J. DIFFERENTIAL GEOMETRY
9 (1974) 565-569

PROPER G-SPACES

WAYNE E. DYDO

Introduction

By a G-space we will mean a completely regular topological space X on which a locally compact topological group G acts continuously on the left. If G is a Lie group, X is a differentiable (C^∞) manifold, and the action is differentiable, then we call X a *differentiable G-space*. We will assume that the reader is familiar with the concepts of *Cartan G-space*, *proper G-space*, and *slice* defined by Palais in [5].

Among the many results of [5] is the fact that if X is a separable metrizable proper G-space with G a Lie group, then each orbit of X is closed, each isotropy group is compact, and there is a metric defined on X with respect to which G acts on X as a group of isometries.

In § 1 we prove the following converse of this result.

Theorem A. *Let X be a connected locally compact metric G-space with G a second countable Lie group acting effectively on X as a group of isometries. If there is a p in X with Gp closed and G_p compact, then X is a proper G-space.*

For a G-space X on which G is a Lie group acting freely, the triple $X(X/G, G)$ is a principal fibre bundle if and only if X is a proper G-space. This result appears in § 4 of [5].

The differentiable version of this result is also true. Specifically, we prove the following theorem in § 2.

Theorem B, *Let X be a differentiable G-space with G acting freely on X and* $\dim G > 0$. *Then X is a proper G-space if and only if $X(X/G, G)$ is a differentiable principal fibre bundle.*

In § 3 we show that the parallelizability theorem of L. Markus (see [4]) is a special case of Theorem B.

0. Notation

Let X be a G-space. For p in X and g in G, let gp denote the image of the pair (g, p) under the action of G. Let $Gp = \{gp \mid g \in G\}$, $G_p = \{g \in G \mid gp = p\}$.

Communicated by Y. Matsushima, June 13, 1973. The results in this paper are part of the author's Ph.D. dissertation written under Professor T. Nagano of the University of Notre Dame.

Call *Gp the orbit of X through p* and G_p the *isotropy group of G at p*. The *orbit space* provided with the quotient topology is denoted by X/G.

1. Proof of Theorem A

In this section X will denote a connected locally compact metric space, G a second countable Lie group, and $I(X)$ the isometry group of X provided with the compact-open topology.

The proof of the following lemma may be found on pp. 47–49 of [3].

Lemma 1. *Let $\{\varphi_n\}$ be a sequence in $I(X)$, and p a point in X. Suppose $\{\varphi_n(p)\}$ converges in X. Then there is a subsequence of $\{\varphi_n\}$ converging in $I(X)$.*

Lemma 2. *Assume that G acts effectively on X as a group of isometries, so that we identify G with a subgroup of $I(X)$. If there is a p in X with Gp closed in X and G_p closed in $I(X)$, then G is closed in $I(X)$ and all orbits of X are closed. Moreover X/G is metrizable.*

Proof. Give $I(X)/G_p$ (left coset space) the quotient topology. Then G_p closed in $I(X)$ implies that $I(X)/G_p$ is Hausdorff. Regard G/G_p as a subset of $I(X)/G_p$. The manifold topology of G contains the subspace topology of G inherited from $I(X)$. It follows that the manifold topology of G/G_p contains the subspace topology of G/G_p inherited from $I(X)/G_p$.

Gp closed implies that $s_p\colon G/G_p \to Gp$ defined by $s_p(gG_p) = pg$ is a homeomorphism.

Let $\{g_n\}$ be a sequence in G, and φ in $I(X)$ with $g_n \to \varphi$ in $I(X)$. Then $g_n p \to \varphi(p)$ in X. Let $\varphi(p) = gp$ for some g in G. Then $s_p^{-1}(g_n p) = g_n G_p \to gG_p = s_p^{-1}(gp)$ in G/Gp with the subspace topology. Therefore $g_n G_p \to gG_p$ in $I(X)/G_p$. But $g_n \to \varphi$ implies $g_n G_p \to \varphi G_p$ in $I(X)/G_p$. Therefore $I(X)/G_p$ Hausdorff implies that $gG_p = \varphi G_p$. In particular φ is in G. Thus G is closed in $I(X)$. This immediately implies that all the orbits are closed.

Let d be the metric of X, and $\pi\colon X \to X|G$ the projection. For p and q in X set $\bar{d}(\pi(p), \pi(q)) = d(Gp, Gq)$. One easily verifies that $\bar{d}$ is a metric for X/G, which induces the quotient topology. q.e.d.

Assume that X is a G-space. For p in X set $J(p) = \{q \in X \mid \exists$ sequences $\{p_n\}$ and $\{g_n\}$ in X and G respectively such that $p_n \to p$, $g_n p_n \to q$, and $\{g_n\}$ has no convergent subsequence in $G\}$. Call $J(p)$ the *prolongational limit set of p*. We use $J(p)$ to characterize Cartan G-spaces.

