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A GENERALIZATION OF THE CLOSED SUBGROUP THEOREM TO QUOTIENTS OF ARBITRARY MANIFOLDS

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1. Introduction

The purpose of this paper is to present a new necessary and sufficient condition for an equivalence relation R on a differentiable manifold M to be regular (i.e., such that the quotient M/R is also a differentiable manifold, and that the canonical projection $M \to M/R$ is a submersion). Our condition is motivated by a problem of system theory. Given a nonlinear system $\dot{x} = f(x, u)$ together with an "output map" $y = \varphi(x)$, one can associate with every "input" $u(t)$, $(0 \leq t \leq T)$, and every "initial state" x^0, an "output" $y(t)$, $(0 \leq t \leq T)$, defined as follows: let $x(t)$ be the solution of $\dot{x}(t) = f(x(t), u(t))$ for which $x(0) = x^0$, and let $y(t) = \varphi(x(t))$. If, for an input u, the outputs which correspond to two initial states x^0 and x^1 are not identical, we say that u *distinguishes* between x^0 and x^1. If there is no input which distinguishes between x^0 and x^1, we say that x^0 and x^1 are *indistinguishable*. If there do not exist states x^0 and x^1 which are indistinguishable but different, we say that the system is *observable*. Given a nonobservable system whose state space is a manifold M, we would like to "make it observable". The obvious way to achieve this is by taking the quotient M/R, where R is the equivalence relation of indistinguishability, and by letting this quotient be the state space of our new system. For this to be possible it is necessary that R be regular. The necessary and sufficient condition given in Serre [2, Part II, Chap. 3, § 12, Theorem 2] is not easy to verify. However, in situations where there is more structure, R turns out to be regular for a different reason. As an example, consider the system

$$(1)\qquad \begin{aligned} \dot{X} &= (A + uB)X\,, \qquad X \in G\,, \\ y &= bX\,. \end{aligned}$$

Here the state space G is a Lie group of $n \times n$ matrices, A and B are matrices in the Lie algebra of G, the inputs are real-valued functions, the variable y takes values in R^n (viewed as a space of row vectors), and $b \in R^n$. For each

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input $u(t)$, $(0 \leq t \leq T)$, let $\Phi_u(t)$ be the solution of (1) for which $\Phi_u(0)$ is the identity matrix. Let H denote the subgroup of G whose elements are the matrices $X \in G$ for which $bX = b$. Let K denote the intersection of the subgroups $\Phi_u(t)^{-1}H\Phi_u(t)$, where u ranges over all inputs and, for each u, t ranges over the interval in which u is defined. In this particular case the relation R can be easily described: X and Y are indistinguishable if and only if $XY^{-1} \in K$. Therefore R is regular, because K is a closed subgroup of G.

The preceding example suggests that the appropriate tool for attacking our system theory problem should be some generalization to arbitrary manifolds of the closed subgroup theorem. We now describe this generalization informally. The rest of the paper is devoted to a precise statement and proof of this result. The application to the observability problem will not be discussed here (cf. Sussmann [5]).

Suppose that M is a connected Lie group, and let L be the Lie algebra of M, viewed as the set of vector fields on M which are infinitesimal generators of one-parameter groups of right multiplications. Let R be an equivalence relation on M which is closed as a subset of $M \times M$ and for which $(x, y) \in R$ implies "$(x \exp(tX), y \exp(tX) \in R$" for every $X \in L$. It is clear that R is necessarily the relation whose equivalence classes are the cosets Kx, where $x \in M$, and K is a closed subgroup of M. Therefore R is regular.

To obtain our generalization we no longer require M to be a Lie group, and allow L to be an arbitrary transitive Lie algebra of vector fields on M. Those vector fields X whose corresponding (local) one-parameter groups of diffeomorphisms map equivalent elements to equivalent elements are called *symmetry vector fields* of R. It turns out that all that is needed for R to be regular is that R be closed in $M \times M$ and that the set of symmetry vector fields of R be "sufficiently large". This is the content of Theorem 8. Moreover, the converse is also true: if R is regular, and M/R is Hausdorff, then R is closed in $M \times M$ and there are sufficiently many symmetry vector fields. However, our proof of this last fact depends on the use of the standard C^∞ machinery (partitions of unity, etc.), and we do not know whether a similar result is valid in the real analytic case (cf. Theorem 10).

Our presentation of the results will first discuss the local problem, i.e., that of characterizing *locally regular* equivalence relations. We present a necessary and sufficient condition in Theorem 5. From this result, we derive a necessary and sufficient condition for regularity in terms of symmetry vector fields which need not be everywhere defined (Theorem 6). Theorem 8 then follows as a corollary. So far, all the results are valid both in the C^∞ and in the real analytic case. In § 7 we prove (in the C^∞ case) that the sufficient condition of Theorem 8 is also necessary (Theorem 10).

In accordance with the standard terminology of Palais [1] or Serre [2], we have chosen to allow in our definition of regularity the possibility of a non-Hausdorff quotient. However, the manifold M on which R is defined will always be

Hausdorff. As the reader will see, integral trajectories of vector fields on M are used throughout. The condition that M be Hausdorff is needed to guarantee the uniqueness of such trajectories.

We have formulated our main "global" results (Theorems 6, 8, 9, and 10) in terms of "almost regularity" rather than "regularity". Equivalently, we are allowing M/R to have connected components of different dimensions. When M is connected, the distinction becomes unnecessary (cf. Theorem 7).

Finally, we show in § 8 how a characterization of the same type is possible for relations R such that the canonical projection π_R is a fibre map. The necessary and sufficient condition is that R be locally closed and have "sufficiently many" complete symmetry vector fields. The classical theorem of Ehresmann on proper submersions is a particular case of our Theorem 11.

2. Notations and definitions

Throughout this paper, the word "manifold" means "finite dimensional paracompact manifold". All manifolds are assumed to be Hausdorff, unless an explicit statement to the contrary is made. All manifolds considered will be C^∞ or C^ω (real analytic). A *submanifold* of a manifold M is a manifold N which is a subset of M and for which the inclusion $N \to M$ is an immersion. If, in addition, N is a topological subspace of M, we shall refer to it as a *regular* submanifold. We use M_x to denote the tangent space of the manifold M at the point x.

