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GENERALIZED SCALAR CURVATURES OF COHOMOLOGICAL EINSTEIN KAEHLER MANIFOLDS

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1. Introduction

In Riemannian geometry all elementary symmetric polynomials of eigenvalues of the Ricci tensor are geometric invariants. In particular, the one of degree 1 is called the scalar curvature.

In this paper, we shall study some properties of the geometric invariants for *cohomological Einstein* Kaehler manifolds. Let M be a Kaehler manifold with fundamental 2-form Φ and Ricci 2-form γ. We say that M is cohomologically Einsteinian if $[\gamma] = a\cdot[\Phi]$ for some constant a, where $[*]$ denotes the cohomology class represented by $*$. It is well-known that the first Chern class $c_1(M)$ is represented by γ.

Let $z_1, \cdots, z_n$ be a local coordinate system in M, $g = \sum g_{\alpha\bar\beta} dz_\alpha d\bar z_\beta$ be the Kaehler metric of M, and $S = \sum R_{\alpha\bar\beta} dz_\alpha d\bar z_\beta$ be the Ricci tensor of M. Define n scalars $\rho_1, \cdots, \rho_n$ by

$$\frac{\det(g_{\alpha\bar\beta} + tR_{\alpha\bar\beta})}{\det(g_{\alpha\bar\beta})} = 1 + \sum_{k=1}^{n} \rho_k t^k ,$$

and denote the scalar curvature of M by ρ. Then it is easily seen that $\rho = 2\rho_1$, and is also clear that $\rho_n = \det(R_{\alpha\bar\beta})/\det(g_{\alpha\bar\beta})$.

We shall prove

Theorem 1. *Let M be an n-dimensional compact cohomological Einstein Kaehler manifold. If $c_1(M) = a\cdot[\Phi]$, then*

$$\int_M \rho_k * 1 = (2\pi a)^k \binom{n}{k} \int_M * 1 ,$$

where $\binom{n}{k}$ denotes the binomial coefficient, and $*1$ the volume element of M.

This results implies that the average of ρ_k, $\int_M \rho_k * 1 \Big/ \int_M *1$, does not depend on the metric too strongly.

Let $P_{n+p}(C)$ be an $(n + p)$-dimensional complex projective space with the

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Fubini-Study metric of constant holomorphic sectional curvature 1. An n-dimensional algebraic manifold imbedded in $P_{n+p}(C)$ is called a *complete intersection manifold* if M is given as an intersection of p nonsingular hypersurfaces $M_1, \cdots, M_p$ in $P_{n+p}(C)$, i.e., if $M = M_1 \cap \cdots \cap M_p$. It is known that the (first) Chern class of a complete intersection manifold M is completely determined by the degrees of $M_1, \cdots, M_p$, and it is easily seen that a complete intersection manifold is cohomologically Einsteinian with respect to the induced Kaehler metric.

Theorem 2. *Let M be an n-dimensional complete intersection manifold in $P_{n+p}(C)$, i.e., let $M = M_1 \cap \cdots \cap M_p$. Then*

$$\int_M \rho_k * 1 = \binom{n}{k}\left[\frac{1}{2}(n + p + 1 - \sum a_\alpha)\right]^k (\prod a_\alpha)\frac{(4\pi)^n}{n!} ,$$

where a_α denotes the degree of M_α, $\alpha = 1, \cdots, p$.

Theorem 3. *Let M be an n-dimensional complete intersection manifold in $P_{n+p}(C)$. If $\rho_k > \binom{n}{k}\left(\frac{n}{2}\right)^k$ for some k, then M is a linear subspace.*

The above theorems can be considered as generalizations of the results in [3]. Theorem 2 is of Gauss-Bonnet type in the sense that it provides a relationship between differential geometric invariants and more primitive invariants: The scalar ρ_k is a differential geometric invariant and depends fully on the equations defining M, but Theorem 2 implies that the integral of ρ_k depends only on (the sum and the product of) the degrees of M. Theorem 3 gives a characterization of a linear subspace among complete intersection manifolds.

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2. Proof of Theorem 1

Let Φ be the fundamental 2-form of M, that is, a closed 2-form defined by

$$\Phi = \frac{\sqrt{-1}}{2} \sum g_{\alpha\bar\beta} dz_\alpha \wedge d\bar z_\beta . \tag{1}$$

Let γ be the Ricci 2-form of M, that is, a closed 2-form defined by

$$\gamma = \frac{\sqrt{-1}}{4\pi} \sum R_{\alpha\bar\beta} dz_\alpha \wedge d\bar z_\beta . \tag{2}$$

Then the first Chern class $c_1(M)$ is represented by γ. We denote [*] to be the cohomology class represented by a closed form * so that, in particular, $c_1(M) = [\gamma]$.

Since $c_1(M) = a \cdot [\Phi]$, there exists a 1-form η satisfying

$$\gamma = a\Phi + d\eta \ .$$

Therefore we obtain

$$(3) \qquad \gamma^k = a^k\Phi^k + \sum_{\ell=1}^{k} (\cdots)\, \Phi^{k-\ell} \wedge (d\eta)^\ell \ ,$$

where $(\cdots)$ is a constant involving ℓ.

