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A GENERALIZATION OF THE TWO-VERTEX THEOREM FOR SPACE CURVES

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1. The classical four-vertex theorem asserts that for a sufficiently smooth simple closed plane curve the curvature has at least four relative extrema. At such a relative extremum of the curvature, or "vertex", the curve has higher contact with its osculating circle. If such a plane curve now is projected stereographically onto the sphere, a vertex goes into a point at which the image curve has higher contact with the osculating plane, i.e., a point at which the torsion vanishes. The four-vertex theorem thus leads naturally to the question: at how many points does the torsion of a closed space curve vanish?

There are closed space curves with nowhere vanishing torsion, for example, a long coil spring which is bent around and connected together. The four-vertex theorem may be interpreted as saying that for a sufficiently smooth simple closed curve on the sphere the torsion vanishes at at least four points. Barner [2] has shown that the conclusion remains true provided only that the curve is simple and lies on a convex surface. A number of similar theorems are known; see Segre [5], [6], and [7].

In this paper we show that a simple closed space curve, which has no tangent lines meeting the curve again (no "cross tangents") and has a point lying on the boundary of the convex hull of the curve though which passes no line meeting the curve in two other points, has at least two points at which the torsion is zero. In fact, if the curve does not lie in a plane, we show that the torsion changes sign.

Two ideas are used in the proof. One is an analysis of certain singularities of the Gauss secant map, and the other is a projection of the curve onto a plane parallel to a plane of support of the convex hull of the space curve. The condition that the curve has a point lying on the boundary of the convex hull of the curve through which passes no line meeting the curve in two other points is required for the projection. The assumption that X has no cross tangents is used in the Gauss secant map. We define three closed one-dimensional submanifolds using singularities of the Stieltjes function

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$$K(t_1, \cdots, t_4) = \frac{2!\,3!\,|X(t_1), X(t_2), X(t_3), X(t_4)|}{\pi_{i<j}(t_j - t_i)}$$

studied by Pohl [4]. These curves are well-behaved if X has no cross tangents.

Throughout this paper, when we speak of a closed curve having some smoothness property we will mean a mapping of the circle having that smoothness property. However, in studying them we will usually find it more convenient to take such a curve as parametrized by an interval. Our curves will have no cusps.

In a Euclidean 3-space E^3, let $X : [0, 2\pi] \to E^3$ be a space curve which is differentiable of class C^4. The *osculating plane* of X at t is the plane through $X(t)$ parallel to $X'(t)$ and $X''(t)$. We assume throughout this paper that $X'(t)$ and $X''(t)$ are in general position at each point so that the curvature never vanishes. An *inflection point* of X is a point t at which the torsion vanishes. At such a point $X'''(t)$ lies in the osculating plane, which is then called an *inflectionary osculating plane*. By a *generic inflection point* of X we mean a point t at which the torsion vanishes, but its derviative does not. At a generic inflection point the torsion changes sign, and the third derivative of X lies in the osculating plane, but as may easily be seen, the fourth derivative does not. By a *double osculating plane* we mean a pair of points (t_1, t_2), $t_1 \neq t_2$, such that the osculating plane at t_1 is the osculating plane at t_2. X has a *cross tangent* at (t_1, t_2), $t_1 \neq t_2$, if the tangent line to X at t_1 meets X at t_2. A *trisecant* of X is any line which meets X at three or more points. Let τ denote the torsion of X, and $\mathscr{K}$ the curvature.