The real line is denoted by $\boldsymbol{R}$, and n-dimensional Euclidean space by $\boldsymbol{R}^n$. The expression C_ε^n denotes the cube

$$\{(t_1, \cdots, t_n) : |t_i| < \varepsilon,\ i = 1, \cdots, n\} .$$

Let M be a C^∞ manifold. We use $V_e^\infty(M)$ to denote the set of all C^∞ vector fields on M (the subscript refers to the fact that they are defined everywhere). We use $V^\infty(M)$ to denote the union of all the sets $V_e^\infty(\Omega)$, where Ω ranges over all open subsets of M. If M is C^ω, then the sets $V_e^\omega(M)$ and $V^\omega(M)$ of real analytic vector fields are defined in a similar way.

If $X \in V^\infty(M)$, and $x \in M$ belongs to the domain of X, then we use $t \to X_t(x)$ to denote the integral curve of X which goes through x when $t = 0$.

Let R be an equivalence relation on M. The equivalence class of an $x \in M$ is denoted by $R(x)$. We shall always view R as a subset of $M \times M$, so that the expressions "$x \in R(y)$", "$(x, y) \in R$" and "x is R-equivalent to y" are used interchangeably. A vector field $X \in V^\infty(M)$ with domain $\Omega \subseteq M$ is said to be a *symmetry vector field* of R if, for every $x \in \Omega$ and $y \in \Omega$, $t \in \boldsymbol{R}$ such that $X_t(x)$ and $X_t(y)$ are both defined, then

$$(x, y) \in R \quad \text{implies} \quad (X_t(x), X_t(y)) \in R .$$

We use $S^\infty(R, M)$ to denote the set of all symmetry vector fields of R. The set of all $X \in S^\infty(R, M)$ which are everywhere defined is denoted by $S_e^\infty(R, M)$. If M is real analytic, then the sets $S^\omega(R, M)$, $S_e^\omega(R, M)$ are defined in an obvious way.

If R is an arbitrary equivalence relation on M, then $S_e^\infty(R, M)$ is a subalgebra of the Lie algebra $V_e^\infty(R, M)$. We state this fact, which will be proved in § 3, as a lemma.

Lemma 1. *The set of everywhere defined symmetry vector fields of an equivalence relation R is a Lie algebra of vector fields.*

If L is a subset of $V^\infty(M)$, and $x \in M$, then we use $L(x)$ to denote the set of all vectors $v \in M_x$ which are of the form $X(x)$ for some $X \in L$. If the linear hull of $L(x)$ is all of M_x, we say that L has *maximal rank* at x. If L has maximal rank at every $x \in M$, we say that L is *transitive*. If A is a subset of $M \times M$ and, for every pair $(x, x') \in A$, the set of all $X \in L$ whose domain contains x' has maximal rank at x, then we say that L is *A-transitive*. If A is an equivalence relation, then A-transitivity is a stronger condition than transitivity.

If R is an equivalence relation on the C^∞ manifold M, let M/R denote the quotient of M by R with the quotient topology, and let $\pi_R: M \to M/R$ denote the canonical projection. If M/R admits a C^∞ structure in such a way that the map π_R (from M onto the not necessarily Hausdorff manifold M/R) is a submersion, we say that R is *C^∞-regular*. It is well known (and trivial), that the differentiable structure on M/R for which π_R is a submersion is unique. Moreover, M/R is Hausdorff if and only if R is closed as a subset of $M \times M$. If M is real analytic, and M/R can be given a C^ω structure such that π_R is a C^ω-submersion, then we shall say that R is *C^ω-regular*.

It is well known that R is regular if and only if R is a regular submanifold of $M \times M$ in such a way that the projection $(x^1, x^2) \to x^1$ from R onto M is a submersion (cf. Serre [2, Part II, Chap. 3, § 12, Theorem 2]).

If Ω is an open subset of M, then the *restriction* of R to Ω is the equivalence relation $R_\Omega = R \cap (\Omega \times \Omega)$. Clearly $S^\infty(R_\Omega, \Omega)$ is the set of all $X \in S^\infty(R, M)$ whose domain of definition is contained in Ω. We say that R is *locally regular* if every $x \in M$ has a neighborhood Ω such that R_Ω is regular. We say that R is *closed* if it is closed as a subset of $M \times M$, and that it is *locally closed* if every $x \in M$ has a neighborhood Ω such that R_Ω is closed in $\Omega \times \Omega$.

3. Proof of Lemma 1

We recall some definitions and results from Sussmann [4] (announced in Sussmann [3]). Let F be a set of vector fields on a manifold N. Let $y \in N$. Let S be the smallest set with the property that $y \in S$ and that, whenever $y' \in S$ and $X \in F$ are such that y' is in the domain of X, then $X_t(y')$ is in S for all t for which it is defined. The set S is called the *F-orbit* of y. The main result of [3] is that if S is an F-orbit, then S can be given a topology and a compatible

C^∞ structure such that, for each $y \in S$, $X \in F$, the map $t \to X_t(y)$ is continuous, and that S is a submanifold of N. From this it follows easily that every $X \in F$ is tangent to S.

If N is a manifold, and B is an arbitrary subset of N, then let F be the set of all vector fields X on N such that

$$\text{“}y \in B,\ X_t(y) \text{ defined” implies } X_t(y) \in B \ .$$

It is clear that if $y \in B$, then the F-orbit of y is contained in B. From this it follows that B is a union of F-orbits. The manifold N is partitioned into F-orbits, and it is clear that the set G of all vector fields X tangent to the F-orbits is a Lie algebra which contains F. Since B is a union of F-orbits, it follows that $G \subseteq F$. Therefore $F = G$, so that F is a Lie algebra.

To prove Lemma 1, we let $N = M \times M$ and $B = R$. Then F consists of all vector fields on $M \times M$ whose integral curves through a point of R are entirely contained in R. If X is a vector field on M, we can associate with it a vector field $X \oplus X$ on $M \times M$ by letting $(X \oplus X)(x, x') = (X(x), X(x'))$ (recall that $(M \times M)_{(x,x')} \simeq M_x \times M_{x'}$ canonically). Clearly, X belongs to $S_e^\infty(R, M)$ if and only if $(X \oplus X) \in F$. Since the map $X \to X \oplus X$ is a Lie algebra homomorphism, and F is a Lie algebra, we conclude that $S_e^\infty(R, M)$ is a Lie algebra, and the proof of Lemma 1 is complete.

