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BUNDLE HOMOGENEITY AND HOLOMORPHIC CONNECTIONS

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1. Let $\xi : G \to P \xrightarrow{\pi} M$ be a holomorphic principal fiber bundle with group G, total space P, base space M and projection π. Let $a(M)$ be the Lie algebra of all holomorphic vector fields on M, and let $b(\xi)$ be the space of all R_g invariant elements of $a(P)$. (By R_g we mean the map $R_g : P \to P$ given by $R_g(p) = p^g$.) Let $\pi_* : b(\xi) \to a(M)$ be the obvious projection. We say that ξ is bundle homogeneous if π_* is onto. The purpose of this paper is to study the relation between the bundle homogeneity of ξ and the existence of a holomorphic connection on ξ.

In § 2 we fix notation, and in § 3 we gather together the various definitions of a holomorphic connection and show that they are equivalent. This equivalence is well-known but does not seem to be written down anywhere.

In § 4 we prove

Theorem 4.1. *If ξ has a holomorphic connection, then ξ is bundle homogeneous.*

We also show that the converse of Theorem 4.1 is false in general, but we prove

Theorem 4.5. *Let M be complex parallelizable. Then ξ is bundle homogeneous if and only if ξ admits a holomorphic connection.*

If M is compact, Theorem 4.1 is due to A. Morimoto [9]. In the case where M is a complex torus, Theorem 4.5 was proven independently by Y. Matsushima [6] and S. Murakami [10].

Recall that a real product bundle is a holomorphic principal fiber bundle which admits a C^∞ cross-section [7]. In § 5, we obtain a necessary condition for a real product bundle to be bundle homogeneous. This condition is also sufficient if M is compact (Theorem 5.2), and we also obtain some information about the kernel of π_* in this case.

Since Dolbeault cohomology is not a homotopy invariant (Corollary 6.1), we are able in § 6 to apply the results of the previous sections to construct an example of a real product bundle with (noncompact) Kähler base which does not admit a holomorphic connection. Because there are no topological obstructions on a real product bundle, this example shows that the Atiyah obstruction

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[1] is not a topological invariant, and also that in general the existence of a holomorphic connection does not depend only on the topological structure of the bundle [8].

2. We now recall some basic definitions and theorems about holomorphic connections. Suppose that G is a complex Lie group, M and P are complex manifolds, and G acts freely and holomorphically on P (on the right). We write p^g for the action of $g \in G$ on $p \in P$, and $R_g : P \to P$ for $R_g(p) = p^g$. We say that $\xi : G \to P \xrightarrow{\pi} M$ is a *holomorphic principal fiber bundle* if P is locally biholomorphically equivalent to $M \times G$. This means (i) M is the quotient space of P under the action of G, (ii) there are an open cover $\{U_r\}$ of M and biholomorphic homeomorphisms $\psi_r : \pi^{-1}(U_r) \to U_r \times G$ which commute with the action of G such that

$$\begin{array}{ccc} \pi^{-1}(U_r) & \xrightarrow{\psi_r} & U_r \times G \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & U_r & \end{array}$$

commutes (where pr_1 is projection in the first coordinate), (iii) π is holomorphic. We shall write T_mM for the complex tangent space of M at m (i.e., $Z_m \in T_mM$ means $Z_m = X_m + iY_m$ where X_m and Y_m are real tangent vectors at m in the usual sense), and ϕ_* for the differential of the map ϕ. We define the vertical $(\ker \pi)_p$ at p by

$$(\ker \pi)_p = \{X_p \in T_p(P) \mid \pi_*(X_p) = 0\} .$$

Let G be a complex Lie group of complex dimension r with complex structure J_G. We denote by $\mathfrak{g}$ the Lie algebra of all left invariant real vector fields on G, considered as a real Lie group, by $\mathfrak{g}_0$ the Lie algebra of all holomorphic left invariant vector fields on G, and by $\mathfrak{g}^C$ the complexification of $\mathfrak{g}$, i.e. $\mathfrak{g}^C$ is the Lie algebra of all left invariant complex vector fields on G. We may also regard $\mathfrak{g}^C$ as a complex manifold with complex structure $\hat{J}$. We shall use $\Lambda^1(M, \mathfrak{g}^C)$ for the vector space of all Lie algebra valued one-forms on M. $\Lambda^1(M, \mathfrak{g}^C)$ may be written as $\Lambda^{(1,0)}(M, \mathfrak{g}^C) \oplus \Lambda^{(0,1)}(M, \mathfrak{g}^C)$ where

$$\Lambda^{1,0}(M, \mathfrak{g}^C) = \{\omega \in \Lambda^1(M, \mathfrak{g}^C) \mid \omega(J_M A) = \hat{J}\omega(A) \text{ for all } A \in TM\} ,$$
$$\Lambda^{0,1}(M, \mathfrak{g}^C) = \{\omega \in \Lambda^1(M, \mathfrak{g}^C) \mid \omega(J_M A) = -\hat{J}\omega(A) \text{ for all } A \in TM\} .$$

