\tilde{W} = W \bigcup_C W''$ where we make the identifications:

$$\tilde{W} = \left\{(y_1, \cdots, y_n) \mid -1 \leq y_1 \leq 1, \sum_2^n y_i^2 \leq 1\right\},$$

i) if $y_1 \leq 0$, then $y_1 = -x$ and $y_i = x_i$, $i \geq 2$,

ii) if $y_1 \geq 0$, then $y_1 = x_1'$ and $y_i = x_i'$, $i \geq 2$.

The functions f and f' define a function g by:

$$g(y_1, \cdots, y_n) = |y_1| - \sum_2^n y_i^2 \quad \text{for} \quad \sum_2^n y_i^2 \leq \tfrac{1}{4},$$

and

$$g(y_1, \cdots, y_n) = -\sum_2^n y_i^2 \quad \text{for} \quad \sum_2^n y_i^2 \geq \tfrac{3}{4}.$$

By definition, we take g as a distinguished function in W. Then $\mathscr{F} \# \mathscr{F}'$ is defined by $\tilde{g}$ in $\tilde{W}$, $\mathscr{F}$ in $M - W$ and $\mathscr{F}'$ in $M - W'$. This is a generic Haefliger structure on $M \# M'$ with one singularity of index one.

Proof of Proposition 3.1. Let τ and τ' be two Reeb components, and $(M, \mathscr{F})$ their connected sum. M is a solid pretzel (figure eight), and $\mathscr{F}$ is a Haefliger structure on M wtih exactly one compact leaf, the ∂M. Embed M in S^3 so that $S^3 - M$ is diffeomorphic to M (the standard embedding works), and put the same Haefliger structure on $S^3 - M$. This structure on S^3 has one compact leaf, the ∂M. We shall perturb this structure to eliminate the compact leaf.

Let γ be the circle on $\partial\tau$ along which we make the connected sum of τ and τ'. The holonomy of γ in $\mathscr{F}_1$ is trivial, i.e., there is a neighborhood U of γ in S^3 such that:

$$U \approx S^1 \times I \times J \quad \text{with} \quad I, J \approx [-1, 1], \quad \gamma = (\theta, 0, 0),\ \theta \in S^1,$$

and $\mathscr{F}_1/U$ is defined by projection on J. We perturb $\mathscr{F}_1$ in U as follows:

i) let $\psi(x)$ be C^∞ decreasing with

$$\psi(x) = 0 \quad \text{for} \quad -1 \leq x \leq -\tfrac{1}{2}, \quad \psi(x) = -\tfrac{1}{5} \quad \text{for} \quad \tfrac{1}{2} \leq x \leq 1,$$

ii) let $\varphi(t)$ be C^∞ with

$$\varphi(t) = 1 \quad \text{for} \quad -\tfrac{1}{4} \leq t \leq \tfrac{1}{4},$$

$\varphi(t) = 0$ for $-1 \leq t \leq -\frac{1}{2}$ and $\frac{1}{2} \leq t \leq 1$,

$$0 \leq \varphi(t) \leq 1 \quad \text{and} \quad |\varphi'(t)| \leq 2.$$

We define $f: U \to R$ by

$$f(\theta, x, t) = t - \varphi(t)\psi(x) .$$

Notice that f is of rank one, $f = t$ outside of $S^1 \times [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$, and the level surfaces of f are cylinders connecting the circles $(\theta, -1, t)$ for fixed t with the circles $(\theta, 1, t')$ for fixed t', with $t' \leq t$ and $t' < t$ if $t \in [-\frac{1}{4}, \frac{1}{4}]$. Let $\mathscr{F}_2$ be the Haefliger structure on S^3 defined by f in U and $\mathscr{F}_1$ in $S^3 - U$. We claim $\mathscr{F}_2$ has no compact leaves. We see this as follows. Let m and m_1 be the points of $U \cap \partial M$ with coordinates $(O, -1, 0)$ and $(0, 1, 0)$ respectively. In $\mathscr{F}_2$ all the leaves have m or m_1 in their closure. If a leaf has m in its closure and not m_1, then it is not compact. In $\mathscr{F}_2$, the leaf of m has m_1 in its closure but does not contain m_1, hence is not compact. The same reasoning holds for leaves having m_1 in their closure. Hence no leaf of $\mathscr{F}_2$ is compact.

It is easy to choose φ and ψ so that $\mathscr{F}_2$ is not stable. However, if a Haefliger structure of S^3 is defined by a Morse function with distinct ciritical values and whose leaves are points or homeomorphic to S^2 (i.e., diffeomorphic to S^2 with one conic singular point), then it is stable. This is an easy consequence of the Reeb stability theorem. This leads us to the end:

Conjecture. *The only stable Haefliger structures of S^3 are those defined by a Morse function having distinct critical values and whose level surfaces are simply connected.*

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