4. Limiting directions

The main result of this section is that if there are sufficiently many symmetry vector fields of R, and R is closed, then every “limiting direction” of the equivalence class of x is in fact the direction of a curve $t \to X_t(x)$, where X is a symmetry vector field of R and $X_t(x)$ is equivalent to x for all t. We begin by giving a precise definition of “limiting direction”.

Let U be an open subset of R^n, and A a subset of U. Let $p \in A$. We say that a vector $v \in R^n$ is a *limiting direction* of A at p if $v = 0$ or there exists a sequence $\{p_j,\ j = 1, 2, \cdots\}$ of points of $A - \{p\}$ such that p_j goes to p as $j \to \infty$ and that

$$\operatorname*{Lim}_{j\to\infty} \frac{p_j - p}{\|p_j - p\|} = \frac{v}{\|v\|} \ .$$

Lemma 2. *Let $\varphi\colon U \to V$ be a diffeomorphism, and let $B = \varphi(A)$. Let v be a limiting direction of A at p, and let J denote the Jacobian matrix of φ at p. Then $J \cdot v$ is a limiting direction of B at $\varphi(p)$.*

Proof. If $v = 0$, the conclusion is trivial. Assume that $v \neq 0$. Then we have

$$\varphi(p + h) = \varphi(p) + J \cdot h + o(\|h\|) \ .$$

Let $\{p_j : j = 1, 2, \cdots\}$ be a sequence such that

$$p_j \neq p \, , \qquad p_j \to p \, , \qquad \operatorname*{Lim}_{j\to\infty} \frac{p_j - p}{\|p_j - p\|} = \frac{v}{\|v\|} \; .$$

Then

$$\varphi(p_j) - \varphi(p) = J\cdot(p_j - p) + o(\|p_j - p\|) \, ,$$

and therefore

$$\operatorname*{Lim}_{j\to\infty} \frac{\varphi(p_j) - \varphi(p)}{\|p_j - p\|} = J\cdot\left(\operatorname*{Lim}_{j\to\infty} \frac{p_j - p}{\|p_j - p\|}\right) = J\cdot\frac{v}{\|v\|} \; .$$

In particular,

$$\operatorname*{Lim}_{j\to\infty} \frac{\|\varphi(p_j) - \varphi(p)\|}{\|p_j - p\|} = \frac{\|Jv\|}{\|v\|} \; .$$

The last two equations imply that

$$\operatorname*{Lim}_{j\to\infty} \frac{\varphi(p_j) - \varphi(p)}{\|\varphi(p_j) - \varphi(p)\|} = \frac{Jv}{\|Jv\|} \, . \qquad \text{q.e.d.}$$

The preceeding lemma shows that the statement that v is a limiting direction of A at x is invariantly defined if A is a subset of a manifold M, $x \in A$, and $v \in M_x$. We shall use $L(A, x)$ to denote the set of limiting directions of A at x.

Lemma 3. *Let $A \subseteq M$, and $x \in A$. Let $\{x_1, \cdots, x_n\}$ be a coordinate chart in a neighborhood of x, such that $x_1(x) = \cdots = x_n(x) = 0$. Then*

$$\frac{\partial}{\partial x_1}(x) \in L(A, x)$$

if and only if there exists a sequence $x^1, x^2, \cdots$ of points of A, different from x, such that $x_1(x^j) > 0$ for all j,

$$\operatorname*{Lim}_{j\to\infty} x_1(x^j) = 0 \, ,$$

and $x_i(x^j) = o(x_1(x^j))$ as $j \to \infty$ for $i = 2, \cdots, n$.

Proof. Immediate.

Theorem 4. *Let $R \subseteq M \times M$ be an equivalence relation. Assume that R is closed in $M \times M$, and that $S_e^\infty(R, M)$ is transitive. Let $X \in S_e^\infty(R, M)$, $x \in M$. Assume that $X(x) \in L(R(x), x)$. Then $X_t(x) \in R(x)$ for all real t for which $X_t(x)$ is defined.*

Proof. The conclusion is trivial if $X(x) = 0$. Assume that $X(x) \neq 0$. Let $X^1 = X$, and let $X^2, \cdots, X^n$ be elements of $S_e^\infty(R, M)$ such that $\{X^1(x), \cdots, X^n(x)\}$ is a basis for M_x. The mapping

$$(t_1, \cdots, t_n) \to X^n_{t_n}(X^{n-1}_{t_{n-1}} \cdots X^1_{t_1}(x) \cdots)$$

defines a diffeomorphism of a cube C^n_ε onto a neighborhood U of x, and the inverse of this mapping defines a chart on U. Clearly

$$X(x) = \frac{\partial}{\partial t_1}(x) .$$

By Lemma 3 there is a sequence $\{x^j\}$ in $U \cap R(x)$ such that $t_1(x^j) \to 0$ for all j, and that, as $j \to \infty$, $t_1(x^j) > 0$ and $t_2(x^j), \cdots, t_n(x^j)$ are $o(t_1(x^j))$. From now on we let $t_i^j = t_i(x^j)$. We choose $\alpha > 0$ so that the following holds:

(*E*) For every n-tuple $(u_1, \cdots, u_n)$ of measurable real-valued functions defined in the interval $[0, \alpha]$ with values in $[-1, 1]$, the equation

$$(\sharp) \qquad \dot{\xi} = \sum u_i X_i$$

has a solution (i.e., an absolutely continuous curve $t \to \xi(t)$ whose tangent vector at t is

$$\sum_{i=1}^{n} u_i(t) X^i(\xi(t))$$

for almost every t) with values in U, and such that $\xi(0) = x$. The domain of definition of this solution is the interval $[0, \alpha]$.