If $h: M \to \mathfrak{g}$ is smooth, then h induces a map $dh: TM \to \mathfrak{g}^C$, i.e., $dh \in \Lambda^1(M, \mathfrak{g}^C)$, so that we may write dh as $dh = \partial h + \bar{\partial} h$ where $\partial h \in \Lambda^{1,0}(M, \mathfrak{g}^C)$ and $\bar{\partial} h \in \Lambda^{0,1}(M, \mathfrak{g}^C)$. If $2\omega_1(A) = dh(A) - \hat{J}dh(J_M A)$ and $2\omega_2(A) = dh(A) + \hat{J}dh(J_M A)$, then $\omega_1 \in \Lambda^{1,0}(M, \mathfrak{g}^C)$, $\omega_2 \in \Lambda^{0,1}(M, \mathfrak{g}^C)$ and $dh = \omega_1 + \omega_2$. Therefore $2\bar{\partial}h(A) = dh(A) + \hat{J}dh(J_M A)$ or

$$(2.1)\qquad 2\hat{J}\bar{\partial}h(A) = \hat{J}dh(A) - dh(J_M A) .$$

If $a \in G$, then $\mathrm{ad}(a)\colon \mathfrak{g}^C \to \mathfrak{g}^C$ will be the usual adjoint map.

If M is a complex manifold, then in a coordinate neighborhood U we know that $\{\partial/\partial x^k, \partial/\partial y^k \mid k = 1, \cdots, n\}$ forms a basis for $T_m M$ at each point $m \in U$. We define

$$\partial/\partial z^k = \tfrac{1}{2}(\partial/\partial x^k - i\partial/\partial y^k) , \qquad \partial/\partial \bar{z}^k = \tfrac{1}{2}(\partial/\partial x^k + i\partial/\partial y^k) .$$

Let $T_m{}^{1,0}M = \{Z \in T_m M \mid JZ = iZ\}$, and $T_m{}^{0,1}M = \{Z \in T_m M \mid JZ = -iZ\}$. Then $T_m M = T_m{}^{1,0}M \oplus T_m{}^{0,1}M$, and $\{(\partial/\partial z^k)_m \mid 1 \le k \le n\}$ (resp. $\{(\partial/\partial \bar{z}^k)_m \mid 1 \le k \le n\}$) forms a basis for $T_m{}^{1,0}M$ (resp. $T_m{}^{0,1}M$) at $m \in U$. A vector field Z is called a *holomorphic vector field* if $Z_m \in T_m{}^{1,0}M$ and in any cordinate chart $Z_m = \sum_{j=1}^n f^j(m)(\partial/\partial z^j)_m$ for some holomorphic functions f^j.

We shall now describe the standard embedding of $\mathfrak{g}^C$ onto the vertical. For $p \in P$ let ${}^p\Phi : G \to P$ be defined by ${}^p\Phi(g) = p^g$. We then define $\Theta_p : \mathfrak{g}^C \to (\ker \pi)_p$ by $\Theta_p(A) = ({}^p\Phi)_*(A)$, where the differential is evaluated at $e \in G$ and we have identified $\mathfrak{g}^C$ and $T_e G$ in the usual manner.

Proposition 2.1. (*a*) $\Theta_p : \mathfrak{g}^C \to (\ker \pi)_p$ *is an isomorphism of vector spaces for each* $p \in P$.

(b) *If* $A \in \mathfrak{g}_0$, *then the vector field* $p \to \Theta_p(A)$ *is a holomorphic vector field.*

Proof. (a) follows as in the C^∞ case [3, p. 51].

(b) The fact that $\Theta_p(A)$ is of type $(1,0)$ follows from [4, p. 179]. If $(w_1, \cdots, w_r)$ and $(z_1, \cdots, z_n)$ are the coordinates about $e \in G$ and $p \in P$ respectively, then we may write

$$\Phi(z_1, \cdots, z_n, w_1, \cdots, w_r) = (\Phi^1(z, w), \cdots, \Phi^n(z, w))$$

with Φ^k holomorphic functions, and so

$$\Theta_p\Big(\frac{\partial}{\partial w_j}\Big)_e = \sum_{k=1}^n \frac{\partial \Phi^k}{\partial w_j}(p, e) \Big(\frac{\partial}{\partial z_k}\Big)_p ,$$

which is clearly a holomorphic vector field because $\dfrac{\partial \Phi^k}{\partial w_j}(p, e)$ is a holomorphic function of p.