2. Theorem I. *Let $X : [0, 2\pi] \to E^3$ be a simple closed nonplanar C^4 space curve. Assume X has nonvanishing curvature (say $\mathscr{K} > 0$) and has no cross tangents, and assume the torsion of X changes sign only a finite number of times. Then, for any $\varepsilon > 0$, there exists $Y : [0, 2\pi] \to E^3$ such that Y is a simple closed nonplanar C^4 space curve which is arbitrarily close to X (i.e., $|X^i(t) - Y^i(t)| < \varepsilon$, $i = 0, \cdots, 4$, $t \in [0, 2\pi]$). Also all inflection points of Y are generic; the torsion of Y has no more sign changes than that of X; if P is an inflectionary osculating plane of Y at t, then P is tangent to Y only at t; and Y has no double osculating planes.*

The proof of this theorem, which is not given here, is a general position argument. "In general", at an inflection point of a curve the torsion vanishes but its derivative does not, an inflectionary osculating plane of a curve at one point is not also tangent to the curve at another point, and the osculating plane of a curve at one point is not also the osculating plane at another point. Such a curve is said to be in general position. Theorem I states that we can replace our curve X by a curve in general position which is arbitrarily close to X and has the properties of X which we desire.

Arguments such as these are illustrated by the Thom transversality theorem and use an approximation approach in the construction of the general position curves. References to the general position theory include the discussion of the Thom transversality theorem in [8, pp. 45–69] and [1]. [1, Chapter 4] is a generalization by Abraham of Thom's work. Two of Thom's articles on the topic are [9] and [10].

3. Let C denote a circle with coordinate t defined modulo integral multiples of 2π, and $X : C \to E^3$ be a simple closed space curve, differentiable of class C^4, with nonvanishing curvature and no cross tangents. We assume that X satisfies the following general position requirements:

i) all inflection points of X are generic,

ii) X has no inflectionary osculating plane which is tangent to the curve at any point other than the point of inflection,

iii) X has no double osculating planes.

We consider the torus $C \times C$ and its representation in a plane. Let (x, y) be the Cartesian coordinates in the plane; we identify all points $(x + 2\pi n, y + 2\pi m)$, m and n; integers, with the point (x, y). In particular, the y-axis of the plane and all lines $x = \pm 2\pi n$ are identified with the set of points $(0, C)$, and the x-axis is identified with the points $(C, 0)$. Thus we speak of the horizontal axis $\{(x, 0)\}$ and of the vertical axis $\{(0, y)\}$. Let $\Delta = \{(t, t)\}$ denote the diagonal of $C \times C$.

We define three closed one-dimensional submanifolds of $C \times C$. Let D^* be the closure in $C \times C$ of the set of pairs of points (t_1, t_2), $t_1 \neq t_2$, such that there is a plane tangent to X at both points, i.e., the secant $X(t_1) - X(t_2)$, the tangent $X'(t_1)$ and the tangent $X'(t_2)$ all lie in a plane. D^* is locally the zero locus of a smooth function. For

$$D^* - \Delta = \{(t_1, t_2) \in C \times C - \Delta \,|\, |X(t_1) - X(t_2), X'(t_1), X'(t_2)| = 0\} ,$$

where $|\,,\,,|$ denotes the determinant.

Let

$$D(t_1, t_2) = |X(t_1) - X(t_2), X'(t_1), X'(t_2)| .$$

At a point (t_1, t_2), $t_1 \neq t_2$, such that $D(t_1, t_2) = 0$ and $(\partial D/\partial t_1)(t_1, t_2) = 0$, the osculating plane of X at t_1 contains the tangent $X'(t_2)$ and the secant $X(t_1) - X(t_2)$. Since X has no double osculating planes, $D(t_1, t_2) = 0$ and $(\partial D/\partial t_1)(t_1, t_2) = 0$ implies $(\partial D/\partial t_2)(t_1, t_2) \neq 0$. Thus the zero locus of $D(t_1, t_2)$ is a curve in $C \times C - \Delta$.