The proof that such an α exists is an easy application of the usual successive approximations methods for existence of solutions of ordinary differential equations. It is clear that the solution ξ referred to in (*E*) is unique. Moreover, the following can be proved easily:

(*C*) Let u_i^j, u_i ($i = 1, \cdots, n$, $j = 1, 2, \cdots$) be measurable functions from $[0, \alpha]$ to $[-1, 1]$, and let ξ_j, ξ be the solutions of ($\sharp$) corresponding to $(u_1^j, \cdots, u_n^j)$, $(u_1, \cdots, u_n)$ respectively. Assume that, for each i,

$$\operatorname*{Lim}_{j \to \infty} u_i^j = u_i \qquad \text{weakly} .$$

Then

$$\operatorname*{Lim}_{j \to \infty} \xi_j(t) = \xi(t) \qquad \text{for } 0 \leq t \leq \alpha .$$

We shall apply (*C*) for a particular sequence which we now define. Let

$$T_j = t_1^j + |t_2^j| + \cdots + |t_n^j| .$$

Let ν_j be the largest integer such that $\nu_j T_j \leq \alpha$. Let $u_1^j(\tau) = 1$, $u_j^2(\tau) = \cdots = u_j^n(\tau) = 0$ for $0 \leq \tau < t_1^j$. Also, let $u_i^j(\tau) = 0$ for $i \neq l$, and $u_l^j(\tau) = \sigma_l^j$ for $t_1^j + \cdots + |t_{l-1}^j| \leq \tau < t_1 + \cdots + |t_l^j|$, where $\sigma_l^j = 1$ or -1 according as $t_l^j > 0$ or $t_l^j < 0$.

In this way, the functions u_i^j are defined on the interval $[0, T_j)$. We extend them to $[0, \nu_j T_j)$ by requiring that they be periodic with period T_j. Finally, we let $u_i^j(\tau) = 0$ for all i if $\nu_j T_j \leq \tau \leq \alpha$. The functions $u_i^j(\tau)$ are now defined for $0 \leq \tau \leq \alpha$, and are certainly measurable with values in $[-1, 1]$. As in the statement of (C), we let ξ_j be the solution of $(\sharp)$ which corresponds to $(u_1^j, \cdots, u_n^j)$, and for which $\xi_j(0) = x$. It is clear from the construction of the u_i^j that if ν is an integer such that $0 \leq \nu < \nu_j$, then

$$\xi_j((\nu + 1)\cdot T_j) = X^n_{t_n^j} X^{n-1}_{t_{n-1}^j} \cdots X^1_{t_1^j}(\xi_j(\nu T_j)) ,$$

from which, together with the fact that X^i are symmetry vector fields of R, we conclude that

$$(\xi_j(\nu T_j), \xi_j((\nu + 1)T_j)) \in R$$

implies

$$(\xi_j(\nu + 1)T_j), \xi_j((\nu + 2)T_j) \in R$$

if $0 \leq \nu \leq \nu_j - 2$. But $\xi_j(T_j)$ is x^j, which is R-equivalent to $\xi_j(0)$ (i.e., x). Therefore $\xi_j(\nu_j T_j) \in R(x)$. Since the u_i^j vanish for $\nu_j T_j \leq \tau \leq \alpha$, we conclude that $\xi_j(\alpha) = \xi_j(\nu_j T_j)$, so that $\xi_j(\alpha) \in R(x)$.

As $j \to \infty$, it is clear that $u_1^j \to 1$ weakly, and that $u_2^j, \cdots, u_n^j$ converge weakly to 0 (in fact, this is also true, for instance, with "weakly" replaced by "in L^1 norm"). If we let $u_1 \equiv 1$, $u_2 \equiv \cdots \equiv u_n \equiv 0$, and ξ denotes the corresponding solution of $(\sharp)$ with $\xi(0) = x$, then it follows from (C) that $\xi_j(\alpha) \to \xi(\alpha)$ as $j \to \infty$. Since $\xi_j(\alpha) \in R(x)$ and $R(x)$ is closed, we conclude that $\xi(\alpha) \in R(x)$. But $\xi(\alpha)$ is simply $X_\alpha(x)$. Moreover, if $\alpha > 0$ is such that (E) holds, then the same is true for any t such that $0 < t \leq \alpha$. Therefore, we have shown that $X_t(x) \in R(x)$ for every t in the interval $[0, \alpha]$. Using the fact that X is a symmetry vector field of R, it follows immediately that $X_t(x) \in R(x)$ for every $t \in R$ for which $X_t(x)$ is defined.

5. Local regularity

We now state and prove our necessary and sufficient condition for local regularity.

Theorem 5. *Let M be a C^∞ manifold, and let $R \subseteq M \times M$ be an equivalence relation. Then R is locally C^∞-regular if and only if* (i) *and* (ii) *below hold*:

(i) *R is locally closed.*

(ii) *$S^\infty(R, M)$ is transitive.*

If M is real analytic, then R is locally C^ω-regular if and only if (i) *holds and* $S^\omega(R, M)$ *is transitive.*

Proof. Suppose that R is locally regular. Then every point has a neighborhood U in which there is a chart $(x_1, \cdots, x_n)$ such that R_U is simply the relation whose equivalence classes are the sets defined by $x_1 = a_1, \cdots, x_k = a_k$, where $a_1, \cdots, a_k$ are constants. From this it is clear that R_U is closed in $U \times U$, and that $S_e^\infty(R_U, U)$ is transitive (it contains the vector fields $\partial/\partial x_1, \cdots, \partial/\partial x_n$). The necessity of conditions (i) and (ii) follows.

To prove the converse, we can replace M, is necessary, by an open subset of M. Therefore we can assume that R is closed, that $S_e^\infty(R, M)$ is transitive, and that M is connected. Let $\mu(x)$ denote, for $x \in M$, the dimension of the linear hull of the set $L(R(x), x)$ of limiting directions of $R(x)$ at x. Since $S^\infty(R, M)$ is transitive and M connected, it is clear that, given any two points x, x' in M, there is a diffeomorphism $\varphi\colon U \to U'$ (where U, U' are neighborhoods of x, x') such that $\varphi(U \cap R(x)) = U' \cap R(x')$. Therefore (cf. Lemma 2) $\mu(x) = \mu(x')$, and the number $\mu(x)$ is in fact a constant μ independent of x. We let $n = \dim M$, $k = n - \mu$.