3. A *connection* on ξ is a distribution $H : p \to H_p$ in P such that (1) $T_p P = (\ker \pi)_p \oplus H_p$, and (2) $(R_a)_* H_p = H_{pa}$. The connection 1-form $\omega \in \Lambda^1(P, \mathfrak{g}^C)$ is defined as follows: Any $X \in TP$ may be written as the sum of $hX \in H$ and $vX \in \ker \pi$. hX is called the horizontal part of X, and vX the vertical part of X. Let $\omega_p(X) = \Theta_p{}^{-1}(vX)$ where Θ is as in Proposition 2.1. The following proposition is quite easy and allows us to call a connection either a distribution as in the definition above or a $\mathfrak{g}^C$-valued 1-form satisfying the two conditions of Proposition 3.1.

Proposition 3.1. *If ω is the connection 1-form of a connection, then*
(1) $\omega_p(\Theta_p(A)) = A$ *for all* $A \in \mathfrak{g}^C$,
(2) $(R_g{}^*\omega)(X) = (\mathrm{ad}\, g^{-1})(\omega(X))$ *for all* $X \in TP$ *and* $g \in G$.
Furthermore, if $\omega \in \Lambda^1(P, \mathfrak{g}^C)$ *satisfies* (1) *and* (2) *above, then* ω *is the connection given by*

$$H_p = \{X \in T_pP \,|\, \omega_p(X) = 0\} .$$

A connection H is of *type* $(1,0)$ if $JH_p = H_p$ for all $p \in P$. This is clearly equivalent to the condition $\omega \in \Lambda^{1,0}(P, \mathfrak{g}^C)$ where ω is the connection 1-form of H. A connection is a *holomorphic connection* if ω is of type $(1,0)$ and $\bar\partial\omega = 0$. The following theorem (which appears to be well-known but not written down) gives the geometric content of the definition of a holomorphic connection. (Recall that if Z is a vector field on M, then the *horizontal lift* $\tilde{Z}$ of Z is the unique vector field on P such that $\pi_*(\tilde{Z}) = Z$ and $\tilde{Z}(p) \in H_p$ for all $p \in P$.)

Theorem 3.2. *If* $\xi : G \to P \xrightarrow{\pi} M$ *is a holomorphic principal fiber bundle, and* H *is a* $(1,0)$ *connection on* ξ, *then the following are equivalent*:
(a) *H is a holomorphic connection.*
(b) *If W is any open subset of P, and X is any holomorphic vector field defined on W, then vX is also a holomorphic vector field on W.*
(c) *If X is holomorphic on W, then hX is holomorphic on W.*
(d) *The horizontal lift of any holomorphic vector field which is defined on any open subset U of M is a holomorphic vector field on* $\pi^{-1}(U)$.

Proof. Let $(w^1, \cdots, w^r)$ be a coordinate chart in G, and $(z^1, \cdots, z^n)$ a coordinate chart in M. We may use $(z^1, \cdots, z^n, w^1, \cdots, w^r)$ as a coordinate in P via the local trivialization. Suppose that ω is the connection 1-form of H. If X is any holomorphic vector field, and $\{e_1, \cdots, e_r\}$ is a basis for $\mathfrak{g}^C$, then we may write locally $\omega = \sum \omega_j{}^k \, dz^j e_k$ and

$$X = \sum f^l(z, w)\frac{\partial}{\partial w^l} + \sum h^k(z, w)\frac{\partial}{\partial z^k} ,$$

where h^k and f^l are holomorphic functions. Therefore

$$vX = \Theta\omega(X) = \sum_{j,k} h^j \omega_j{}^k \Theta(e_k) . \qquad (1)$$

Using Proposition 2.1 (b), it follows from (1) that $\Theta(\omega(X))$ is holomorphic for all X if and only if $\omega_j{}^k$ are holomorphic; hence (a) $\Leftrightarrow$ (b).

The equivalence of (b) and (c) follows from $X = vX + hX$.

Assume (c), and suppose that X is a holomorphic vector field on U which we may assume is small enough so that $\pi^{-1}(U)$ is trivial. We now regard X as the vector field $(X, 0)$ on $U \times G$, and clearly $\tilde{X} = h(X, 0)$; hence (c) $\Rightarrow$ (d).

We now complete the proof by showing that (d) $\Rightarrow$ (a). Because

$$\widetilde{\frac{\partial}{\partial z^j}} = \left(\frac{\partial}{\partial z^j} - \sum_k \omega_j{}^k \Theta(e_k)\right)$$

must be holomorphic for each j by assumption, we see that $\omega_j{}^k$ must be holomorphic; hence (d) $\Rightarrow$ (a). q.e.d.

There is an alternate formulation due to Atiyah [1]. Because we shall not need it explicitly, we shall not go into it except to say that in his formulation a holomorphic connection exists on ξ if and only if a certain element (called the *Atiyah obstruction*) is zero in a certain cohomology set. To see that this is equivalent to our definition, see [7, Proposition 3.12].