Lemma 3. *X is a Cartan G-space if and only if $p \notin J(p)$ for all p in X.*

Proof. Assume that X is a Cartan G-space. Let $p \in J(p)$ with $p_n \to p$, $g_n p_n \to p$, and $\{g_n\}$ having no convergent subsequence in G. Let U be an open neighborhood of p with $(U, U) = \{g \in G \mid gU \cap U \neq \phi\}$ relatively compact. For large n, p_n and $g_n p_n$ are in U so that g_n is in (U, U) and hence $\{g_n\}$ contains a convergent subsequence, a contradiction.

Conversely assume that $p \notin J(p)$ for all p in X. For p in M suppose G_p is not compact. Then there is a sequence $\{g_n\}$ in G_p having no convergent subsequence in G_p and hence in G since G_p is closed in G. But this implies that $p \in J(p)$ by letting $p_n = p$, a contradiction. Thus for all p in X, G_p is compact.

If X is not a Cartan G-space, then there are a p in M and a sequence $\{U_n\}$ of open neighborhoods of p with $U_{n+1} \subset U_n$, (U_n, U_n) not relatively compact, and $\bigcap_{n=1}^{\infty} U_n = \{p\}$. Choose an open neighborhood U of e in G (where e is the identity) so that $G_p \subset U$ and U is relatively compact. Then there is a g_n in $(U_n, U_n) - U$. g_n in (U_n, U_n) implies that there is a p_n in U_n such that $g_n p_n$ is in U_n. $\bigcap_{n=1}^{\infty} U_n = \{p\}$ implies that $p_n \to p$ and $g_n p_n \to p$. Since $p \notin J(p)$, $\{g_n\}$ has a convergent subsequence, say $\{g_{n_k}\}$ with $g_{n_k} \to g$. Then $g_{n_k} p_{n_k} \to p$, $p_{n_k} \to p$, $g_{n_k} \to g$ imply that $p = gp$. Thus g is in G_p, and hence g_{n_k} is in U for large n_k, a contradiction.

Proof of Theorem A. Identify G with a subgroup of $I(X)$. Let T_m be the manifold topology of G, and T_s the subspace topology inherited from $I(X)$. Then the identity $\iota : (G, T_m) \to (G, T_s)$ is a continuous homomorphism. Thus by [2, Corollary 3.3, p. 111] ι is also open, so that $T_s = T_m$. Hence G_p compact implies that G_p is closed in $I(X)$. By Lemma 2, G is closed in $I(X)$ and X/G is Hausdorff.

Suppose that for q in X, $q \in J(q)$ with $q_n \to q$, $g_n q_n \to q$ and $\{g_n\}$ having no convergent subsequence in G. Let d be the metric of X. Then $d(q, g_n q) \leq d(q, g_n q_n) + d(g_n q_n, g_n q) = d(q, g_n q_n) + d(q_n, q) \to 0$. Thus $g_n q \to q$. By Lemma 1, $\{g_n\}$ contains a subsequence convergent in $I(X)$ and hence in G since G is closed and $T_s = T_m$. This contradicts the assumption on $\{g_n\}$. Thus $q \notin J(q)$ for all q in X.

Hence Theorem A follows from Lemma 3 and Theorem 1.2.9 of [5].

2. Proof of Theorem B

Proof. Assume that X is a proper G-space. Since G acts freely on X, there exist complete vector fields V_i on X, $i = 1, \cdots, m = \dim G$, such that for all p in M, $\{V_i(p)\}$ is a basis for the tangent space $T_p(Gp)$. Therefore given p in M we can find a coordinate chart $(U, y = y_1, \cdots, y_n))$, $n = \dim X$, about p with $y(p) = 0$ and $V_i(p) = \partial/\partial y_i(p)$, $i = 1, \cdots, m$. Let $S_p^* = \{q \in U \mid y_i(q) = 0, i = 1, \cdots, m\}$. Then S_p^* is a submanifold of X, $p \in S_p^*$, and by making U smaller if necessary we may assume that for all q in S_p^*, $T_q(X) = T_q(Gq) \oplus T_q(S_p^*)$. By § 2.2 and Proposition 2.1.7 of [5] there exists an open set S_p in S_p^* such that S_p is a slice at p. It is easily verified that the map $\alpha_p : G \times S_p \to GS_p$ defined by $\alpha_p(g, q) = gq$ is a diffeomorphism.

Let $\pi : X \to X/G$ be the projection. Then π is open. For each p in X, choose S_p and α_p as above. It is readily verified that $\pi|_{S_p}$ maps S_p homeomorphically onto the open set $\pi(S_p)$, and if we set $\psi_p = \pi|_{S_p}{}^{-1}$, then for p and q in X with $\pi(S_p) \cap \pi(S_q) \neq \phi$, $\psi_q \circ \psi_p^{-1} : \psi_p(\pi(S_p) \cap \pi(S_q)) \to \psi_q(\pi(S_p) \cap \pi(S_q))$ is a diffeomorphism. Since $\{\pi(S_p)\}$ covers X/G, by choosing as coordinate charts parirs of the form (U, φ) where (V, ψ) is a coordinate chart in some S_p,

$U = \pi(V)$ and $\varphi = \psi \circ \psi_p$ we have a C^∞ atlas on X/G such that each ψ_p is a diffeomorphism. By Theorem 1.2.9 of [5] X/G is Hausdorff. Thus X/G is a differentiable manifold. It is easily verified that π is C^∞ and $\pi^{-1}(\pi(S_p)) \approx \pi(S_p) \times G$ by $gq \to (\pi(q), g)$ where $g \in G$ and $q \in S_p$. Hence $X(X/G, G)$ is a differentiable principal fibre bundle.

