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WINDING NUMBERS AND THE SOLVABILITY CONDITION (Ψ)

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Introduction

In [3] R. Moyer has proposed a new formulation of the solvability conditions (P) and (Ψ) for pseudodifferential equations of principal type (see [4], [5]). Moyer relates these conditions to an index or winding number whose meaning is very clear and natural, when the principal symbol of the operator under study has the property that the Poisson bracket $\{p, \bar{p}\}$ does not vanish at any point where p itself does ($\bar{p}$ is the complex conjugate of p). In this case, Property (Ψ) simply says that $(1/i)\{p, \bar{p}\}$ should be >0 at any such point.

In the present paper we show that Property (Ψ) for an arbitrary symbol p without critical points is equivalent to the fact that p is the limit, in the local C^1 topology, of symbols having the above property[1]. Such a result points to a new definition of (Ψ). In our view the new definition has a two-fold advantage: first, it shows that the principal symbols of the pseudodifferential equations of principal type, whose solvability has been established so far (and which do not yet include all those satisfying (Ψ)), are limits of symbols of the kind alluded to above, and whose solvability has been well-understood (cf. [2]); secondly and perhaps most importantly, it is totally independent of the concept of *bicharacteristic*, and thus lends itself perfectly to generalization to arbitrary symbols with an arbitrary multiplicity of the characteristics or even degenerating on certain subsets. This of course leads to a new general conjecture on the necessity of (Ψ), redefined as indicated, for local solvability of any linear differential or pseudodifferential equation (see § 3).

1. Noninvolutive functions and their signatures

We shall first explain the notation used throughout the article. We shall deal with an even-dimensional Euclidean space $\boldsymbol{R}^{2n} = \boldsymbol{C}^n$, where the variable is denoted by (x, y), $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$, or by $z = x + \sqrt{-1}y = (z_1, \cdots, z_n)$. In application to partial differential equations, $\boldsymbol{R}^{2n}$ serves as

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[1] That a fact of this kind might be true was suggested approximately ten years ago to L. Nirenberg and the author by J. Moser.

"local model" for the cotangent bundle T^*M over a smooth (i.e., C^∞) manifold M (of dimension n). The real inner product on $\boldsymbol{R}^{2n}$ will be denoted by $xx' + yy' = \operatorname{Re} z\bar{z}'$. Since we have in mind the case of a cotangent bundle T^*M, we shall use the *symplectic form* $\omega(z, z') = \operatorname{Im} z\bar{z}' = x'y - xy'$. If f is a continuously differentiable function, the *Hamiltonian field* of f is defined in the standard fashion:

$$H_f = \sum_{j=1}^{n} \frac{\partial f}{\partial y_j}\frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial}{\partial y_j}\,, \tag{1.1}$$

and if g is another C^1 function, the *Poisson bracket* of f and g is given by

$$\{f, g\} = H_f g\ . \tag{1.2}$$

In the applications to partial differential equations, one of the variables, either x or y, is taken to be the "vertical" variable, which along the fibres, that is to say in the cotangent spaces, is then usually denoted by ξ or p. Because of the way we have chosen the sign conventions in what follows, the reader should think of y as the vertical variable.

We are going to deal systematically with a *bounded* open subset Ω of $\boldsymbol{R}^{2n}$, whose boundary $\partial\Omega$ is a C^∞ hypersurface, and with the space $C^1(\bar{\Omega})$ of the *complex*-valued functions in $\bar{\Omega}$, which can be extended as C^1 functions to $\boldsymbol{R}^{2n}$, equipped with its standard topology, the topology of uniform convergence on the closure $\bar{\Omega}$, of the functions and their gradients; $C^1(\bar{\Omega})$ is a complete normable space.

Definition 1.1. We shall say that $f \in C^1(\bar{\Omega})$ is *noninvolutive* if

$$\forall z \in \bar{\Omega}, f(z) = 0 \Rightarrow \{f, \bar{f}\}(z) \neq 0\ , \tag{1.3}$$

and if, moreover, when $n > 1$, $d(\operatorname{Re} f)$, $d(\operatorname{Im} f)$ and the normal to $\partial\Omega$ are linearly independent at every point of $\partial\Omega$ where $f = 0$. The set of noninvolutive functions in $C^1(\bar{\Omega})$ will be denoted by $\mathscr{P}(\bar{\Omega})$ (or simply by $\mathscr{P}$ if there is no risk of confusion).

Remark 1.1. Going to the theory of partial differential equations (and therefore replacing Ω by an open subset of a cotangent bundle T^*M), we note that *principal symbols* $p(x, \xi)$ which are noninvolutive have been much studied, locally and globally, by Hörmander (in [2]) and Sjöstrand (in [6]) and others. Their microlocal prototype is the symbol of the so-called Mizohata operator:

$$L = \frac{\partial}{\partial x_1} - i x_1 \frac{\partial}{\partial x_2} \qquad (i = \sqrt{-1})\ , \tag{1.4}$$

that is to say, the function

$$p = \xi_2 x_1 + i\xi_1\ . \tag{1.5}$$

From the viewpoint of the properties of L, the "interesting" points in (x, ξ)-space are the zeros of p, i.e., the characteristics of L, which lie away from the zero section of the cotangent bundle, in other words, the zeros of p corresponding to large frequencies ξ. The latter requires $\xi_2 \neq 0$, otherwise $p = 0$ implies $\xi = 0$, and because of the homogenity of p it suffices to look at the two cases $\xi_2 = 1$, $\xi_2 = -1$.

Noting that a Fourier transformation with respect to x_2 transforms L into

$$\hat{L} = \frac{\partial}{\partial x_1} + x_1\xi_2 \, , \tag{1.6}$$

we see that the case $\xi_2 = 1$ (or, if one prefers, $\xi_2 > 0$) corresponds to *solvability points* of L, whereas the other case $\xi_2 = -1$ corresponds to nonsolvability points. But, on the other hand, the case $\xi_2 = -1$ corresponds to *hypoellipticity points* of L, whereas the case $\xi_2 = 1$ does not. On this subject the reader is referred to [2] and, for a simple description, to [9, § 1].