4. Let $\xi : G \to P \xrightarrow{\pi} M$ be a holomorphic principal fiber bundle. Let $a(M)$ be the Lie algebra of all holomorphic vector fields on M, and let $b(\xi) = \{X \in a(P) \mid (R_g)_* X = X \text{ for all } g \in G\}$. We call $X \in b(\xi)$ an *infinitesimal bundle automorphism of* ξ. If $X \in b(\xi)$, then by $\pi_*(X)$ we mean $\pi_*(X)_m f = X_p\,(f \circ \pi)$ for any $m \in M$ and $p \in \pi^{-1}(m)$. This is well-defined because $(R_g)_* X = X$ for all $g \in G$, and is holomorphic because of the local product structure. We say that ξ is *bundle homogeneous* if $\pi_* : b(\xi) \to a(M)$ is onto.

Theorem 4.1. *If ξ has a holomorphic connection, then ξ is bundle homogeneous.*

Proof. If $X \in a(M)$, then by Theorem 3.2 the horizontal lift $\tilde{X}$ with respect to the holomorphic connection is holomorphic. On the other hand, if $\tilde{X}(p)$ is horizontal, then so is $(R_g)_* \tilde{X}(p)$; hence $(R_g)_* \tilde{X}(p) = \tilde{X}(p^g)$. We therefore have $(R_g)_* \tilde{X} = \tilde{X}$ and so $\tilde{X} \in b(\xi)$. Clearly $\pi_*(\tilde{X}) = X$ and so π_* is onto. q.e.d.

By [1, p. 188] we have

Corollary 4.2. *Any holomorphic principal fiber bundle whose base space is a Stein manifold is bundle homogeneous.*

Let M be compact, and let $A(M)$ denote the identity component of the complex Lie group of biholomorphic homeomorphisms of M, and $B(\xi)$ the identity component of the group of holomorphic bundle automorphisms (i.e., $B(\xi)$ is the identity component of $\{\phi \in A(P) \mid \pi \circ \phi = \pi \text{ and } \phi \circ R_a = R_a \circ \phi \text{ for all } a \in G\}$). Then $\pi : B(\xi) \to A(M)$ is defined by $\pi(\phi)(m) = \pi(\phi(p))$ for any $p \in \pi^{-1}(m)$.

Proposition 4.3. (*Morimoto* [9]). (a) *If ξ is bundle homogeneous, then $\pi : B(\xi) \to A(M)$ is onto.*

(b) *If M is compact, then $B(\xi)$ is a Lie group, and so π is onto if and only if ξ is bundle homogeneous.*

Proof. If $f_t \in A(M)$ is a 1-parameter subgroup for all $0 \leq t \leq 1$, then f_t induces an element X of $a(M)$. Let $\tilde{X} \in b(\xi)$ such that $\pi_*(\tilde{X}) = X$, and let ϕ_t be the local 1-parameter subgroup generated by $\tilde{X}$ at $p \in P$. To prove (*a*), we need only to show that ϕ is a global 1-parameter subgroup because clearly $\pi(\phi_t) = f_t$ and $\phi_t \in B(\xi)$. To do this we show that ϕ_t is the horizontal lift of f_t with respect to some (not necessarily holomorphic) connection Γ on ξ.

Let g be any right G-invariant Riemannian metric on P, and $\tilde{H}_p$ the orthogonal subspace in T_pP of $V_p + C\tilde{X}_p$. If $\Gamma : p \to H_p$ is defined by $H_p = \tilde{H}_p + C\tilde{X}_p$, then Γ is the desired connection.

The statement that $B(\xi)$ is a Lie group if M is compact is Morimoto's theorem. He also proved that the Lie algebra map induced by π is π_*, and so we have (b). q.e.d.

For compact M Theorem 4.1 is due to Morimoto [9, p. 166] who also proved

Theorem 4.4. *If M is a compact Kähler manifold whose first Betti number is zero and G is nilpotent, then the holomorphic principal fiber bundle $\xi : G \to P \to M$ is bundle homogeneous.*

Both of these theorems of Morimoto are proven by using the Atiyah viewpoint. Applying Theorem 4.4 to the canonical C^* bundle ξ over CP^n we see that the converse of Theorem 4.1 is false. We can also do this constructively as follows: $\phi \in B(\xi)$ if and only if $\phi : C^{n+1} - \{0\} \to C^{n+1} - \{0\}$ is a holomorphic homeomorphism and $\phi(\lambda z) = \lambda\phi(z)$ for all $\lambda \in C^*$ and $z \in C^{n+1} - \{0\}$. By [2, p. 21] ϕ can be extended to a map of $C^{n+1} \to C^{n+1}$ such that $\phi(\lambda z) = \lambda\phi(z)$ for all $\lambda \in C$ and $z \in C^{n+1}$. By the standard trick this means that $\phi \in Gl(n+1, C)$. Clearly any $\phi \in Gl(n+1, C)$ restricts to an element of $B(\xi)$, and hence $B(\xi) = Gl(n+1, C)$. By using a result of Lichnerowicz [5] to give us all $A(CP^n)$, we see that π is onto. Recall that a complex parallelizable n-manifold is one on which there are n holomorphic vector fields which are linearly independent at each point (see [12]). The following theorem gives a converse to Theorem 4.1.

