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# AN INCLUSION THEOREM FOR OVALOIDS WITH COMPARABLE SECOND FUNDAMENTAL FORMS

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The basic question we study is under what conditions on the curvature of two ovaloids can we guarantee that one fits inside the other. The result obtained is that if at points with equal exterior normals the second fundamental from of one ovaloid is greater or equal to that of a second ovaloid then the first fits inside the second.

## 1. Notation

A smooth closed hypersurface in $\boldsymbol{R}^k$ ($k \geq 3$) with strictly positive scalar curvature is called an *ovaloid*. Hadamard's theorem asserts that an ovaloid is the boundary of a bounded open strictly convex set. For an ovaloid $M$ and $x \in M$, let $n(x)$ be the unit outward normal at $x$. The Gauss map $x \to n(x)$ is a diffeomorphism of $M$ onto $S$, the unit sphere in $\boldsymbol{R}^k$. The map $\gamma = n^{-1}$ gives a parametrization of $M$ by $S$ which is important for our work. If $M'$ is a second ovaloid, and $w \in S$, then $\gamma(w)$ and $\gamma'(w)$ are the points on $M$ and $M'$ whose outward normals are equal.

$D$ denotes the directional derivative operator. That is, if $F\colon \mathcal{O} \to \boldsymbol{R}^m$ is a smooth map on an open set $\mathcal{O} \subset \boldsymbol{R}^k$, and $v = (v_1, \cdots, v_k) \in \boldsymbol{R}^k$, then $D_vF(x) = \sum_1^k v_j \frac{\partial F}{\partial x_j}(x)$. This still makes sense if $F$ is defined on a submanlfold such that $v$ is in the tangent space at $x$. We view the tangent space as the linear subspace of $\boldsymbol{R}^k$ consisting of tangential directions. With this convention, the tangent space $M_x$ to $M$ at $x$ is the set of vectors in $\boldsymbol{R}^k$ orthogonal to $n(x)$.

The Weingarten map $L_x\colon M_x \to M_x$ is defined by $L_xv = D_vn(x)$. With the scalar product on $M_x$ inherited from $\boldsymbol{R}^k$, $L_x$ is a self-adjoint operator on $M_x$ whose eigenvalues are the principal curvatures at $x$. The second fundamental form $II_x$ is a quadratic form on $M_x$ defined by $II_x(v) = v \cdot L_xv$. With the above conventions the tangent space to $M$ at $x$ is identical to the tangent space of $S$ at $w = n(x)$, and $L$ is the Jacobian $n_*$ of the Gauss map. Since $M_x = S_w$, $L_x$ may be viewed as an operator in $S_w$ and $II_x$ as a quadratic form on $S_w$. We abuse notation by denoting these objects by $L_w$ and $II_w$. The relation $\gamma = n^{-1}$

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implies $\gamma_* = (n_*)^{-1}$, so with the above conventions this shows that if $w = n(x)$ then $D_v\gamma(w) = (L_w)^{-1}v$ for all $v \in S_w$.

## 2. Main result

Our main theorem asserts that, if we have two convex surfaces one of which is more curved than the other, then the more curved one fits inside the less curved. Such a result is very natural from the intuitive point of view, but there are several pitfalls. One must choose the correct notion of curvature and the right way to compare the curvatures of the two surfaces.

For example, there are nice surfaces in $\boldsymbol{R}^3$ with Gauss curvature and mean curvature both bounded below by 1, which nevertheless do not fit inside a sphere of radius one. Notice that if the Gauss curvature is larger than one, and if the appropriate orientation is taken, then the mean curvature is automatically larger than one. Thus the example sketched by Spruck [4] performs the desired trick. On the other hand, Blaschke [1] has obtained positive results when one surface is a sphere of radius $R$ and the principal curvatures of the second surface are always larger or smaller than $1/R$. These results were extended by Koutroufiotis [2] who also treated noncompact surfaces. For compact surfaces our theorem contains the above results as special cases.

To guess a correct notion of curvature consider the problem locally. Suppose we have two ovaloids $M, M'$ which are tangent at a point $x$ and have the same outward normal at $x$ (we say $M$ and $M'$ are *internally tangent* at $x$). A necessary condition for $M$ to be inside $M'$ near $x$ is that $II_x \geq II'_x$, that is, $II_x(v) \geq II'_x(v)$ for all $v \in M_x (= M'_x)$. This condition is equivalent to the requirement that the osculating ellipsoid of $M$ at $x$ be inside the osculating ellipsoid of $M'$ at $x$.