To represent D^* on a set which includes points of the diagonal we use the function studied by Pohl [4], which he calls the Stieltjes function of four variables. Choose homogeneous coordinates (x_0, x_1, x_2, x_3) in E^3, and let $(x_1(t), x_2(t), x_3(t))$ be a local representation of X in nonhomogeneous coordinates so that $\tilde{X}(t) = (1, x_1(t), x_2(t), x_3(t))$ is local representation of X in the homogeneous coordinates. We suppose that the x_i are differentiable of class C^4 and defined on an open interval I. Consider the expression

$$K(t_1, \cdots, t_4) = \frac{2!\,3!\,|\tilde{X}(t_1), \tilde{X}(t_2), \tilde{X}(t_3), \tilde{X}(t_4)|}{\pi_{i<j}(t_j - t_i)} .$$

It is not hard to see that

$$K(t, t, t, t) = |\tilde{X}(t), \tilde{X}'(t), \tilde{X}''(t), \tilde{X}'''(t)| .$$

Pohl [4, p. 359] shows that K is differentiable of class C^1. We consider

$$K_D(t_1, t_2) = K(t_1, t_2, t_1, t_2) = \frac{2!\,3!\,|\tilde{X}(t_1), \tilde{X}'(t_1), \tilde{X}(t_2), \tilde{X}'(t_2)|}{(t_2 - t_1)^4} ,$$

which is the Stieltjes function with pairs of coincident points.

For $t_1 \neq t_2$, $K_D(t_1, t_2) = 0$ if and only if $D(t_1, t_2) = 0$. At a point (t_1, t_2), $t_1 \neq t_2$, such that $K_D(t_1, t_2) = 0$, we have

$$(\partial K_D / \partial t_i)(t_1, t_2) = \frac{2!\,3!\,(\partial D/\partial t_i)(t_1, t_2)}{(t_2 - t_1)^4} , \qquad i = 1, 2 .$$

For $t_1 = t_2 = t$,

$$(dK_D/dt)(t, t) = |\tilde{X}(t), \tilde{X}'(t), \tilde{X}''(t), \tilde{X}''''(t)| = |X'(t), X''(t), X''''(t)| .$$

Thus $K_D(t, t) = 0$ if and only if X has an inflection point at t. Since all inflection points are generic, $dK_D/dt \neq 0$, so that if $K_D(t, t) = 0$, $(\partial K_D/\partial t_1)(t, t) \neq 0$. Thus we see that the zero loci of $K_D(t_1, t_2)$ and of $D(t_1, t_2)$ are locally curves, and that these loci agree at all points in $C \times C$ for which they are both defined. So D^*, the closure in $C \times C$ of the set of pairs of points (t_1, t_2), $t_1 \neq t_2$, such that $D(t_1, t_2) = 0$, is a one-dimensional submanifold of $C \times C$.

Similarly, we define J^* to be the closure in $C \times C$ of the set of pairs of points (t_1, t_2), $t_1 \neq t_2$, such that the osculating plane of X at t_1 contains the secant $X(t_1) - X(t_2)$. As above, we can show that the set J^* is a one-dimensional submanifold of $C \times C$. To see this we consider the zero loci of the following functions:

$$J(t_1, t_2) = |X(t_1) - X(t_2), X'(t_1), X''(t_1)| , \qquad t_1 \neq t_2 ,$$

$$K_J(t_1, t_2) = \frac{3!\,|\tilde{X}(t_1), \tilde{X}'(t_1), \tilde{X}''(t_1), \tilde{X}(t_2)|}{(t_2 - t_1)^3} .$$

We define N^* to be the closure in $C \times C$ of the set of pairs of points (t_1, t_2), $t_1 \neq t_2$, such that the osculating plane of X at t_2 contains the secant $X(t_1) - X(t_2)$. N^*, also, is a one-dimensional submanifold of $C \times C$ as may be seen by considering the zero loci of the functions:

$$N(t_1, t_2) = |X(t_1) - X(t_2), X'(t_2), X''(t_2)| , \qquad t_1 \neq t_2 ,$$

$$K_N(t_1, t_2) = \frac{3!\,|\tilde{X}(t_1), \tilde{X}(t_2), \tilde{X}'(t_2), \tilde{X}''(t_2)|}{(t_2 - t_1)^3} .$$

Notice that $N(t_1, t_2) = -J(t_2, t_1)$, and that $K_N(t, t) = 0$ if and only if X has an inflection point at t.