Theorem 4.5. *Suppose that $\xi : G \to P \to M$ is a holomorphic fiber bundle, and M is complex parallelizable. Then ξ is bundle homogeneous if and only if ξ admits a holomorphic connection.*

Proof. We need only to assume that ξ is bundle homogeneous, and to show that ξ admits a holomorphic connection. Let $X_1, \cdots, X_n \in a(M)$ be linearly independent. Let $X_j{}^*$ be any element of $b(\xi)$ such that $\pi_* X_j{}^* = X_j$, and let $\bar{X}_j{}^*$ denote the complex conjugate of $X_j{}^*$. We claim that if $H_p =$ span of $\{X_1{}^*(p), \cdots, X_n{}^*(p), \bar{X}_1{}^*(p), \cdots, \bar{X}_n{}^*(p)\}$, then $H : p \to H_p$ is a holomorphic connection on ξ. Since $JX_j{}^* = iX_j{}^*$ and $J\bar{X}_j{}^* = -i\bar{X}_j{}^*$, we see that H_p is of type $(1, 0)$. Since $X_j{}^*$ is of type $(1, 0)$, there is a real tangent vector A such that $X_j{}^* = A - iJA$. Hence $(R_g)_* X_j{}^* = (R_g)_* A - iJ(R_g)_* A$ and $\bar{X}_j{}^* = A + iJA$, which imply that $(R_g)_* \bar{X}_j{}^* = (R_g)_* A + iJ(R_g)_* A$, so that $(R_g)_* \bar{X}_j{}^* = \overline{(R_g)_* X_j{}^*}$ for all $g \in G$. Because $X_j{}^* \in b(\xi)$, we have that $(R_g)_* X_j{}^* = X_j{}^*$ and $(R_g)_* \bar{X}_j{}^* = \overline{(R_g)_* X_j{}^*} = \bar{X}_j{}^*$, so $(R_g)_* H_p = H_{pg}$. By a dimension argument, to show that $T_pP = (\ker \pi)_p \oplus H_p$ we need only to show that $(\ker \pi)_p \cap H_p = (0)$, but this is clear because π_* is one to one on a basis of H_p by definition. Hence H is a connection of type $(1, 0)$.

If X is any (local) holomorphic vector field on M, then there are (local) holomorphic functions f^j on M such that $X = \sum_{j=1}^n f^j X_j$, but then $\sum_{j=1}^n (f^j \circ \pi) X_j{}^*$ is clearly the horizontal lift of X with respect to H and is a holomorphic vector field. Hence H is a holomorphic connection by Theorem 3.2. q.e.d.

5. A holomorphic principal fiber bundle ξ is called a *real product bundle* if ξ admits *a* C^∞ section (i.e., *a* C^∞ map $s: M \to P$ such that $\pi \circ s = 1_M$). From [7, Theorems 1.2.6 and 2.3.5] we know that every real product bundle must take the form $\xi: G \to (M \times G)_{J^\eta} \to M$ where $\eta \in \Lambda^{0,1}(M, \mathfrak{g}^C)$ and (for $z \in M$, $\lambda \in G$, $A \in T_z M$, $B \in T_\lambda G$)

$$J^\eta_{z,\lambda}(A, B) = (J_M A, J_G B + (dR_\lambda)_e \eta(A)) ,$$

and $\bar\partial \eta = \frac{1}{4} i [\eta, \eta]$. We shall ask when $\pi: B(\xi) \to A(M)$ is onto. This will give us conditions for ξ to be bundle homogeneous (see Proposition 4.3). $\phi: M \times G \to M \times G$ is a C^∞ bundle automorphism if and only if for $z \in M$ and $g \in G$, ϕ takes the form

$$\phi(z, g) = (f(z), s(z)g) \tag{5.1}$$

for some $f \in A(M)$ and $s: M \to G$ (not necessarily holomorphic). ϕ is a bundle automorphism in this case because

$$\tilde\phi(z, g) = (f^{-1}(z), ((s \circ f^{-1})(z))^{-1} g)$$

is a C^∞ bundle map which is the inverse of ϕ. It is clear from (5.1) that $\pi(\phi) = f$, so we must only find conditions on $f \in A(M)$ such that there is an $s: M \to G$ for which ϕ defined by (5.1) is holomorphic with respect to J^η. Let $\alpha: M \times G \to G$ be defined by $\alpha(z, \lambda) = s(z)\lambda$. Then $\phi(z, \lambda) = (f(z), \alpha(z, \lambda))$, and so (using upper dot "$\cdot$" to denote the differential), for $A \in T_Z M$ and $B \in T_\lambda G$,

$$\dot\phi_{z,\lambda}(A, B) = (\dot f_z(A), \dot\alpha_{z,\lambda}(A, B)) \tag{5.2}$$