Proposition 1.1. *The set $\mathscr{P}(\bar{\Omega})$ is open in $C^1(\bar{\Omega})$, and is stable under multiplication by any element of $C^1(\bar{\Omega})$ which does not vanish anywhere in $\bar{\Omega}$.*

Proof. Evident.

Let f be an arbitrary element of $\mathscr{P}$. The zero-set of f,

$$Z_f = \{z \in \bar{\Omega} \,;\, f(z) = 0\} \, , \tag{1.7}$$

is a C^1 *noninvolutive* submanifold (regarded as a manifold with boundary), of codimension *two* in $\bar{\Omega}$, which means that the restriction of the sympletic form ω to every tangent space to Z_f is nondegenerate. Note that this makes sense even at the boundary of Ω, for f can be extended as a C^1 function $f^\#$ in the whole of $\boldsymbol{R}^{2n}$. The zero-set $Z_{f\#}$ of $f^\#$ in some open neighborhood of $\bar{\Omega}$ is a C^1 noninvolutive submanifold of codimension two.

Now, because of the compactness of $\bar{\Omega}$, Z_f consists of a finite number of connected components $Z_f^{(j)}$ $(j = 1, \cdots, r)$; unless, of course, $Z_f = \emptyset$. Incidentally, note that some or all of these components might intersect the boundary $\partial\Omega$. At any rate, on each of these components the sign of $(1/i)\{f, \bar{f}\}$ remains constant. Let us write $f = a + ib$ and observe that

$$\{f, \bar{f}\} = -2i\{a, b\} \, . \tag{1.8}$$

We shall denote by $m^+(f)$ (resp. $m^-(f)$) the number of connected components $Z_f^{(j)}$ on which $(1/2i)\{f, \bar{f}\} > 0$ (resp. <0).

Definition 1.2. The pair of nonnegative integers $(m^+(f), m^-(f))$ will be called the *signature* of $f \in \mathscr{P}$ in $\bar{\Omega}$.

Example 1.2. Take $n = 1$, and Ω to be the unit disk in the plane. The signature of the function $f(z) = z$ in $\bar{\Omega}$ is $(1, 0)$, and that of $f(z) = \bar{z}$ is $(0, 1)$. If $\alpha_1, \cdots, \alpha_s, \beta_1, \cdots, \beta_t$ are $r = s + t$ *distinct* points in Ω, then

$$f(z) = \left(\prod_{j=1}^{s} (z - \alpha_j)\right)\left(\prod_{k=1}^{t} (\bar{z} - \beta_k)\right) \tag{1.9}$$

has signature (s, t) in $\bar{\Omega}$.

Remark 1.2. Let us return to the symbol (1.5) of the Mizohata operator. We see that it reads $z_1 = x_1 + iy_1$ when we take $\xi_2 = 1$ (and write y_1 instead of ξ_1), whereas it reads $\bar{z}_1 = x_1 - iy_1$ if we take $\xi_2 = -1$. In both cases the zero-set is given by $z_1 = 0$. Let us take Ω to be a bounded convex set in $\boldsymbol{C}^2$ whose closure is contained in the complement of the origin and which intersects the plane $z_1 = 0$. The signature of (1.5) will be $(1, 0)$ if Ω intersects this plane at points where $y_2 > 0$, which are solvability, but nonhypoellipticity points for (1.4). It will be $(0, 1)$ if $y_2 < 0$ on the intersection, which consists then of hypoellipticity, but nonsolvability points for (1.4).

Proposition 1.2. *If $f, g \in \mathscr{P}$ and $Z_f \cap Z_g = \emptyset$, then*

$$(m^+(fg), m^-(fg)) = (m^+(f) + m^+(g), m^-(f) + m^-(g)) \,. \tag{1.10}$$

In particular, if g does not vanish anywhere in $\bar{\Omega}$, the signature of fg in $\bar{\Omega}$ is equal to that of f.

Proof. Evident.

We shall denote by $\mathscr{P}^{p,q}(\bar{\Omega})$, or simply by $\mathscr{P}^{p,q}$ if there is no risk of confusion, the subset of functions $f \in \mathscr{P}$ whose signature in $\bar{\Omega}$ is (p, q) (p, q are any two nonnegative integers). Note that $\mathscr{P}^{0,0}$ consists of the C^1 functions f in $\bar{\Omega}$ which do not vanish anywhere; in the language of partial differential equations, these would be the *elliptic* symbols.

In Ω is the union of r connected components $\Omega^{(j)}$, $j = 1, \cdots, r$, and (p_j, q_j) is the signature of $f \in \mathscr{P}$ in the closure of $\Omega^{(j)}$, then the signature of f in the closure of Ω is equal to $(p_1 + \cdots + p_r, q_1 + \cdots + q_r)$. This is evident. It is also evident that if Ω' is any open subset of Ω, in general the restriction of $f \in \mathscr{P}^{p,q}(\bar{\Omega})$ to $\bar{\Omega}'$ will *not* belong to $\mathscr{P}^{p,q}(\bar{\Omega}')$, unless, of course, $p = q = 0$. Let us introduce the following subsets of $\mathscr{P}$:

$$\mathscr{P}^+ = \bigcup_{p=0}^{+\infty} \mathscr{P}^{p,0} \,, \qquad \mathscr{P}^- = \bigcup_{q=0}^{+\infty} \mathscr{P}^{0,q} \,.$$

Note that $\mathscr{P}^{0,0} = \mathscr{P}^+ \cap \mathscr{P}^-$. Complex conjugation $f \mapsto \bar{f}$ is an isomorphism of $\mathscr{P}^{p,q}$ onto $\mathscr{P}^{q,p}$.