Let $x \in M$. Let $X^1, \cdots, X^\mu$ be vector fields in $S_e^\infty(R, M)$ such that $\{X^1(x), \cdots, \cdots, X^\mu(x)\}$ is a maximal linearly independent subset of $L(R(x), x)$. Choose $X^{\mu+1}$, X^n also in $S_e^\infty(R, M)$ such that $(X^1(x), \cdots, X^n(x))$ is a basis for M_x. Choose ε so small that the mapping Φ defined by

$$(t_1, \cdots, t_n) \to X^n_{t_n} \cdots X^1_{t_1}(x)$$

is a diffeomorphism from the cube C_ε^n onto a neighborhood U of x. Identify C_ε^n with $C_\varepsilon^\mu \times C_\varepsilon^k$ in the obvious way. We shall show that if $\delta > 0$ (and $\leq \varepsilon$) is sufficiently small, then the restriction of Φ to $C_\varepsilon^\mu \times C_\delta^k$ has the property that $\Phi(y, z)$ and $\Phi(y', z')$ are R-equivalent if and only if $z = z'$. From this it is clear that the restriction of R to $\Phi(C_\varepsilon^\mu \times C_\delta^k)$ is regular, and the sufficiency of (i) and (ii) for local regularity follows.

To show that a δ with the desired property exists, we first show that $\Phi(y, 0) \in R(x)$ for $y \in C_\varepsilon^\mu$. Let $y = (t_1, \cdots, t_\mu)$. By Theorem 4, $X^1_{t_1}(x)$ is R-equivalent to x. Since X^2 is a symmetry vector field for R, we conclude that $X^2_{t_2}X^1_{t_1}(x)$ is R-equivalent to $X^2_{t_2}(x)$. But, from Theorem 4 again, it follows that $X^2_{t_2}(x) \in R(x)$, so that $X^2_{t_2}X^1_{t_1}(x) \in R(x)$. Repeating this reasoning $\mu - 2$ more times, we conclude that $X^\mu_{t_\mu}X^{\mu-1}_{t_{\mu-1}} \cdots X^1_{t_1}(x)$ (which is precisely $\Phi(y, 0)$) belongs to $R(x)$.

It is an immediate consequence of the preceding paragraph that $\Phi(y, 0)$ and $\Phi(y', 0)$ are R-equivalent whenever $y \in C_\varepsilon^\mu, y' \in C_\varepsilon^\mu$. Using the fact that $X^{\mu+1}, \cdots, X^n$ are symmetry vector fields of R, we conclude that $(\Phi(y, z), \Phi(y', z)) \in R$ whenever $y \in C_\varepsilon^\mu$, $y' \in C_\varepsilon^\mu$, $z \in C_\varepsilon^k$.

There remains to be shown that if δ is sufficiently small, then $\Phi(y, z)$ and $\Phi(y', z')$ $(y, y' \in C_\varepsilon^\mu, z, z' \in C_\delta^k)$ cannot be equivalent unless $z = z'$. It this were not the case, there would exist sequences y_j^1, y_j^2 of points in C_ε^μ, and z_j^1, z_j^2 of points in C_ε^k such that $z_j^1 \to 0$, $z_j^2 \to 0$ as $j \to \infty$, $z_j^1 \neq z_j^2$, and that $(\Phi(y_j^1, z_j^1), \Phi(y_j^2, z_j^2)) \in R$. Since $(\Phi(y_j^i, z_j^i), \Phi(0, z_j^i)) \in R$ for $i = 1, 2$, we see that $(\Phi(0, z_j^1), (\Phi(0, z_j^2)) \in R$. Let $z_j^1 = (\tau_j^1, \cdots, \tau_j^k)$. If j is sufficiently large, then the diffeomorphism $\Psi_j = X_{-\tau_j^1}^{\mu+1} \cdots X_{\tau_j^{k-1}}^{n-1} X_{-\tau_j^k}^n$ is defined on a fixed neighborhood $U' \subseteq U$ of x, and $\Psi_j(U') \subseteq U$. Therefore for sufficiently large j there exist $y_j^3 \in C_\varepsilon^\mu$ and $z_j^3 \in C_\varepsilon^k$ such that

$$(*) \qquad \Psi_j(\Phi(0, z_j^2)) = \Phi(y_j^3, z_j^3) .$$

It is clear that $\Psi_j(\Phi(0, z_j^1)) = x$. Since $\Phi(0, z_j^1)$ and $\Phi(0, z_j^2)$ are R-equivalent, it follows that $\Phi(y_j^3, z_j^3) \in R(x)$. Therefore $\Phi(0, z_j^3) \in R(x)$. We show that z_j^3 cannot vanish if j is sufficiently large. Indeed, if $z_j^3 = 0$, then $(*)$ implies that

$$\Phi(0, z_j^2) = X_{\tau_j^k}^n X_{t_j^{k-1}}^{n-1} \cdots X_{\tau_j^1}^{\mu+1} \Phi(y_j^3, 0) = \Phi(y_j^3, z_j^1) .$$

Therefore $z_j^2 = z_j^1$, which is a contradiction. Thus we have found a sequence $\{z_j^3\}$ which converges to zero, and is such that $z_j^3 \neq 0$, $\Phi(0, z_j^3) \in R(x)$. Replacing it by a subsequence, if necessary, we can assume that the limit of $z_j^3/\|z_j^3\|$ exists. Let v be this limit. Then $w = d\Phi(v)$ is a nonzero limiting direction of $R(x)$ at x. Moreover, w is clearly tangent to the manifold $\Phi(\{0\} \times C_\varepsilon^k)$ at x. Therefore w cannot belong to the linear hull of $X^1(x), \cdots, X^\mu(x)$, and this is a contradiction.

The proof of our theorem is complete, except for the final remarks on real analyticity. However, the preceding proof goes through without any change if "C^∞" is replaced throughout by "C^ω".

6. Regularity

We now give necessary and sufficient contitions for regularity. However, we shall state our theorem in terms of "almost regularity" rather than regularity. We say that the equivalent relation R on M is *almost C^∞-regular* if every connected component M' of M/R can be given a C^∞ differentiable structure in such a way that the canonical projection from $\pi_R^{-1}(M')$ onto M' is a submersion. In other words, R is almost regular if the quotient M/R is a manifold, whose connected components are allowed to be of different dimensions.

In the statement of our main theorem we shall refer to the following "homogeneity condition" (H). Suppose that R is locally regular. It is then clear that the equivalence classes of R are regular submanifolds. If $x \in M$, two regular submanifolds S and T will be said to be *transversal at* x if $S_x \oplus T_x = M_x$. We shall say that R *satisfies condition* (H) if, given any two points x, y, such that $(x, y) \in R$, there exist regular submanifolds S, T transversal to $R(x)$ at x and y, respectively, and a diffeomorphism Φ from S onto T such that $\Phi(x) = y$ and that $(z, \Phi(z)) \in R$ for all $z \in S$.