for $z \in M$. Let ${}^z\alpha: G \to G$ be ${}^z\alpha(\lambda) = \alpha(z, \lambda) = L^s{}_{(z)}\lambda$, and $\alpha^\lambda: M \to G$ be $\alpha^\lambda(z) = \alpha(z, \lambda) = R_\lambda \circ s(z)$. The Leibniz formula [3] says:

$$\dot\alpha_{z,\lambda}(A, B) = (\dot\alpha^\lambda)_z(A) + ({}^z\dot\alpha)_\lambda(B) = \dot L_{s(z)}(B) + \dot R_\lambda \dot s(A) ,$$

which, together with (5.2), gives

$$\dot\phi_{z,\lambda}(A, B) = (\dot f_z(A), \dot L_{s(z)}(B) + \dot R_\lambda \dot s(A)) . \tag{5.3}$$

Therefore

$$\begin{aligned} &J^\eta_{f(z), s(z)\lambda} \dot\phi_{z,\lambda}(A, B) \\ &\qquad = (J_M \dot f_z(A), J_G(\dot L_{s(z)} B + \dot R_\lambda \dot s(A)) + \dot R_{s(z)\lambda} \eta(\dot f_z(A)) . \end{aligned} \tag{5.4}$$

On the other hand, (5.3) implies

$$
\begin{aligned}
\dot{\phi}_{z,\lambda}(J^{\eta}_{z,\lambda}(A,B)) &= \dot{\phi}_{z,\lambda}(J_M A, J_G B + \dot{R}_\lambda \eta(A)) \\
&= (\dot{f}_z(J_M A), \dot{L}_{s(z)}(J_G B + \dot{R}_\lambda \eta(A)) + \dot{R}_\lambda \dot{s}(J_M A)) \; .
\end{aligned}
\tag{5.5}
$$

Comparing (5.4) with (5.5) we see that ϕ is holomorphic if and only if

$$
\begin{aligned}
J_G \dot{L}_{s(z)} B + J_G \dot{R}_\lambda \dot{s}(A) + \dot{R}_\lambda \dot{R}_{s(z)}(f_* \eta)(A) \\
= \dot{L}_{s(z)}(J_G B + \dot{R}_\lambda \eta(A)) + \dot{R}_\lambda \dot{s}(J_M A) \; ,
\end{aligned}
$$

and so we may conclude

Proposition 5.1. *Let $\phi(z,\lambda) = (f(z), s(z)\lambda)$. Then $\phi : M \times G \to M \times G$ is holomorphic if and only if*

$$
J_G \dot{s}(A) - \dot{s}(J_M A) = \dot{L}_{s(z)} \eta(A) - \dot{R}_{s(z)} f^* \eta(A)
\tag{5.6}
$$

for all $z \in M$ and $A \in T_z M$.

Proceeding as in [7], we assume for the moment that there is a C^∞ function $h : M \to \mathfrak{g}$ such that

$$
\begin{array}{ccc}
 & & \mathfrak{g} \\
 & {}^{h}\nearrow & \downarrow \exp \\
M & \xrightarrow[s]{} & G
\end{array}
\tag{5.7}
$$

commutes. Let $\hat{J}$ be the complex structure of $\mathfrak{g}^C$ viewed as a manifold. If $X = h(z)$ where $z \in M$ is fixed, then (5.6) becomes

$$
d(\exp)_X(\hat{J} dh(A) - dh(J_M A)) = \dot{L}_{\exp X} \eta(A) - \dot{R}_{\exp X} f^* \eta(A) \; ,
$$

since exp is a holomorphic map for Lie groups. Using (2.1) we thus obtain

$$
2 J_G d(\exp)_X \bar{\partial} h(A) = \dot{L}_{\exp X}(\eta(A)) - \dot{R}_{\exp X} f^* \eta(A) \; ,
$$

and therefore, by the expression for $d(\exp)$ [7],

$$
2 J_G d(L_{\exp X})_e \circ \frac{I - e^{-\operatorname{ad} X}}{\operatorname{ad} X} \bar{\partial} h(A) = d(L_{\exp X}) \eta(A) - d R_{\exp X} f^* \eta(A) \; ,
$$

or

$$
2 J_G \frac{I - e^{-\operatorname{ad} X}}{\operatorname{ad} X} \bar{\partial} h(A) = \eta(A) - d(L_{\exp(-X)} \circ R_{\exp X}) f^* \eta(A) \; .
$$

Since $d(L_{\exp(-X)} \circ R_{\exp X}) = \operatorname{ad} \exp(-X) = e^{-\operatorname{ad} X}$, we have

$$2J_G \frac{I - e^{-\mathrm{ad}\, h(z)}}{\mathrm{ad}\, h(z)} (\bar{\partial} h(A)) = \eta(A) - e^{-\mathrm{ad}\, h(z)} f^* \eta(A) \ . \tag{5.8}$$

We say that for $\omega, \eta \in \Lambda^{0,1}(M, \mathfrak{g}^C)$, ω is *exponentially cohomologous* to η (and write $\omega \tilde{_{\exp}} \eta$) if there is a C^∞ map $h : M \to \mathfrak{g}$ such that

$$2J_G \frac{I - e^{-\mathrm{ad}\, h(z)}}{\mathrm{ad}\, h(z)} (\bar{\partial} h(A)) = \eta(A) - e^{-\mathrm{ad}\, h(z)} \omega(A) \ . \tag{5.9}$$

We say that M has the *exponential lift property* with respect to G if for any $s : M \to G$ there is an $h : M \to \mathfrak{g}$ such that the diagram (5.7) is commutative.