It is evident that $\mathscr{P}^{0,0}$ is an open subset of $\mathscr{P}$. It is less evident that this is also true of every $\mathscr{P}^{p,q}$, but it follows from the next result:

Proposition 1.3. *The signature is a locally constant function in $\mathscr{P}$.*

Proof. We suppose that $f \in \mathscr{P}(\bar{\Omega})$ has been extended as a C^1 function in some open neighborhood of $\bar{\Omega}$, Ω', and that $f \in \mathscr{P}(\bar{\Omega}')$, which is of course permitted. We suppose also that $Z_f \neq \emptyset$. We can construct a tubular neighborhood U of Z_f ($= Z_f \cap \bar{\Omega}$) in Ω' in the following manner. For each $z \in Z_f$ let P_z denote the two-dimensional (real) plane through z which is orthogonal, for the symplectic form ω, to the tangent plane to Z_f at z. Since the restriction of ω to this tangent plane is nondegenerate, the same is true of the restriction of ω to P_z (which, in particular, can be canonically oriented). On every plane P_z we use the Euclidean metric $|z|^2$ induced by the surrounding space $\boldsymbol{R}^{2n}$, and we call σ_z the open disk of radius $r > 0$, and c_z the circumference of radius $r/2$, both centered at z; c_z will be oriented counterclockwise. We may choose r so small as to achieve a number of properties: #1) as z ranges over Z_f, the union of the disks σ_z is equal to U, which is contained in Ω'; #2) U is not self-intersecting, which implies that each σ_z does not contain any other point of Z_f besides z; #3) and most importantly, if g is any element of $C^1(\bar{\Omega})$ such that

$$\forall z \in \bar{\Omega}, |df(z) - dg(z)| < \tfrac{1}{2}|df(z)| , \tag{1.11}$$

then, whatever $z \in Z_f$, g can vanish at most once in σ_z.

Once all this is achieved we set

$$I_f(z) = \frac{1}{2i\pi} \oint_{c_z} \frac{1}{f}\, df . \tag{1.12}$$

It is checked at once that

$$I_f = \frac{1}{i}\{f, \bar{f}\}/|\{f, \bar{f}\}| \quad on \quad Z_f . \tag{1.13}$$

Let $Z_f^{(j)}$ $(j = 1, \cdots, r)$ be the connected components of Z_f, and let us select arbitrarily a point $z^{(j)}$ of $Z_f^{(j)}$ for each j. Then

$$m^+(f) = \sum_{j=1}^{r} \sup\,(I_f(z^{(j)}), 0) , \qquad m^-(f) = \sum_{j=1}^{r} \inf\,(0, I_f(z^{(j)})) . \tag{1.14}$$

It is clear that there is an open neighborhood of f in $\mathscr{P}$ in which any element g has the following properties: #1) g does not vanish at any point of $\bar{\Omega}\backslash U$ nor at any point of c_z whatever $z \in Z_f$; #2) g satisfies (1.11); #3) $I_f = I_g$ throughout Z_f. These properties imply that g vanishes once and only once in the interior of c_z for every $z \in Z_f$. In other words, for each $j = 1, \cdots, r$, the union of the disks σ_z, $z \in Z_f^{(j)}$, contains a unique connected component $Z_g^{(j)}$ of Z_g, and the sign of $-i\{g, \bar{g}\}$ is equal to that of $-i\{f, \bar{f}\}$ on $Z_g^{(j)}$. Since g does not vanish in the complement of U, this completes the proof of Proposition 1.3.

2. Functions without critical points, Condition (Ψ) and its invariance

In the present section we look at the smooth (i.e., C^∞ or only C^1) *complex-valued* functions f in $\bar{\Omega}$, which do not have critical points:

$$\textit{whatever } z \textit{ in } \bar{\Omega},\ f(z) = 0 \Rightarrow df(z) \neq 0 . \tag{2.1}$$

In the applications to the theory of partial differential equations this would correspond to symbols of *principal type*, except that it is not their total differential which is required not to vanish on the zero-set (i.e., the *characteristic* set), but actually their differential with respect to the fibre variable ξ. In the present notation, $f = 0$ should imply $d_y f \neq 0$. Here, however, we shall disregard this fact and restrict ourselves to Condition (2.1).

Let us first assume that f is *real*-valued (note that a real function f cannot be noninvolutive in the sense of Definition 1.1 unless its zero-set is empty). We shall refer to the integral curves of the Hamiltonian field H_f as the *bicharacteristics* of f. In view of (2.1) they are "true" curves; through each point of $\bar{\Omega}$ there passes one and only one of them. Since $H_f f = 0$, the function f itself is constant along any one of its bicharacteristics. Consequently, if one of these meets the zero-set Z_f, then it lies entirely in Z_f. To such a bicharacteristic we shall refer as a *null bicharacteristic* of f (in $\bar{\Omega}$).

Let us return to complex-valued functions f satisfying (2.1). Let $z_0 \in Z_f$. By virtue of (2.1) there must be a complex number θ and an open neighborhood U_0 of z_0 such that the following holds:

(2.2) $d(\mathrm{Re}\,(\theta f))$ *does not vanish at any point of* U_0.

Remark 2.1. Suppose that f is noninvolutive (Definition 1.1). Then (2.1) is automatically satisfied. As a matter of fact, we may choose the neighborhood U_0 of $z_0 \in Z_f$ so as to have (2.2) *whatever* $\theta \in \boldsymbol{C}$, $\theta \neq 0$. In this case, $d(\mathrm{Re}\, f)$ and $d(\mathrm{Im}\, f)$ are linearly independent at, and therefore near, z_0; they span the plane $P^{\#}_{z_0}$ through the origin (in the cotangent space to $\boldsymbol{R}^{2n}$ at z_0) which is the orthogonal of the tangent plane $T_{z_0}Z_f$ to Z_f at z_0 in the sense of the symplectic form ω.

The solvability theory for linear partial differential equations of principal type has led to the introduction of the following property (see [4], [5]):

Definition 2.1. We say that f satisfies the condition $(\Psi)_\theta$ at $z_0 \in Z_f$ if there is an open neighborhood U_0 of z_0 in $\bar{\Omega}$ such that (2.2) and the following property are true:

(2.3) *if the restriction of* $\mathrm{Im}\,(\theta f)$ *to any null bicharacteristic* Γ *of* $\mathrm{Re}\,(\theta f)$, *contained in* U_0, *is* <0 *at some point, then it is* ≤ 0 *at every later point of* Γ.