Remark. D^*, N^* and J^* can be shown to be closed one-dimensional submanifolds of $C \times C$ even if X has cross tangents.

Suppose that the osculating plane of X at t_2 contains the tangent $X'(t_1)$, $t_1 \neq t_2$. Then it is easy to show that D^* intersects N^* at (t_1, t_2), D^* has a vertical tangent, and N^* has a horizontal tangent at this point. In the same way, if the osculating plane of X at t_1 contains the tangent $X'(t_2)$, then D^* intersects J^*. D^* has a horizontal tangent, and J^* has a vertical tangent. The inflectionary osculating plane of X at t_2 contains the secant $X(t_1) - X(t_2)$ if and only if N^* has a vertical tangent.

Suppose $D(t_1, t_2) = 0$ and $(\partial D/\partial t_2)(t_1, t_2) \neq 0$. Then in some neighborhood of (t_1, t_2) the slope m_D of D^* is given by

$$m_D = -\frac{\partial D}{\partial t_1} \Big/ \frac{\partial D}{\partial t_2} .$$

Since $D(t_1, t_2) = 0$, we can write $X'(t_1) = A(X(t_1) - X(t_2)) + BX'(t_2)$ where A and B are nonzero. Then

$$m_D = \frac{J(t_1, t_2)}{B^2 N(t_1, t_2)} .$$

For each of the curves D^*, J^* and N^*, the curve is entirely on one side of the tangent in some neighborhood of a point at which the curve has a horizontal or vertical tangent. That is to say, for example, if the D^* curve has a vertical tangent at a point, then at this point the curve changes direction either from left to right or from right to left. We say the curve has an extremum at such a point.

Let $D(t_1, t_2) = 0$ and $(\partial D/\partial t_1)(t_1, t_2) = 0$. Then

$$(\partial^2 D/\partial t_1^2)(t_1, t_2) = |X(t_1) - X(t_2), X'''(t_1), X'(t_2)| \neq 0 ,$$

since X has no inflectionary osculating plane tangent to the curve at a point other than the inflection point. Thus D^* has an extremum whenever it has a horizontal tangent and, by the symmetry of the arguments, whenever it has a vertical tangent. Similarily, since X has only generic inflection points, J^* has an extremum whenever it has a vertical tangent. Since X has no double osculating planes, J^* has an extremum whenever it has a vertical tangent. Since $J(t_1, t_2) = -N(t_1, t_2)$, N^* also changes direction at horizontal or vertical tangents.

Suppose X has no inflection points. Let T, $0 < T < 2\pi$, be the smallest t such that $N(t, 0) = 0$. Consider the set of points $(t_i, 0)$ such that $0 < t_i < T$

and $D(t_i, 0) = 0$; *the* D^* *curve has either negative or positive slope at each of these points. Then there are at least as many points where the slope of* D^* *is positive as where it is negative.*

To see this recall that if X has no inflection points, then D^* and N^* never meet the diagonal and N^* has no vertical tangents. So D^* cannot have a vertical tangent in the triangle bounded by $y = 0$, $x = T$ and $x = y$. Suppose the opposite, let $D(\bar{x}, \bar{y}) = 0$ and $(\partial D/\partial y)(\bar{x}, \bar{y}) = 0$ where $0 < \bar{x} < T$ and $\bar{y} < x$. Then $N(\bar{x}, \bar{y}) = 0$. N^* has no vertical tangents, so the branch of N^* through $(\bar{x}, \bar{y})$ can be written as $g(x) = y$. Since N^* cannot cross the diagonal, $g(x) < x$. Now $g(\bar{x}) > 0$ since N^* crosses the x-axis transversally. Thus there is a value a, $0 < a < \bar{x} < T$, such that $N(a, 0) = 0$. But this contradicts the choice of T.