Theorem 6. *Let R be an equivalence relation in the n-dimensional C^∞ manifold M. Then the following conditions are equivalent:*

(i) *R is almost C^∞-regular;*

(ii) *R is locally C^∞-regular and condition (H) holds,*

(iii) *R is locally closed and $S^\infty(R, M)$ is R-transitive.*

Proof. We first show that (i) implies (iii). If (i) holds, it is clear that R is locally regular, so that R is locally closed and $S^\infty(R, M)$ is transitive. To show that $S^\infty(R, M)$ is R-transitive, let $(x, y) \in R$, $v \in M_x$. We must find $X \in S^\infty(R, M)$ such that X is defined both at x and y, and that $X(x) = v$. If $x = y$, this is trivially possible, because $S^\infty(R, M)$ is transitive. Assume that $x \neq y$. Let $z = \pi_R(x)\mathscr{D}$. Choose coordinates $z'_1, \cdots, z'_k$ in a neighborhood U' of z in M/R, and let $U = \pi_R^{-1}(U')$, $z_i = z'_i \circ \pi_R$, $i = 1, \cdots, k$. Then the functions $z_1, \cdots, z_k$ are defined on U, and the differentials $dz_1, \cdots, dz_k$ are linearly independent. Clearly, it is possible to define C^∞ functions $z_{k+1}, \cdots, z_n$ in an open set V which contains both x and y, such that $dz_1, \cdots, dz_n$ are linearly independent at every point of V. By letting X be an appropriate linear combination with constant coefficients of the elements of the basis dual to $\{dz_1, \cdots, dz_n\}$, it is possible to have $X(x) = v$. Moreover, with X so defined it is clear that $\langle X, dz_i \rangle$ is a constant for every i. Therefore $X \in S^\infty(R, M)$. This completes the proof that (i) $\Rightarrow$ (iii).

We now show that (iii) $\Rightarrow$ (ii). If (iii) holds, it follows from Theorem 5 that R is locally C^∞-regular. We must show that condition (H) holds. Let $(x, y) \in R$. Let P be a linear subspace of M_x such that $P \oplus R(x)_x = M_x$. By the R-transitivity of $S^\infty(R, M)$, there exist vector fields $X^1, \cdots, X^k$ in $S^\infty(R, M)$ which are defined both at x and y such that $\{X^1(x), \cdots, X^k(x)\}$ is a basis for P. We now show that $X^1(y), \cdots, X^k(y)$ are linearly independent modulo $R(y)_y$. Let $c_1, \cdots, c_k$ be such that $c_1X^1(y) + \cdots + c_kX^k(y)$ belongs to $R(y)_y$. If we let $X = c_1X^1 + \cdots + c_kX^k$, then it follows that $X \in S^\infty(R, M)$ and that $X(y)$ is a limiting direction of $R(y)$ at y. By Theorem 4, $X_t(y) \in R(y)$ for all t for which it is defined. Since X is a symmetry vector field for R, and $(x, y) \in R$, we conclude that $X_t(x) \in R(x)$ for sufficiently small t. Therefore $X(x) = c_1X^1(x) + \cdots + c_kX^k(x)$ belongs to $R(x)_x$. Since $X^1(x), \cdots, X^k(x)$ are linearly independent modulo $R(x)_x$, we obtain that $c_1 = \cdots = c_k = 0$, as we wanted to show. The preceding considerations imply, in particular, that $\dim (M_x/R(x)_x) \leq \dim (M_y/R(y)_y)$. Since the roles of x and y can clearly be reversed, it follows that both dimensions are equal. If we let Q be the linear hull of $X^1(y), \cdots, X^k(y)$, we have $Q \oplus R(y)_y = M_y$. Moreover, $X^1(y), \cdots, X^k(y)$ form a basis for Q. If $\varepsilon > 0$ is small enough, then the mappings F, G defined by

$$F(t_1, \cdots, t_k) = X^1_{t_1} \cdots X^k_{t_k}(x) , \qquad G(t_1, \cdots, t_k) = X^1_{t_1} \cdots X^k_{t_k}(y)$$

are diffeomorphisms of the cube C^k_ε onto regular submanifolds S, T through x, y respectively. Clearly $S_x = P$, $T_y = Q$ so that the transversality requirement of condition (H) holds. Finally, if we let $\Phi = G \circ F^{-1}$, then Φ is the desired diffeomorphism, and (H) holds.

We now complete the proof by showing that (ii) implies (i). Assume that (ii) holds. We first show that π_R maps open sets onto open sets. To prove this, it is sufficient to show that if $(x, y) \in R$, U is open, and $x \in U$, then there is an open set V such that $y \in V$ and that every point of V is equivalent to some point of U. Let S, T be transversal to $R(x)$, $R(y)$ at x, y respectively, and let $\Phi : S \to T$ be such that $(s, \Phi(s)) \in R$ for all $s \in S$. Moreover, we can assume that S is contained in U. Since R is locally regular, there is a coordinate chart $\{y^1, \cdots, y^n\}$ in a neighborhood W of y, such that two points of W are R-equivalent if and only if they have the same $y_{k+1}, \cdots, y_n$ coordinates. Moreover, we can assume that $y_1(y) = \cdots = y_n(y) = 0$. Since T is transversal to $R(y)$, there is a neighborhood Z of y in T such that the mapping

$$z \to (y^1(z), \cdots, y^k(z))$$

is a diffeomorphism from Z onto the cube C^k_ε. If V is the subset of W whose elements are the points for which $|y^i| < \varepsilon$ for $i = 1, \cdots, k$, then V is open, $y \in V$, and every point of V is R-equivalent to a point of T and hence to a point of S. Since $S \subseteq U$, the assertion that π_R is open is proved.