The meaning of "later point" is defined by the natural orientation on the bicharacteristics, which itself is defined by the Hamiltonian field.

In [4] it has been conjectured that the *local solvability* of a pseudodifferential operator of principal type on a C^∞ manifold M is equivalent to the validity of $(\Psi)_\theta$ at every point of its characteristic set for some θ depending on the point. This conjecture has been proved under various additional hypotheses. One of the first cases in which it was proved (in [2]) was that of a principal symbol which is noninvolutive (Definition 1.1). Concerning these symbols we make the following observation:

Proposition 2.1. *Let $f \in \mathscr{P}(\bar{\Omega})$ and $z_0 \in Z_f$. In order that f satisfy $(\Psi)_\theta$ at z_0 for some complex number θ it is necessary and sufficient that $(1/i)\{f, \bar{f}\}(z_0)$ be >0.*

Proof. Let us take $|\theta| = 1$ and set $a = \operatorname{Re}(\theta f)$, $b = \operatorname{Im}(\theta f)$. We have

$$\frac{1}{i}\{f, \bar{f}\} = \frac{1}{i}\{\theta f, \overline{\theta f}\} = -2\{a, b\} .$$

Let then Γ be the bicharacteristic of a through z_0. It suffices to observe that the sign of the first derivative of b at z_0 along Γ is equal to that of $-(1/i)\{f, \bar{f}\}(z_0)$.

Corollary 2.1. *Let $f \in \mathscr{P}(\bar{\Omega})$. In order that f satisfy $(\Psi)_\theta$ at every point z_0 of Z_f for some $\theta \in \boldsymbol{C}$ (depending on z_0) it is necessary and sufficient that $f \in \mathscr{P}^+(\bar{\Omega})$.*

We recall that $\mathscr{P}^+$ is the set of functions $f \in \mathscr{P}$ with signature of the form $(p, 0)$, $p \in \boldsymbol{Z}_+$, i.e., such that $m^-(f) = 0$.

The main result of the present section will be the following:

Theorem 2.1. *Let $z_0 \in \Omega$ be a zero of f, and let $\theta \in \boldsymbol{C}$ be such that $d(\operatorname{Re}(\theta f))(z_0) \neq 0$. In order that f satisfy Condition $(\Psi)_\theta$ at z_0 it is necessary and sufficient that there be an open neighborhood U of z_0 in Ω such that $f|\bar{U}$ belongs to the clocure of $\mathscr{P}^+(\bar{U})$ in $C^1(\bar{U})$.*

Proof of Theorem 2.1. We may take z_0 to be the origin and also, by virtue of Proposition 1.2, $\theta = 1$. Let us write $f = a + ib$; we may assume that (2.2) holds for a suitable choice of the open neighborhood U_0 of 0, hence that $da \neq 0$ in U_0. Possibly after shrinking U_0, we may perform a canonical (i.e., preserving the symplectic form ω) change of variables in $\boldsymbol{R}^{2n}$ such that the expression of a in U_0 becomes y_n. Throughout the proof we shall write $x' = (x_1, \cdots, x_{n-1})$, $y' = (y_1, \cdots, y_{n-1})$, $z' = x' + \sqrt{-1}y'$.

I. *Proof of the necessity.* It suffices to show that, in a suitable open neighborhood $U \subset U_0$ of the origin, the function $b_0(z', x_n) = b(x', x_n, y', 0)$ is the limit, in $C^1(\bar{U})$, of a sequence of functions $\beta_j(z', x_n)$ satisfying the following condition:

$$\forall z \in \bar{U},\ \beta_j(z', x_n) = 0 \Rightarrow (\partial/\partial x_n)\beta_j(z', x_n) < 0 . \tag{2.5}$$

Indeed, $f = y_n + i(b_0(z', x_n) + b(z) - b(x', x_n, y', 0)) = y_n + i(b_0(z', x_n) + h(z)y_n)$ will then be the limit, in $C^1(\bar{U})$, of the sequence of functions $f_j = y_n + i(\beta_j(z', x_n) + h(z)y_n)$. We note that $f_j(z) = 0$ is equivalent to

$$y_n = 0 , \qquad \beta_j(z', x_n) = 0 , \tag{2.6}$$

and that, for such z's,

$$\frac{1}{2i}\{f_j, \bar{f}_j\}(z) = -\{y_n, \beta_j(z', x_n) + h(z)y_n\} = -(\partial/\partial x_n)\beta_j(z', x_n) > 0 ,$$

and therefore $f_j \in \mathscr{P}^+(\bar{U})$.

In other words, we may assume that $a = y_n$, and $b(z) = b(z', x_n)$ is independent of y_n. We shall study b in the (z', x_n)-projection of U_0, which we take to be of the form

$$W_0 = U_0' \times \{x_n \in \boldsymbol{R}; |x_n| < T\} , \tag{2.7}$$

where U_0' is an open neighborhood of the origin in $\boldsymbol{R}^{2(n-1)}$., and T is a positive number. Since the null bicharacteristics of a are the straight lines parallel to the x_n-axis and lying in the hyperplane $y_n = 0$. Condition (Ψ) may be translated in the present set-up as

(2.8) *$\forall z' \in U_0'$, if $b(z', x_n) < 0$ for some x_n, $|x_n| < T$, then we have $b(z', t) \leq 0$ for all t, $x_n < t < T$.*

For convenience we are going to assume that all the above properties of $b(z', x_n)$ hold in a neighborhood of the closure of W_0.