The intersections of D^* with the x-axis are finite in number since each intersection is transversal. Thus it is easy to give a relationship such that for each intersection for which the slope of D^* is negative there is at least one intersection for which the slope of D^* is positive.

4. Let $X : [0, 2\pi] \to E^3$ be a simple closed space curve, differentiable of class C^4, with nonvanishing curvature and no cross tangents. Let X have an arc A on the surface of the convex hull of X such that through a point on the arc, say 0, X has no trisecants. We assume that X does not lie in a plane and that:

i) X does not have an inflection point at 0,

ii) $X(0)$ has a plane P of support which meets the curve at no other point,

iii) the determinant $|X(t) - X(0), X'(0), X''(0)| > 0$ for $0 < t < \varepsilon$,

iv) for $t \in [0, 2\pi]$ if one of the functions $D(t, 0)$, $J(t, 0)$ or $N(t, 0)$ is zero, then the other two are not zero at this point.

The assumptions i), ii), iii) and iv) do not restrict the curve. It is easy to see that some point in the neighborhood of 0 satisfies i) and iv). After a reflection of X through the osculating plane of X at 0, some point in the neighborhood of 0 satisfies i), iii) and iv). Suppose at the point t_0 satisfying i), iii) and iv) a plane P of support meets X at other points. P meets X tangentially at these points since X is on the surface of its convex hull at t_0. A small deformation of X in the direction of the outer normal to P at $X(t_0)$ gives a curve which satisfies all assumptions.

For convenience of representation we suppose $X(0)$ is the origin in E^3, the plane P of support is the plane $z = 0$, and $X(t)$ lies in the half space $z \geq 0$. Consider the plane $P' : z = C > 0$. The osculating plane of X at 0 and the plane P' intersect in a line L which we picture as a directed horizontal line in P' increasing to the left. Let $\underset{\sim}{X}(t)$ denote the central projection of $X(t)$ through $X(0)$ onto the plane P'. Then we may write

$$\underset{\sim}{X}(t) = A(t)X(t) \qquad \text{where } A(t) > 0 \,.$$

(See, for example, [3, pp. 112–114].)

$\underset{\sim}{X}(t)$ is simple since X has no trisecants at 0. The osculating plane crosses X at 0 and so crosses X at an odd number of other points. Thus $\underset{\sim}{X}(t)$ crosses the line L an odd number of times. $\underset{\sim}{X}(t)$ is asymptotically tangent to L as $t \to 0$, for projection preserves the order of contact. We have $\underset{\sim}{X}(t)$, $0 < t < \varepsilon$, above the line L and going left, for the determinant $|X(t) - X(0), X'(0), X''(0)| > 0$ for $0 < t < \varepsilon$. The points t at which D^* meets the horizontal axis of the torus, i.e., $D(t, 0) = 0$, are also the points t at which $\underset{\sim}{X}(t)$ has a horizontal tangent. Since the projection preserves the order of contact, $|X(t), X'(t), X'(0)| = 0$ if and only if $\underset{\sim}{X}'(t)$ is parallel to L. $\underset{\sim}{X}$ has an inflectionary tangent at t if and only if $J(t, 0) = |X(t), X'(t), X''(t)| = 0$, that is, if and only if the osculating plane at t projects into the tangent line of $\underset{\sim}{X}$ at t. $\underset{\sim}{X}$ meets the line L at t if and only if $N(t, 0) = 0$. $\underset{\sim}{X}(t)$ crosses L transversally, and has an extremum whenever it has a horizontal tangent, by assumption iv); see Fig. 4.1.