Since R is locally regular, we can cover M by open sets U_α such that R, restricted to U_α, is regular. Let $V_\alpha = \pi_R(U_\alpha)$. By the preceding paragraph, the V_α are open. Moreover, we have a canonical homeomorphism of V_α with U_α / R_{U_α}. Since this space is a C^∞ manifold, we conclude that M/R is covered by the open sets V_α, and that each V_α has a C^∞ structure such that π_R, restricted to U_α, is a submersion onto V_α. To complete our proof, we must show that the differentiable structures of the V_α are compatible. Let $z \in V_\alpha \cap V_\beta$. We show that the identity map I from $V_\alpha \cap V_\beta$, considered as an open submanifold of V_α, into $V_\alpha \cap V_\beta$, considered as an open submanifold of V_β, is C^∞ in a neighborhood of z. Let $z = \pi_R(x) = \pi_R(y)$ with $x \in U_\alpha$, $y \in U_\beta$. Let S, T be as in the statement of condition (H). It is clear that, by shrinking S and T if necessary, we can assume that $S \subseteq U_\alpha$, $T \subseteq U_\beta$, that π_R is a diffeomorphism of S onto an open neighborhood Z of z, considered as a submanifold of V_α, and that a similar statement is true for T and V_β. Since $I = \pi_R \circ \Phi \circ \pi_R^{-1}$, the conclusion that I is C^∞ follows. q.e.d.

When M is connected, it is clear that regularity is equivalent to almost regularity. The following is, therefore, a trivial corollary of Theorem 6.

Theorem 7. *Let R be an equivalence relation on the connected C^∞ manifold M. Then R is regular if and only if R is locally closed and $S^\infty(R, M)$ is R-transitive.*

We have a very important particular case of Theorem 7 when there are sufficiently many symmetry vector fields of R which are everywhere defined. We state this result separately.

Theorem 8. *Let M be a C^∞ manifold, and R an equivalence relation on M. Assume that R is locally closed, and that the set $S^\infty_e(R, M)$ of everywhere* *defined C^∞ symmetry vector fields of R is transitive. Then R is almost C^∞ regular. If M is connected, then R is regular.*

As we shall show in § 7, the converse of Theorem 8 is also true, but cf. Remark (c) below.

We can get a seemingly stronger form of Theorem 8 by observing that $S_e^\infty(R, M)$ is a Lie algebra (Lemma 1). Let us say that a set S of vector fields is *weakly transitive* if the Lie algebra generated by S is transitive. The following is then an obvious consequence of Theorem 8.

Theorem 9. *Let M be a C^∞ manifold, and R an equivalence relation on M. Assume that R is locally closed, and that there is a set of everywhere defined symmetry vector fields of R which is weakly transitive. Then R is almost C^∞-regular. If M is connected, then R is regular.*

Remarks. (a) If M is the real line, and R is the equivalence relation which identifies the points -1 and 1, then R is locally regular but not regular. This trivial example shows the role of the conditions of Theorem 6. It is easy to see that $S^\infty(R, M)$ is not R-transitive (but, of course, it is transitive). Also, condition (H) does not hold. In fact, the canonical quotient map π_R does not map open sets to open sets.

(b) Since manifolds whose connected components have different dimensions appear anyhow in the conclusions of Theorems 6, 8 and 9, it would perhaps be desirable to generalize our results so that the original manifold M is also allowed to have components of different dimensions. This generalization offers no difficulty, and all our proofs apply without any change.

(c) The results of this section only involve those properties of C^∞ functions which are also valid for real analytic functions. It follows that Theorems 6, 7, 8 and 9 are also true with "C^∞" replaced throughout by "C^ω" (and S^∞, S_e^∞ replaced by S^ω and S_e^ω). However, we do not know whether Theorem 10 is also true in the C^ω case.

7. Converse of Theorem 8

In this section we show that the sufficient condition of Theorem 8 is also necessary. As we remarked before, the proof depends on properties of C^∞ functions which are not shared by C^ω functions.

Theorem 10. *Let M be a C^∞ manifold, and let R be an equivalence relation on M which is almost C^∞-regular. Then the set $S_e^\infty(R, M)$ of everywhere defined symmetry vector fields of R is transitive.*

Proof. We can assume that M is connected. Let M' be the quotient M/R. Let $n = \dim M$, and $k = \dim M'$. Let $x^0 \in M$. Then there exists a coordinate chart $(x_1, \cdots, x_n)$ defined in a neighborhood U of x^0 such that: (a) $x \to (x_1(x), \cdots, x_n(x))$ is a diffeomorphism from U onto C_1^n; (b) $x_1(x^0) = \cdots = x_n(x^0) = 0$, and (c) two points x^1, x^2 in U are R-equivalent if and only if $x_1(x^1) =$

$x_1(x^2), \cdots, x_k(x^1) = x_k(x^2)$. We shall show that, for $j = 1, \cdots, n$, there exists a vector field $X_j \in S_e^\infty(R, M)$ such that

$$X_j(x^0) = \frac{\partial}{\partial x_j}(x^0) .$$

For $k + 1 \leq j \leq n$ this is trivial. Indeed, let f be a C^∞ function on M, such that $f(x^0) = 1$ and that the support of f is a compact subset of U. We let

$$X_j = f\frac{\partial}{\partial x_j} .$$

It is clear that X_j is tangent to all the equivalence classes of R, and therefore $X_j \in S_e^\infty(R, M)$.

For $1 \leq j \leq k$, the preceding argument does not work, and a slightly more involved reasoning is needed. Let $U' = \pi_R(U)$, and $V = \pi_R^{-1}(U')$. Clearly, we can define functions $x_1', \cdots, x_k'$ on U' such that $x_j'(\pi_R(x)) = x_j(x)$ for $x \in U$, $1 \leq j \leq k$. The preceding equation can then be used to define $x_j(x)$ for every $x \in V$. It is clear that the functions $x_1', \cdots, x_k'$ define a coordinate chart for U'. Since π_R is a submersion, it follows that the differentials $dx, \cdots, dx_k$ are linearly independent at every point of V. The standard proof that every manifold has a Riemannian metric can be used to show the existence of a Riemannian metric on V with respect to which

$$(\#) \qquad \langle dx_i, dx_j \rangle = \delta_{ij} , \qquad 1 \leq i, j \leq k .$$

Moreover, we can assume that, in a neighborhood of x^0, (#) holds for $1 \leq i, j \leq n$.