Let ε be an arbitrary number >0. We introduce a function $w = w(z', t, \varepsilon)$, defined and C^∞ in a neighborhood of $\overline{W}_0$, and valued in $\boldsymbol{C}^{n-1}$, as the unique solution (see [7, Lemma 2.1]) of the problem:

$$\dot{w} = -(\partial_{\bar{z}'}b)(z' + \varepsilon w, t) , \qquad w|_{t=0} = 0 . \tag{2.9}$$

We set

$$b^\varepsilon(z', t) = b(z' + \varepsilon w, t) . \tag{2.10}$$

Lemma 2.1. *If $\varepsilon > 0$ is sufficiently small, the following two properties hold:*

(2.11) *db^ε vanishes at every point of $\overline{W}_0$ where both b^ε and $\partial_t b^\varepsilon$ vanish.*

(2.12) *Assertion* (2.8) *is true with b^ε substituted for b.*

Proof of (2.11). By (2.9) we have $\partial_t b^\varepsilon(z', t) = (\partial_t b - \varepsilon|\partial_{z'}b|^2)(z' + \varepsilon w, t)$ (provided that ε is small enough). By (2.8) we know that wherever $b = 0$, we must have $b_t \leq 0$. Therefore, if $b^\varepsilon(z', t) = 0$ we shall have $\partial_t b^\varepsilon(z', t) < 0$, unless $db(z' + \varepsilon w, t) = 0$. But then

$$\partial_{x'} b^\varepsilon(z', t) = (\partial_{x'} b)(z' + \zeta w, t)(I + \varepsilon\partial_{x'} w) = 0 ,$$

and similarly for $\partial_{y'} b^\varepsilon$.

Proof of (2.12). For the sake of clarity, let us use real coordinates in $\boldsymbol{R}^{2n-1}$ and make the following change of coordinates:

$$\lambda = (x' + \varepsilon \operatorname{Re} w, y' + \varepsilon \operatorname{Im} w)\,, \qquad s = t. \tag{2.13}$$

We have, by (2.9),

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial s} - \varepsilon b_\lambda(\lambda, s)\cdot\frac{\partial}{\partial \lambda}\,. \tag{2.14}$$

We may apply a result of Brézis [1, Theorem 2] to the functions $b(\lambda, s)$ and $Y(\lambda, s) = -\varepsilon b_\lambda(\lambda, s)$. The hypotheses (4), (5), (6) of Brézis are clearly satisfied in our case ((4) is nothing else but our hypothesis (2.8)). The conclusion in Theorem 2 of [1] is exactly (2.12).

It is obvious that the functions b^ε converges to b in $C^\infty(\overline{W}_0)$ as $\varepsilon \to +0$. It will therefore suffice to approximate each b^ε in $C^1(W_0)$ by elements of $C^\infty(W_0)$, $\{\beta_j^\varepsilon\}$ $(j = 1, 2, \cdots)$ satisfying (2.5). But then we may as well and we shall, in the remainder of the proof, assume that b itself is one of the b^ε, in other words, that (2.11) is true for $\varepsilon = 0$.

Let us introduce the set F_0 of points (z', x_n) of W_0 such that for some t satisfying $-T < t < x_n$ we have $b(z', t) < 0$; we shall denote by F the closure and by $\dot{F}$ the boundary of F_0 in W_0. It is seen at once that $b = 0$ on $\dot{F}$. By (2.11) (for $\varepsilon = 0$), we have $\dot{F} = G_0 \cup G_1$, G_0 being the set of points where $db = 0$ and G_1 the set of points where $\partial_t b < 0$.

For each $z' \in U_0'$, we denote by $t^+(z')$ the infimum of the numbers t, $|t| < T$, such that $(z', t) \in F_0$, and by $+T$ if there are no such numbers t. We denote by $t^-(z')$ the supremum of the numbers t, $|t| < T$, such that $(z', t) \notin F$, and by $-T$ if there are no such numbers t. The function t^+ is upper-semicontinuous, and the function t^- is lower-semicontinuous in U_0'. They are equal and C^∞ in the z'-projection of G_1, and their singular supports are contained in the closure of the z'-projection of G_0. Let us extend them to the whole of $\boldsymbol{R}^{2(n-1)}$ by setting $t^+ = +T$ and $t^- = -T$ in the complement of U_0'.

Let $\delta > 0$ be arbitrary. We shall denote by S_δ the set of points z' whose distance to the singular support of t^- (regarded as a function in $\boldsymbol{R}^{2(n-1)}$) is $\leq \delta$. Let then $\alpha \in C^\infty(\boldsymbol{R}^{2(n-1)})$ vanish outside S_δ and be equal to 1 in $S_{\delta/2}$, and let us denote by U_δ' the set of points z' in U_0' whose distance to the complement of U_0' is $>\delta$.

Let $\rho \in C_c^\infty(\boldsymbol{R}^{2(n-1)})$, $\rho \geq 0$ everywhere, $\int \rho dx'dy' = 1$, and set, as customarily done, $\rho_\varepsilon(z') = \varepsilon^{-2(n-1)}\rho(z'/\varepsilon)$. We then define:

$$t_\varepsilon = (1 - \alpha)t^- + \rho_\varepsilon * (\alpha t^-) \qquad (\text{in } \boldsymbol{R}^{2(n-1)})\,. \tag{2.15}$$

Note that $t_\varepsilon = t^-$ in the open set $U_{\delta+\varepsilon}' \backslash S_{\delta+\varepsilon}$. Furthermore:

(2.16) *given any* $\delta > 0$ *there is* $\varepsilon > 0$ *such that, if* $z' \in U'_{2\delta}$ *and* $|t_\varepsilon(z')| < T - 2\delta$, *then the distance of* $(z', t_\varepsilon(z'))$ *to* $\dot{F}$ *is* $< 2\delta$.

Proof of (2.16). It is immediately seen that $\dot{F}$ is exactly equal to the union of the closed sets $\dot{F}_{z'} = \{(z', x_n)\,;\, t^-(z') \leq x_n \leq t^+(z')\}$. By semicontinuity, given any $\eta > 0$ there is $\varepsilon > 0$ such that if $|z' - \zeta'| < \varepsilon$ then

$$(\alpha t^-)(z') - \eta < (\alpha t^-)(\zeta') \leq (\alpha t^+)(\zeta') < (\alpha t^+)(z') + \eta \,. \tag{2.17}$$

We derive, from (2.15) and (2.17),

$$t^-(z') - \eta \leq t_\varepsilon(z') \leq t^+(z') + \eta \,. \tag{2.18}$$

By choosing $\eta \leq \delta$ we see that this implies (2.16).