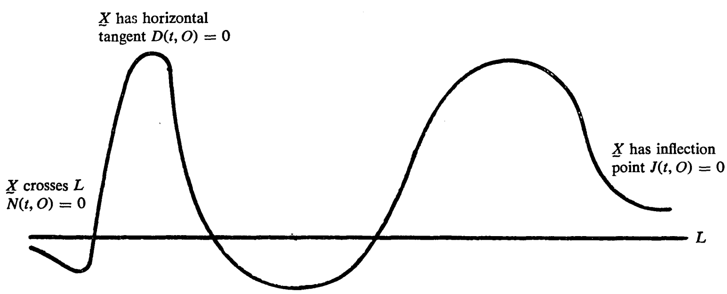

Fig. 4.1

If $\underset{\sim}{X}$ has a horizontal tangent at t, then $\underset{\sim}{X}$ has a relative maximum or relative minimum at t, that is, $\underset{\sim}{X}$ has no inflectionary horizontal tangents. For $\underset{\sim}{X}$ has a horizontal tangent at t if and only if $D(t, 0) = 0$, and $\underset{\sim}{X}$ has an inflection point at t if $J(t, 0) = 0$; by construction $D(t, 0) = 0$ implies $J(t, 0) \neq 0$. There are four possibilities for a horizontal tangent of X at a point t:

(a) $\underset{\sim}{X}(t)$ is a maximum and $\underset{\sim}{X}'(t)$ is parallel to L,
(b) $\underset{\sim}{X}(t)$ is a maximum and $\underset{\sim}{X}'(t)$ is opposite to L,
(c) $\underset{\sim}{X}(t)$ is a minimum and $\underset{\sim}{X}'(t)$ is opposite to L,
(d) $\underset{\sim}{X}(t)$ is a minimum and $\underset{\sim}{X}'(t)$ is parallel to L.

As before, let T be the smallest value so that $N(t, 0) = 0$ for $0 < t < 2\pi$. Thus T is the smallest value such that $\underset{\sim}{X}(t)$ meets L. Let a, b, c, and d be the numbers of horizontal tangents of type (a), (b), (c), and (d) respectively, for $0 < t < T$. Thus these are the numbers of horizontal tangents of $\underset{\sim}{X}$ before $\underset{\sim}{X}$ meets L. With this notation we state a proposition:

Proposition. *Let $X : [0, 2\pi] \to E^3$ be a simple closed space curve, which does not lie entirely in a plane, is differentiable of class C^4, and has nonvanishing curvature and no cross tangents. Also let X have an arc on the surface of* *the convex hull of X such that at some point on the arc X has no trisecants. Then*

$$a + c - (b + d) = 1 .$$

Also, for $0 < t < T$, *the horizontal tangents of types* (a) *and* (c) *correspond to points in* $C \times C$ *at which* D^* *meets the x-axis with negative slope, and horizontal tangents of types* (b) *and* (d) *correspond to* D^* *intersections with positive slope.*

Proof. The line L divides the plane into two half-planes. We say that $\underset{\sim}{X}(0 + \varepsilon)$, $\varepsilon > 0$, is above the line L, and we assume $\underset{\sim}{X}$ does not have a horizontal extremum for $0 < t < \varepsilon$. Let Y be a line perpendicular to L through the point $\underset{\sim}{X}(0 + \varepsilon)$, and consider the sectionally smooth simple closed curve Z given by the three arcs, $\underset{\sim}{X}(t)$, $\varepsilon < t < T$, L between $\underset{\sim}{X}(T)$ and Y, and Y between L and $\underset{\sim}{X}(\varepsilon)$. For example Z might look like the following:

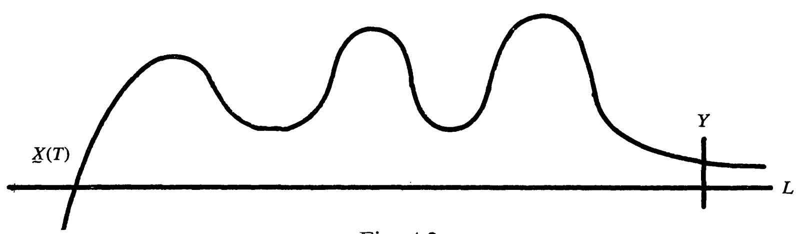

Fig. 4.2

Z is a sectionally smooth closed curve with rotation index 1.