Using the Riemannian metric, define Y_j to be the gradient of x_j for $j = 1, \cdots, k$. Then Y_j is a C^∞ vector field on V. Let g be a C^∞ function in the cube C_1^k, whose support is compact, and for which $g(0) = 1$. Define

$$X_j(x) = g(x_1(x), \cdots, x_k(x))Y_j(x)$$

for $x \in V$, $1 \leq j \leq k$ (and, of course, $X_j(x) = 0$ for $x \notin V$). It is clear that $X_j \in S_e^\infty(R, M)$, and that

$$X_j(x^0) = \frac{\partial}{\partial x_j}(x^0) .$$

The proof is complete.

8. Characterization of fibre spaces

In this section we give a characterization in terms of symmetry vector fields of those equivalence relations R on a manifold M such that the canonical projection π_R is a fibre map. The condition is that R be locally closed and that there exist sufficiently many *complete* symmetry vector fields of R.

A submersion $\pi: M \to N$ is called a C^∞ *fibre map* if every point $x \in N$ has an open neighborhood U such that there is a C^∞ diffeomorphism Φ from $\pi^{-1}(x) \times U$ onto $\pi^{-1}(U)$ with the property that $\pi\Phi(y, u) = u$ for $y \in \pi^{-1}(x)$, $u \in U$.

We shall allow N to have connected components of different dimensions.

Theorem 11. *Let R be an equivalence relation on a C^∞ manifold M. Then the following two conditions are equivalent*:

(i) *R is almost C^∞-regular and π_R is a C^∞ fibre map,*

(ii) *R is locally closed and there exists a transitive set of everywhere defined complete C^∞ symmetry vector fields of R.*

Proof. Assume that (i) holds. Then R is locally closed. Let $y \in M$, and $v \in M_y$. We want to show that there exists a symmetry vector field X of R which is complete and defined on all of M, and for which $X(y) = v$. Let $x = \pi(y)$, and $F = \pi^{-1}(x)$. Let U, Φ be as in the definition of fibre map. Let $V = \pi^{-1}(U)$, and identify V with $F \times U$ by means of Φ. If Y, Z are C^∞ vector fields on F, U respectively, then there is a vector field $Y \oplus Z$ on $F \times U$, defined in an obvious way. Moreover, $Y \oplus Z$ is easily seen to be a symmetry vector field of R. Clearly, it is possible to choose Y and Z so that $(Y \oplus Z)(y) = v$, and that both have compact support. We can then extend $Y \oplus Z$ to a vector field X defined on all of M, by letting X vanish outside V. Thus $X(y) = v, X$ is a symmetry vector field of R, and X is complete (because it has a compact support).

We now prove that (ii) implies (i). If (ii) holds, then in particular the conditions of Theorem 6 hold, so that R is almost regular. We prove that π_R is a fibre map. Take $x \in M/R$. Let $F = \pi_R^{-1}(x)$, and take $y \in F$. Let $X^1, \cdots, X^k$ be elements of $S_e^\infty(R, M)$ which are complete, and such that $X^1(y), \cdots, X^k(y)$ are linearly independent modulo F_y and, together with F_y, span all of M_y. If $\varepsilon > 0$ is sufficiently small, then the map $\Psi: C_\varepsilon^k \to M$ defined by $\Psi(t_1, \cdots, t_k) = X_{t_1}^1 \cdots X_{t_k}^k(y)$ is a diffeomorphism of C_ε^k onto a k-dimensional submanifold S of M, which is transversal to F at y. By making ε even smaller if necessary, we can assume that π_R maps S diffeomorphically onto a neighborhood U of x. Let ρ denote the inverse of this diffeomorphism. Define $\Phi: F \times C_\varepsilon^k \to M$ by

$$\Phi(z, t_1, \cdots, t_k) = X_{t_1}^1 \cdots X_{t_k}^k(z) .$$

Clearly, Φ is C^∞. Moreover, the fact that the X^i are symmetry vector fields of R implies that $\Phi(z, t_1, \cdots, t_k)$ is R-equivalent to $\Phi(y, t_1, \cdots, t_k)$, which belongs to S. Therefore Φ maps $F \times C_\varepsilon^k$ into $\pi_R^{-1}(U)$. Now define $\Sigma: \pi_R^{-1}(U) \to F \times C_\varepsilon^k$ as follows: if $z \in \pi_R^{-1}(U)$, let $\Psi^{-1}\rho\pi_R(z) = (t_1, \cdots, t_k)$ and $\sigma(z) = X_{-t_k}^k \cdots X_{-t_1}^1(z)$. Then let

$$\Sigma(z) = (\sigma(z); t_1, \cdots, t_k) .$$

It is clear that Σ is also C^∞, and that Σ and Φ are inverses of each other. By means of Φ, identify $\pi_R^{-1}(U)$ with $F \times C_\varepsilon^k$. Also, identify C_ε^k with U by means of $\pi_R \Psi$. It is then clear that π_R is the projection $(z, t_1, \cdots, t_k) \to (t_1, \cdots, t_k)$. Therefore π_R is a fibre map.

Remarks. (a) The proof that (ii) implies (i) is equally correct when "C^∞" is replaced throughout by "C^ω".

(b) Theorem 11 does not admit a generalization similar to Theorem 9, in which "transitivity" is replaced by "weak transitivity". The reason, of course, is that the set of complete vector fields is not a Lie algebra.

(c) A particular case when it is easy to show that condition (ii) of Theorem 11 holds is when π_R is a proper submersion (here "proper" means that $\pi_R^{-1}(K)$ is compact whenever K is compact). Indeed, let $y \in M$ and $v \in M_y$. Let $X \in S_e^\infty(R, M)$ be such that $X(y) = v$. Let f be a C^∞ function on M/R with compact support such that $f(\pi_R(y)) = 1$. It is easy to verify that the vector field $Y = (f_{\circ}\pi_R)\cdot X$ is also a symmetry vector field for R. Clearly, $Y(y) = v$. Since π_R is proper, the support of Y is compact, so that Y is complete.

Therefore Theorem 11 implies the well known fact (due to Ehresmann) that a proper submersion is necessarily a fibre map.

(d) Another case in which Theorem 11 can be applied is the one where R is the relation whose equivalence classes are the cosets Kx, where K is a closed subgroup of the Lie group M. As in the introduction, let L be the Lie algebra of M. Then R is closed, and every $X \in L$ is a symmetry vector field for R. Therefore M/R is a (Hausdorff) manifold, and the projection $x \to Xx$ is a submersion. Moreover, since every $X \in L$ is complete, we recover the well known fact that M is a fibre space over M/R.

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