For convenience let us assume that b has been extended as a C^∞ function to the whole of $\boldsymbol{R}^{2n-1}$. We now construct a Whitney's partition of unity in $\boldsymbol{R}^{2n-1}\backslash\bar{G}_0$ in the manner of [10, Appendix]. It consists of a sequence of non-negative C^∞ functions $\{\phi_j\}$ $(j = 1, 2, \cdots)$ with compact support in $\boldsymbol{R}^{2n-1}\backslash\bar{G}_0$ such that, for some constant $C > 0$,

$$\sum_{j=1}^{+\infty} |d\phi_j| \leq C(1 + 1/d_0) \qquad \text{in } \boldsymbol{R}^{2n-1}\backslash\bar{G}_0 \,, \tag{2.19}$$

where $d_0(z', x_n)$ denotes the distance of (z', x_n) to $\bar{G}_0$ (which is compact). Then we set, for $J = 1, 2, \cdots$,

$$b_J = \sum_{j=1}^{J} \phi_j b \,.$$

Lemma 2.2. *As* $J \to +\infty$, b_J *converges to* b *in* $C^1(\boldsymbol{R}^{2n-1})$.

Proof. It suffices to reason in a bounded neighborhood $\tilde{\mathcal{O}}$ of $\bar{W}_0$ (or $\bar{G}_0$) and to prove there that $h_J = b - b_J = \sum_{j>J} \phi_j b$ converges to zero in $C^1(\tilde{\mathcal{O}})$. Note that the support of h_J is contained in an arbitrarily "small" neighborhood of $\bar{G}_0$, provided that J is large enough. Since $|h_J| \leq |b|$ for whatever J and $b = 0$ on G_0, we see that $h_J \to 0$ in C^0. By the same token, since $db = 0$ on G_0,

$$dh_J - \sum_{j>J} (d\phi_j) b = \sum_{j>J} \phi_j db$$

tends to zero in C^0. By virtue of (2.19) we have

$$\Big|\sum_{j>J} (d\phi_j) b\Big| \leq C \sup_{\operatorname{supp} h_J} \{(1 + 1/d_0)\,|b|\} \,. \tag{2.20}$$

But

$$|b(z', x_n)| \leq d_0(z', x_n)\left[\sup_{d_0(\zeta', t) \leq d_0(z', x_n)} |db(\zeta', t)|\right] .$$

Since $|db(z', x_n)|$ tends to zero with $d_0(x', x_n)$, we see that the right-hand side of (2.20) goes to zero as $J \to +\infty$. q.e.d.

By virtue of Lemma 2.2, if we want to approximate b in the announced manner, it suffices to approximate each b_J. Let us therefore fix J arbitrarily. We know that b_J is a C^∞ function, vanishing in some open neighborhood $\mathcal{O}_J$ of $\bar{G}_0$. We reintroduce the function t_ε (in U_0') defined and studied earlier. By virtue of (2.16) and the remark which precedes, we can choose δ and ε sufficiently small so that the set

(2.21) $$z' \in U'_{2\delta} , \quad |t| < T - 2\delta , \qquad t = t_\varepsilon(z') ,$$

is identical with $\dot{F}$ except possibly in some compact subset of $\mathcal{O}_J$. It follows at once that there is a nonnegative C^∞ function g_J in $U'_{2\delta} \times]-T+2\delta, T-2\delta[$ such that for this same set

(2.22) $$b_J(z', x_n) = g_J(z', x_n)(t_\varepsilon(z') - x_n) .$$

Then

(2.23) $$b_{J,k}(z', x_n) = \left[g_J(z', x_n) + \frac{1}{k}\right][t_\varepsilon(z') - x_n]$$

converges to b_J in C^∞ as $k \to +\infty$; it clearly satisfies (2.5) if U is chosen small enough (but independently of J and k).

II. *Proof of the sufficiency.* Let U be an arbitrary open subset of U_0 containing the origin, and let $\{f_j\}$ $(i = 1, 2, \cdots)$ be a sequence of elements of $\mathcal{P}^+(\bar{U})$ converging to $f = y_n + ib(z)$ in $C^1(\bar{U})$. We may assume that the f_j belongs to $C^\infty(\bar{\Omega})$. We shall assume that f does not satisfy $(\Psi)_\zeta$ (with $\zeta = 1$, cf. remark at the beginning of the proof) at the origin, and show that this leads to a contradiction.

In the language of the natural topology on subsets, we may assert the following: in $\bar{U}$ and for j sufficiently large, the zero-set of $a_j = \operatorname{Re} f_j$ is arbitrarily close to that of $\operatorname{Re} f$, i.e., to the hyperplane $y_n = 0$, and the null bicharacteristics of a_j are arbitrarily close to those of $\operatorname{Re} f$, that is to say, to the x_n-lines lying in the hyperplane $y_n = 0$. Suppose then that $b(z^{(1)}) < 0$, $b(z^{(2)}) > 0$ with

$$z'^{(1)} = z'^{(2)} , \qquad x_n^{(1)} < x_n^{(2)} , \qquad y_n^{(1)} = y_n^{(2)} = 0 ,$$

the segment joining $z^{(1)}$ to $z^{(2)}$ being entirely contained in $\bar{U}$. Then, as soon as j is large enough, there is a null bicharacteristic of a_j along which b_j must change sign from minus to plus and therefore vanish at a point z of $\bar{U}$ where $H_{a_j} b_j > 0$, contrary to the hypothesis that $f_j \in \mathcal{P}^+(\bar{U})$. q.e.d.

The so-called "invariance of Property (Ψ)" follows immediately from Theorem 2.1.