Now we can "smooth out" the corners of Z to obtain a smooth simple closed curve which we shall also denote by Z. This construction can be performed without altering the number of horizontal tangents of $\underset{\sim}{X}$ for $t < T$.

Let $\Gamma : Z \to S^1$ be the tangent map, which maps a point of Z into the unit vector through the origin parallel to the tangent vector of Z. The index of rotation of Z is also the degree of the map Γ, and Γ is a differentiable map of Z into S^1.

By [8, p. 127, Theorem 4.2] the local degree of the map Γ is also one. The curve Z has no more horizontal tangents directly parallel to L than $\underset{\sim}{X}$ has. Thus since $\Gamma^{-1}(\pi) = 1$ we have $a - d = 1$. The curve Z has one more horizontal tangent oppositely parallel to L, namely, the strip $-L$ which is in the curve Z. Thus since $\Gamma^{-1}(0) = 1$ we have $-b + c + 1 = 1$.

Combining these equations we have

$$a + c - (b + d) = 1 .$$

Recall that the slope of $D(t, 0) = 0$ is given by

$$m_D = \frac{J(t, 0)}{B^2 N(t, 0)} ,$$

where B is a constant. At a tangent of type (a) $m_D < 0$. To see this we apply an affine transformation to E^3 so that the support plane P is perpendicular to the osculating plane Q of X at 0. We define three vectors $\underset{\sim}{A} = X'(0)$, $\underset{\sim}{B} =$ projection of $X(t)$ into Q, and $\underset{\sim}{C} =$ projection of $X(t)$ into P. $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$ are independent since $X(t)$ lies in neither P or Q. We apply an affine transformation with positive determinant so that $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$ are orthonormal. Now we consider the following in terms of the vectors $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$: $X'(0) = \alpha \underset{\sim}{A}$; $X(t) = \beta \underset{\sim}{B} + \gamma \underset{\sim}{C}$; α, β and γ are positive, by our choice of transformation.

$X''(0) = \mathscr{H} \underset{\sim}{B}$; where $\mathscr{H} > 0$ if we use the arc length as the parameter. $X'(t) = \delta X(t) + \varepsilon \underset{\sim}{A}$; ε is negative for the tangents of type (a). $X''(t) = \eta X(t) + \theta \underset{\sim}{C} + \lambda \underset{\sim}{A}$; θ is negative also for type (a). Thus the slope

$$m_D = \frac{(-\beta)\varepsilon\theta}{B^2\gamma\mathscr{H}\alpha} ,$$

which is negative.

Likewise for horizontal tangents of type (c) $m_D < 0$ and for (b) and (d) $m_D > 0$. This completes the proof of the proposition.

Now we state and prove the main theorem of this paper.

Theorem II. *Let $X : [0, 2\pi] \to E^3$ be a simple closed space curve, which does not lie entirely in a plane, is differentiable of class C^4, and has non-vanishing curvature and no cross tangents. Also let X have an arc on the surface of the convex hull of X such that at some point on the arc X has no trisecants. Then τ has at least two sign changes.*

The proof is immediate. By Theorem I we may assume that X satisfies the general position requirements. If X has no inflection points, then preceding the first N^* intersection the number of D^* intersections with negative slope is always less than or equal to the number with positive slope. (See § 3.) However, by the proposition in this section we see that preceding the first N^* intersection the number of D^* intersections with negative slope is always one larger than the number with positive slope. Thus X must have at least one inflection point. However, by the general position argument the torsion changes sign at this point. So, since X is closed it must have at least two inflection points. This completes our proof.

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