Conversely, if $X(X/G, G)$ is a differentiable principal fibre bundle with G acting on X on the left, then X/G is Hausdorff; and if p is in X, choose U to be an open neighborhood of $\pi(p)$ in X/G with $\beta\colon \pi^{-1}(U) \approx U \times G$. Let $\beta(p) = (\pi(p), g)$ and $S_p = \beta^{-1}(U \times \{g\})$. It is readily verified that S_p is a slice at p, and from Theorems 1.2.9 and 2.3.3 of [5] it follows that X is a proper G-space.

Corollary. *Let X be a paracompact differentiable manifold, and R^m a Euclidean m-space. Then X is a differentiable proper R^m-space if and only if X is diffeomorphic to a product $N \times R^m$.*

Proof. If X is a differentiable proper R^m-space, then from Proposition 1.1.4 of [5] R^m acts freely on X. By Theorem B, $X(X/R^m, R^m)$ is a differentiable principal fibre bundle. By Theorem 4.3.1 of [5] R^m acts on X as a group of isometries with respect to a Riemannian metric. From Lemma 2 it follows that X/R^m is paracompact. Thus the corollary follows from the following theorem whose proof may be found on pp. 58–59 of [3].

Theorem. *If $X(X/R^m, R^m)$ is a differentiable principal fibre bundle with X/R^m paracompact, then $X(X/R^m, R^m)$ admits a cross section. If s is a cross section, then $f\colon X/R^m \times R^m \to X$ defined by $f(y, t) = ts(y)$ is a diffeomorphism.*

The converse is obvious.

Corollary. *Let X be a Riemannian manifold, and V a complete Killing vector field on X. Assume that the action of R $(= R^1)$ on X induced by the one-parameter group of V is free, and that one integral curve of V is closed. Then X is diffeomorphic to a product $N \times R$ by a diffeomorphism f with $f_*(V) = \partial/\partial x$ where $\{x\}$ is the usual coordinate system on R.*

Proof. From Theorem A it follows that X is a proper differentiable R-space where the action is given by the one-parameter group of V. The above corollary yields the existence of an $f\colon X \approx X/R \times R$ where f^{-1} is of from $(y, t) \to ts(y)$ for s a cross section of $X(X/R, R)$. An easy computation shows that $f_*^{-1}(\partial/\partial x) = V$.

3. Parallelizability

In this section X will be a connected paracompact differentiable manifold, and V a complete vector field on X. Via the one-parameter group of V, X is a differentiable R-space denoted by $X_{(V)}$.

In [4] Markus defined the concepts of a *completely unstable* complete vector field and a complete vector field *without separatricies*. From Theorem 2 of [4] and Theorem 1.2.9 of [5] it follows that V is completely unstable if and only if $X_{(V)}$ is a Cartan R-space, and that V is completely unstable without separatricies if and only if $X_{(V)}$ is a proper R-space (see [1] for details). Thus the first corollary to Theorem B and the proof of the second corollary to Theorem B give the parallelizability theorem of Markus (Theorem 4 of [4]).

Theorem. *Assume that V is completely unstable and without separatricies. Then X/R is a differentiable manifold, and there is an $f: X \approx X/R \times R$ such that $f_*(V) = \partial/\partial x$ where $\{x\}$ is the usual coordinate system on R.*

Remark. Let g be a Riemannian metric on X, and $L_V g$ the Lie derivative of g with respect to V. Assume that $L_V g$ is again a Riemannian metric and that V never vanishes. Then $X_{(V)}$ is a proper R-space. If X vanishes at some point p, then p is unique, $X - \{p\}_{(V)}$ is a proper R-space, and X is diffeomorphic to a Euclidean space. For details see [1].

References

[1] W. Dydo, *Group dynamics and instability,* Ph.D. Dissertation, University of Notre Dame, 1973.

[2] S. Helgason, *Differential geometry and symmetric spaces,* Academic Press, New York, 1962.

[3] S. Kobayashi & K. Nomizu, *Foundations of differential geometry,* Vol. I, Interscience, New York, 1963.

[4] L. Markus, *Parallel dynamical systems,* Topology **8** (1969) 47–57.

[5] R. S. Palais, *On the existence of slices for actions of non-compact Lie groups,* Ann. of Math. **73** (1961) 295–323.

UNIVERSITY OF NOTRE DAME