Corollary 2.1. *Let $z_0 \in V_f$ and let $\zeta \in \boldsymbol{C}$ be such that $d[\mathrm{Re}\,(\zeta f)](z_0) \neq 0$. Let $g \in C^\infty(\bar{\Omega})$ be such that $d[\mathrm{Re}\,(\zeta\, gf)](z_0) \neq 0$. If $(\Psi)_\zeta$ holds for f at z_0, it also does for fg.*

In particular, we have

Corollary 2.2. *Let z_0 and ζ be as in Theorem 2.1. If $(\Psi)_\zeta$ holds for f at z_0, so does $(\Psi)_\theta$ where θ is any complex number such that $d[\mathrm{Re}\,(\theta f)](z_0) \neq 0$.*

The results in Corollaries 2.1 and 2.2 have been originally proved in [5, Appendix].

Definition 2.2. Let $f \in C^1(\bar{\Omega})$ satisfy (2.1). For any point z_0 of Ω we say that f satisfies the condition (Ψ) at z_0 if either $f(z_0) \neq 0$ or $f(z_0) = 0$ with f satisfying $(\Psi)_\theta$ at z_0 for some θ (and then this is true for all θ such that (2.2) holds).

Theorem 2.1 then implies

Corollary 2.3. *Let z_0 be any point of Ω. In order that f satisfy (Ψ) at z_0 it is necessary and sufficient that there be an open neighborhood U of z_0 in Ω such that $f|_{\bar{U}}$ belongs to the closure of $\mathscr{P}^+(\bar{U})$ in $C^1(\bar{U})$.*

Definition 2.3. Let f, z_0 be as in Definition 2.2. We say that f satisfies the condition (P) at z_0 if (Ψ) holds at z_0, both for f and $\bar{f}$.

Corollary 2.4. *Let f, z_0 be as in Corollary 2.3. In order that f satisfy (P) at z_0 it is necessary and sufficient that there be an open neighborhood U of z_0 in Ω such that $f|_{\bar{U}}$ belongs to the intersection of the closures in $C^1(\bar{U})$ of $\mathscr{P}^+(\bar{U})$ and $\mathscr{P}^-(\bar{U})$.*

We recall that $\mathscr{P}^-$ is the union of the $\mathscr{P}^{(0,q)}$, $q = 0, 1, \cdots$, and that $f \in \mathscr{P}^-$ is equivalent to $\bar{f} \in \mathscr{P}^+$.

We ought perhaps to recall the "other" meaning of Condition (P) (at z_0): either $f(z_0) \neq 0$ or if $f(z_0) = 0$ then, for a suitable open neighborhood U_0 of z_0 in Ω and for some (or any) complex number θ such that (2.2) holds,

(2.24) *the restriction of* $\mathrm{Im}\,(\theta f)$ *to any null bicharacteristic Γ of* $\mathrm{Re}\,(\theta f)$ *contained in U_0 does not change sign on Γ.*

Finally we should underline the fact that the condition in Corollary 2.4 involves the intersection of the closures (of $\mathscr{P}^+$ and $\mathscr{P}^-$) and not the closure of the intersection, which would be the closure of $\mathscr{P}^{0,0}$, the set of functions h which do not vanish anywhere in $\bar{\Omega}$. The closure of $\mathscr{P}^{0,0}$ is easy to characterize.

Definition 2.4. Let z_0 be any point in Ω. We say that f satisfies the condition (R) at z_0 if there are a complex number θ and an open neighborhood U_0 of z_0 in Ω such that $d[\mathrm{Re}\,(\theta f)]$ does not vanish anywhere in U_0 and that the following holds:

(2.25) *the restriction of* $\mathrm{Im}\,(\theta f)$ *to the zero-set of* $\mathrm{Re}\,(\theta f)$ *in U_0 does not change sign.*

Theorem 2.2. *Let z_0 be any point in Ω. In order that f satisfy (R) at z_0 it is necessary and sufficient that there be an open neighborhood U of z_0 in Ω such that $f|_{\bar{U}}$ belongs to the closure in $C^1(\bar{U})$ of the set of functions h which do not vanish at any point of $\bar{U}$.*

The proof is easy and we leave it to the reader.

Condition (R) occurs in the theory of partial differential equations, and in [8] it has been shown that if it holds at every point of the cotangent bundle T^*M, then a simple construction of local parametrices is possible.

3. A few remarks in the general case

The condition that $f \in C^\infty(\bar{\Omega})$ belong to the closure of $\mathscr{P}^+$ in $C^1(\bar{\Omega})$ makes sense even when f has critical points. Going back to the theory of partial differential equations and taking Theorem 2.1 into account, one is led to the natural generalization of the conjecture made in [4] that Property (Ψ) is equivalent to local solvability in the case of operators of principal type. Since it is well known that lower-order terms in a differential operator can affect its solvability properties, the only aspect of the conjecture which one can hope to generalize is the "necessity" (of Condition (Ψ) in the principal type case). Let therefore $P(x, D)$ be a pseudodifferential operator in a C^∞ manifold M, and $p(x, \xi)$ its principal symbol (we are tacitely assuming that the *total* symbol of P is an asymptotic sum of terms which are homogeneous with respect to the variables ξ, with homogeneity degrees decreasing by integral values; it is likely that more general situations than this one could be considered).

Conjecture. *If $P(x, D)$ is locally solvable at every point of M, then every point (x_0, ξ^0) of the cotangent bundle T^*M such that $\xi^0 \neq 0$ has an open neighborhood U such that $p|_{\bar{U}}$ belongs to the closure of $\mathscr{P}^+(\bar{U})$ in $C^1(\bar{U})$.*

Such a statement makes it important to find out whether a symbol does belong (locally) to the closure (in C^1) of $\mathscr{P}^+$. It should be noted that this property is *open*, i.e., it cannot hold at a point unless it also holds at every point of some neighborhood of it. In view of this the next proposition might be useful.