Theorem 5.2. *Let $\eta \in \Lambda^{0,1}(\mathrm{M}, \mathfrak{g}^C)$ with M connected, $\xi : G \to (M \times G)_{J^\eta} \to M$ be a real product bundle with J^η as above, $\pi : B(\xi) \to A(M)$, and $f \in A(M)$.*

(a) *If $f^* \eta \tilde{_{\exp}} \eta$, then $f \in \pi(B(\xi))$.*

(b) *Suppose that G has the exponential lift property. Then $f \in \pi(B(\xi))$ if and only if $f^* \eta \tilde{_{\exp}} \eta$.*

(c) *If G is abelian and $\pi_1(M)$ is a torsion group, then* $\dim_C \ker \pi_* = 1$.

(d) *Suppose $G = C^*$, and M is compact. Then*

(i) *$f^* \eta \tilde{_{\exp}} \eta$ if and only if $f \in \pi(B(\xi))$, and*

(ii) $\dim \ker \pi_* = 1$.

Proof. (a) If $f^* \eta \tilde{_{\exp}} \eta$, then there is an $h : M \to \mathfrak{g}$ satisfying (5.8). If $s : M \to G$ is $s = \exp \circ h$, then s satisfies (5.6), and hence $f \in \pi(B(\xi))$.

(b) We need only to prove if $f \in \pi(B(\xi))$ then $f^* \eta \tilde{_{\exp}} \eta$. By Proposition 5.1, we have a map $s : M \to G$ satisfying (5.6). If $h : M \to \mathfrak{g}$ is the map of diagram (5.7) (which exists by exponential lift), then by the above computation, h satisfies (5.8), and hence $\eta \tilde{_{\exp}} f^* \eta$.

(c) Under the hypotheses of (c), (5.8) yields that $\pi(\phi)$ equals the identity (i.e., $f = 1_M$) if and only if there is $h : M \to \mathfrak{g}$ such that $2\bar{\partial} h = \eta - \eta = 0$, which happens if and only if h is a constant. Thus $s : M \to G$ of (5.1) must be the constant map at $\lambda = \exp X$ for some $X \in \mathfrak{g}$, and therefore

$$\ker \pi = \{\phi : M \times G \to M \times G \,|\, \phi(z, g) = (z, \lambda g) \text{ for some } \lambda \in \exp(\mathfrak{g})\} \ ,$$

which implies that $\dim \ker \pi_* = 1$.

(d) Follows from the following proposition and lemma.

Lemma. *If G is abelian, then for each $g \in G$ the map $\beta : (M \times G)_{J^\eta} \to (M \times G)_{J^\eta}$ given by $\beta(z, x) = (z, L_g x)$ is holomorphic.*

Proof. $\dot{\beta}_{z,x}(A, B) = (A, \dot{L}_g B)$ for $A \in T_z M$ and $B \in T_x G$, hence

$$J^\eta \dot{\beta}_{z,x}(A, B) = (J_M A, J_G \dot{L}_g B + \dot{R}_{gx} \eta(A)) \ ,$$

$$\dot{\beta}_{z,x} J^\eta(A, B) = (J_M A, \dot{L}_g(J_G B + \dot{R}_x \eta(A))) \ ,$$

and so $\dot{\beta} J^\eta = J^\eta \dot{\beta}$ if G is abelian.

Proposition 5.3. *Suppose that M is compact and $G = C^*$. Then $s : M \to G$ satisfies* (5.6) *if and only if there is $\tilde{s} : M \to G$ defined by $\tilde{s} = L_{2r} \circ s$ and satisfying* (5.6) *such that $\tilde{s}$ factors through the exponential map as in diagram* (5.7).

Proof. Let $B_r(g) = \{z \in C^* \,|\, |z - g| < r\}$, and assume that $s : M \to G$ satisfies (5.6). Let $r > 0$ be any real number such that $s(M) \subset B_r(0)$. If $\tilde{s} = L_{2r} \circ s$, then $\tilde{s}(M) \subset L_{2r}B_r(0) = B_r(2r)$. This means that $\tilde{s}(M)$ never winds around the origin; that is, $\tilde{s}(M)$ is a simply-connected subspace of C^*. Because the logarithm is well-defined on any simply-connected region in C^*, $\tilde{s}$ factors through the exponential map. By the above lemma, the map $\tilde{\beta}(z, \lambda) = (f(z), \tilde{s}(z)\lambda)$ is holomorphic in the J^η structure on $M \times G$ if and only if $\beta(z, \lambda) = (f(z), s(z)\lambda)$ is holomorphic. q.e.d.