Let Λ be the operator of interior product by Φ. Then it follows from (1) and (2) that

$$\Lambda^k\Phi^k = \frac{k!\,n!}{(n-k)!}\ , \qquad \Lambda^k\gamma^k = \frac{k!\,k!}{(2\pi)^k}\rho_k \ .$$

These, together with (3), imply

$$\frac{k!\,k!}{(2\pi)^k}\rho_k = a^k\frac{k!\,n!}{(n-k)!} + \sum_{\ell=1}^{k} (\cdots)\Lambda^k\Phi^{k-\ell} \wedge (d\eta)^\ell \ ,$$

so that

$$(4) \qquad \rho_k = (2\pi a)^k\binom{n}{k} + \sum_{\ell=1}^{k} \{\cdots\}\, \Lambda^\ell(d\eta)^\ell \ ,$$

where $\{\cdots\}$ is a constant involving ℓ.

Let δ be the codifferential operator, and C the operator defined by $C\alpha = (\sqrt{-1})^{r-s}\alpha$, where α is a form of bidegree (r, s). Then $\delta\Lambda = \Lambda\delta$, $C\Lambda = \Lambda C$ and $d\Lambda - \Lambda d = C^{-1}\delta C$ (cf. for example [1]). We can prove inductively that $d\Lambda^\ell - \Lambda^\ell d = \ell C^{-1}\delta C\Lambda^{\ell-1}$, from which it follows that $\Lambda^\ell(d\eta)^\ell = \Lambda^\ell d(\eta \wedge (d\eta)^{\ell-1}) = -\ell C^{-1}\delta C\Lambda^{\ell-1}(\eta \wedge (d\eta)^{\ell-1})$, and hence $\int_M \Lambda^\ell(d\eta)^\ell * 1 = 0$. Therefore from (4) we have

$$\int_M \rho_k * 1 = (2\pi a)^k\binom{n}{k}\int_M * 1 \ .$$

3. Proof of Theorems 2 and 3

Let $\tilde{h}$ be the generator of $H^2(P_{n+p}(C), Z)$ corresponding to the divisor class of a hyperplane in $P_{n+p}(C)$. Then the first Chern class $c_1(P_{n+p}(C))$ of $P_{n+p}(C)$ is given by

$$(5) \qquad c_1(P_{n+p}(C)) = (n + p + 1)\tilde{h} \ .$$

Let $j\colon M \to P_{n+p}(C)$ be the imbedding, and h the image of $\tilde{h}$ under the homomorphism $j^*\colon H^2(P_{n+p}(C), Z) \to H^2(M, Z)$. Then the first Chern class $c_1(M)$ of M is given by

(6) $$c_1(M) = (n + p + 1 - \sum a_\alpha)h \ .$$

Let $\tilde{\Phi}$ be the fundamental 2-form of $P_{n+p}(C)$. Since the Fubini-Study metric $\tilde{g}$ and the Ricci tensor $\tilde{S}$ of $P_{n+p}(C)$ are related by

$$\tilde{S} = \tfrac{1}{2}(n + p + 1)\tilde{g} \ ,$$

the Ricci 2-form $\tilde{\gamma}$ of $P_{n+p}(C)$ satisfies

$$\tilde{\gamma} = \frac{n + p + 1}{4\pi}\tilde{\Phi} \ .$$

Therefore we have

(7) $$c_1(P_{n+p}(C)) = \frac{n + p + 1}{4\pi}[\tilde{\Phi}] \ .$$

Since $\Phi = j^*\tilde{\Phi}$, it follows from (5), (6) and (7) that

$$c_1(M) = \frac{n + p + 1 - \sum a_\alpha}{4\pi}[\Phi] \ ,$$

which implies that M is cohomologically Einsteinian. Therefore from Theorem 1 we have

(8) $$\int_M \rho_k * 1 = \left[\frac{1}{2}(n + p + 1 - \sum a_\alpha)\right]^k \binom{n}{k} \int_M * 1 \ .$$

Let $P_p(C)$ be a p-dimensional linear subspace of $P_{n+p}(C)$, and ν the number of points in $M \cap P_p(C)$. Then the dimension theory for algeraic manifolds states that ν does not depend on the choice of $P_p(C)$ if $P_p(C)$ is in general position. By a theorem of Wirtinger [4], the volume of M is given by

$$\int_M * 1 = \nu \frac{(4\pi)^n}{n!} \ .$$

On the other hand, since M is a complete intersection maifold, we have [2]

$$\nu = \prod a_\alpha \ .$$

Therefore it follows that

$$\int_M * 1 = (\prod a_\alpha)\frac{(4\pi)^n}{n!} \ ,$$

which, combined with (8), completes the proof of Theorem 2.

If $\rho_k > \binom{n}{k}\left(\frac{n}{2}\right)^k$, then it follows from (8) that

$$\binom{n}{k}\left(\frac{n}{2}\right)^k \int_M *1 < \left[\frac{1}{2}(n + p + 1 - \sum a_\alpha)\right]^k \binom{n}{k} \int_M *1 ,$$

which implies $\sum a_\alpha < p + 1$, that is, $a_1 = \cdots = a_p = 1$. This proves Theorem 3.

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