Proposition 3.1. *Let $f \in C^\infty(\bar{\Omega})$, and let z_0 be a point of Ω such that $f(z_0) = 0$ and that for a suitable open neighborhood U_0 of z_0 in Ω the following is true:*

(3.1) *there is a noninvolutive submanifold W of codimension 2 in U_0 which contains $U_0 \cap Z_f$ (Z_f: zero-set of f).*

Let p_0 be the winding number of f about z_0 in the two-dimensional plane P_{z_0} through z_0, which is orthogonal to the tangent plane $T_{z_0}W$ in the sense of the symplectc form ω. Consider the following property (for a pair of nonnegative integers p, q):

(3.2) *there is an open neighborhood $U \subset U_0$ such that $f|_{\bar{U}}$ belongs to the closure of $\mathscr{P}^{p,q}(\bar{U})$ in $C^1(\bar{U})$.*

Then (3.2) *holds when* $p = p_0$, $q = 0$ *if* $p_0 \geq 0$, *or when* $p = 0$, $q = -p_0$ *if* $p_0 < 0$. *Furthermore, if* U_0 *is small enough, whenever* (3.2) *holds for some pair* (p, q) *of nonnegative integers, we must have* $p = q + Np_0$ *for some integer* $N \geq 1$ *in the first case, and* $q = p - Np_0$ *in the second case.*

Proof. We may perform a canonical change of variables such that z_0 becomes the origin and W becomes the piece of hyperplane $z_n = 0$ defined by $|z'| < r'$. We may also assume that $p_0 \geq 0$; the result for $p_0 < 0$ is then derived by exchanging f and $\bar{f}$.

We shall denote by $I_f(z)$ the winding number of f about $z \in W$ in the plane P_z.

Let us first suppose $p_0 = 0$. If r' and $r_n > 0$ are small enough, and U denotes the set $\{z \in C^n; |z'| < r', |z_n| < r_n\}$ for each $k = 1, 2, \cdots$, then we can find a smooth complex-valued function $\lambda_k(z')$ of $z' \in C^{n-1}$, $|z'| \leq r'$, converging uniformly to zero in this set together with its first partial derivatives (with respect to x', y'), as $k \to +\infty$, such that, for every k and each z' with $|z'| < r'$, $\lambda_k(z')$ does not belong to the range of $f(z', z_n)$ when $|z_n| < r_n$. Thus $f_k(z) = f(z) - \lambda_k(z') \in \mathscr{P}^{0,0}(\bar{U})$ converges to $f(z)$ in $C^1(\bar{U})$.

Suppose now $p_0 > 0$. Then we may write $f(z) = z_n^{p_0} g(z)$ with $Z_g \subset W$, $I_g = 0$ throughout W (we recall that I_f is locally constant on W). We apply the first part to the function g. We may form a sequence of elements $g_1, g_2, \cdots$ in $\mathscr{P}^{0,0}(\bar{U})$ (for U open, containing the origin, sufficiently small) converging to g in $C^1(\bar{U})$. On the other hand, let $\theta_1, \cdots, \theta_{p_0}$ denote the p_0-th roots of unity and set $f_k(z) = (z_n - \theta_1/k) \cdots (z_n - \theta_{p_0}/k) g_k(z)$. It is clear that $f_k \to f$ in $C^1(\bar{U})$, and that $f_k \in \mathscr{P}^{p_0,0}(\bar{U})$ as soon as $k > 1/r_n$.

Let now U be an open neighborhood of the origin, contained in Ω, and let f_j be a sequence of elements of $\mathscr{P}^{p,q}(\bar{U})$ converging to f in $C^1(\bar{U})$. Let us decompose $\bar{U} \cap W$ into N connected components $W^1, \cdots, W^N$, and for each $\alpha = 1, \cdots, N$ let U^α be a tubular neighborhood of the compact set W^α in the fashion of those considered in the proof of Proposition 1.3. The cross section of each U^α is a disk of fixed radius centered at $z \in W^\alpha$ and contained in the plane P_z. It is clear that for j large enough f_j will not vanish except possibly in $U^1 \cup \cdots \cup U^N$. Consequently, if c_z denotes the circumference (oriented counter-clockwise) centered at z which bounds the cross-section of U^α through z ($z \in W^\alpha$), then $I_{f_j}(z) = (2\pi i)^{-1} \oint_{c_z} df_j/f_j$ is equal to $I_f(z)$, i.e., to p_0. This implies that U^α contains p_α (resp. q_α) connected components of the zero-set of f_j on which $-i\{f_j, \bar{f}_j\}$ is positive (resp. negative), and that $p_\alpha - q_\alpha = p_0$. But of course $p = p_1 + \cdots + p_N$ and $q = q_1 + \cdots + q_N$, hence $p = q + Np_0$.

Corollary 3.1. *With the same hypotheses as in Proposition* 3.1 *we further assume that* z_0 *belongs to the closure of* $W \backslash Z_f$ *in* W. *Then* (3.2) *holds with* $p = q = 0$.

There are other cases in which f will belong (locally) to the closure (in C^1) of $\mathscr{P}^{0,0}$. A notable one is that of the *real*-valued functions as we can see in the following proposition.

Proposition 3.2. *Suppose that $f \in C^1(\bar{\Omega})$ is real-valued. Then every point z_0 of Ω has an open neighborhood U such that $f|_{\bar{U}}$ belongs to the closure of $\mathscr{P}^{0,0}(\bar{U})$ in $C^1(\bar{U})$.*

Proof. Real-valued functions $f \in C^1(\bar{\Omega})$ are limits, in this space, of real-valued smooth functions in $\bar{\Omega}$ which have no critical point. It suffices then to apply Theorem 2.2.

In connection with these considerations it is perhaps worth mentioning that if f and g are locally in the C^1 closure of $\mathscr{P}^+$, so is their product fg. We shall leave the proof of this fact to the reader. Note that fg might well belong to the closure of $\mathscr{P}^+$ in the neighborhood of a given point, without this being true of neither f nor g; e.g., it is always true, according to Proposition 3.2, of the products $f\bar{f}$, though of course it is not always true of f.

Finally, we wish to mention that one might state the same conjecture for (determined) systems of pseudodifferential equations as the one we have stated for a single (scalar) such equation, provided that we interpret $p(x, \xi)$ as the *determinant* of the principal symbol of the system. In our opinion, it is a reasonable conjecture and perhaps the only one which can be made, bearing solely on the principal symbol.

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