We remark that the above proposition can be used to strengthen some results in [7], e.g., for compact M with $G = C^*$, Exp $D(M, G) = 0$ if and only if Pic $(M, G) = 0$.

6. Combining Theorem 5.2 (b) and Proposition 4.3 yields

Corollary 6.1. *If $\xi : C^* \to (M \times C^*)_{J^\eta} \to M$ is bundle homogeneous, and M has the exponential lift property with respect to C^*, then for all $f \in A(M)$*

$$f^*\eta - \eta = \bar{\partial}h \tag{6.1}$$

for some $h : M \to C$. If M is compact, then the converse holds.

Observe that (6.1) says that $A(M)$ must "act" as the identity on $\mathscr{D}_{0,1}(M, C)$; however, it is known that if f is homotopic to g through complex analytic maps and $\bar{\partial}\omega = 0$, it is not necessarily true that $f^*\omega - g^*\omega = \bar{\partial}l$ for some $l : M \to C$ [11]! The example in [11] is on the Iwasawa manifold. We shall now present a different example.

If $M = C^2 - \{(0, 0)\}$, A and B are complex numbers with nonzero imaginary parts such that $AB \neq 1$, and we define $f_t : M \to M$

$$f_t(z_1, z_2) = \left(\frac{Az_1}{1 + (1 - t)A}, \frac{Bz_2}{1 + (1 - t)B}\right),$$

then $f_t \in A(M)$, and so in particular $f_1 = f : M \to M$ is an element of $A(M)$. We define $\eta \in A^{0,1}(M, C)$ by

$$\eta_{(z_1, z_2)} = \begin{cases} \bar{\partial}(\bar{z}_2/(z_1 r^2)) & \text{when } z_1 \neq 0, \\ -\bar{\partial}(\bar{z}_1/(z_2 r^2)) & \text{when } z_2 \neq 0, \end{cases} \tag{6.2}$$

where $r^2 = |z_1|^2 + |z_2|^2$. η is well-defined (but not $\bar{\partial}$-cohomologous to zero) by [2, p. 30]. We now calculate $f^*\eta - \eta$. If $z_1 \neq 0$, then

$$f^*\eta_{(z_1, z_2)} = \bar{\partial}(\bar{z}_2/(z_1 r^2) \circ f),$$

and therefore

$$(6.3)\qquad f^*\eta_{(z_1,z_2)} = \bar\partial\left(\frac{\overline{Bz_2}}{Az_1(|Az_1|^2+|Bz_2|^2)}\right), \qquad \text{if } z_1 \neq 0\ .$$

If $f^*\eta - \eta = \bar\partial h$ for some $h: M \to C$, then for $z_1 \neq 0$, (6.2) and (6.3) imply

$$(6.4)\qquad \bar\partial h = \bar\partial\left(\frac{\overline{Bz_2}}{Az_1(|Az_1|^2+|Bz_2|^2)} - \frac{\bar z_2}{z_1(|z_1|^2+|z_2|^2)}\right).$$

If we let $g: M \to C$ be given by

$$(6.5)\qquad g(z_1,z_2) = z_1 h(z_1,z_2) - \left(\frac{\overline{Bz_2}}{A(|Az_1|^2+|Bz_2|^2)} - \frac{\bar z_2}{|z_1|^2+|z_2|^2}\right),$$

then for ($z_1 \neq 0$) we have, from (6.4),

$$\bar\partial(g/z_1) = \bar\partial h - \bar\partial h = 0\ .$$

$g(z_1,z_2)$ is therefore holomorphic for $z_1 \neq 0$. Since g is locally bounded on $M - X$ where $X = \{(z_1,z_2) \in C^2 \mid z_1 = 0\}$ and X is thin, we may apply the Riemann extension theorem [2, p. 19] and conclude that $g: M \to M$. Since a point is a removable singularity in C^n ($n > 1$), g must be a holomorphic map of C^2 to C^2. However, by the form of g given by (6.5) we have

$$g(0,z_2) = \frac{1}{z_2} - \frac{1}{ABz_2},$$

which is not holomorphic at $z_2 = 0$ since $AB \neq 1$. Therefore (6.1) cannot hold in this case. Because M is simply connected, M has the exponential lift property with respect to C^* [7, Proposition 2.2.2], and so Corollary 6.1 implies

Corollary 6.2. *There exists a real product bundle which does not have a holomorphic connection; in particular, the Atiyah obstruction is not a topological invariant.*

Note also that $C^2 - \{0,0\}$ is a Kähler manifold, so compactness cannot be dropped from [7, Theorem 3.1.7].

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