**Theorem.** *Suppose $M$ and $M'$ are ovaloids such that $II_x \geq II_{x'}$ for all $x \in M$, $x' \in M'$ with $n(x) = n'(x')$. If $M$ and $M'$ are internally tangent at one point, then $M$ is contained in the closed bounded region determined by $M'$.*

The conclusion of this theorem asserts that $M$ is inside $M'$. If we drop the hypothesis that $M$ and $M'$ are internally tangent at one point, it still follows that $M$ can fit inside $M'$; for, given any $x' \in M'$ we may translate $M$ so that $M$ is internally tangent to $M'$ at $x'$, and then the theorem implies that $M$ is inside $M'$.

*Proof.* We make essential use of the support functions of the convex sets determined by $M$ and $M'$. A similar device was used by Blaschke [1] to prove the plane curve analogue of the theorem. For our purpose the support function $p$ is viewed as a function on the unit sphere $S$ defined by $p(w) = w \cdot \gamma(w)$. The corresponding functions for $M'$ are denoted with a prime. Thus $M$ is inside $M'$ if and only if $p(w) \leq p'(w)$ for all $w \in S$.

Suppose $M$ is internally tangent to $M'$ at $x$ and $w_0 = n(x)$ $(= n'(x))$. We prove $p \leq p'$ by showing that $p \leq p'$ on each great circle $\Gamma$ on $S$, which passes through $w_0$. Fix such a great circle $\Gamma$ and parametrize $\Gamma$ by the arc length $\theta$

measured from $w_0$. Functions on $\Gamma$, for example $p$ restricted to $\Gamma$, can then be viewed as functions of $\theta$ periodic with period $2\pi$. We now derive a differential equation for $p$ on $\Gamma$. Differentiating $p(w) = w \cdot \gamma(w)$ with respect to $\theta$ yields

$$\frac{dp}{d\theta} = \frac{dw}{d\theta} \cdot \gamma(w(\theta)) + w(\theta) \cdot \frac{d\gamma}{d\theta}(w(\theta)) .$$

If $T$ is the unit tangent field to $\Gamma$, then $dw/d\theta = T$, and $d\gamma/d\theta = D_T\gamma(w(\theta)) \in M_{\gamma(w(\theta))}$ and so is perpendicular to $w(\theta)$. Therefore $dp/d\theta = T \cdot \gamma$. Differentiating a second time yields

$$\frac{d^2p}{d\theta^2} = \gamma \cdot \frac{dT}{d\theta} + \frac{d\gamma}{d\theta} \cdot T .$$

As above $d\gamma/d\theta = D_T\gamma$, so as computed at the end of section one $d\gamma/d\theta = (L_{w(\theta)})^{-1}T$. In addition $dT/d\theta = -w(\theta)$, so we have

$$\frac{d^2p}{d\theta^2}(\theta) + p(\theta) = T(\theta) \cdot (L_{w(\theta)})^{-1}T(\theta) . \tag{1}$$

This identity is the heart of the proof. For $p'(\theta)$ we have

$$\frac{d^2p'}{d\theta^2}(\theta) + p'(\theta) = T(\theta) \cdot (L'_{w(\theta)})^{-1}T(\theta) . \tag{2}$$

Since $M$ and $M'$ are internally tangent at $x$, we have $x = \gamma(w_0) = \gamma'(w_0)$, so that

$$p(0) = w_0 \cdot \gamma(w_0) = w_0 \cdot \gamma'(w_0) = p'(0) , \tag{3}$$

$$\frac{dp}{d\theta}(0) = T(w_0) \cdot \gamma(w_0) = T(w_0) \cdot \gamma'(w_0) = \frac{dp'}{d\theta}(0) . \tag{4}$$

Next we need an elementary result from linear analysis.

**Lemma.** *If $A$ and $B$ are strictly positive bounded operators on a Hilbert space with $A \geq B$, then $A^{-1} \leq B^{-1}$.*

*Proof of lemma.* Let $C(t) = [A + t(B - A)]^{-1}$. Then $C(0) = A^{-1}$, and $C(1) = B^{-1}$, and $\dfrac{d}{dt}C = C(A - B)C$ is a nonnegative operator so that $C(1) - C(0) = \int_0^1 \left(\dfrac{dC}{dt}(t)\right)dt \geq 0$. q.e.d.

We apply the lemma to $L_{w(\theta)} \geq L'_{w(\theta)}$ to conclude that

$$T \cdot (L_w)^{-1}T \leq T \cdot (L'_w)^{-1}T \qquad \text{for all } \theta . \tag{5}$$

If $\Delta(\theta) = p'(\theta) - p(\theta)$, then (1), $\cdots$, (5) imply

$$\frac{d^2\Delta}{d\theta^2} + \Delta = \phi(\theta) \geq 0 \ , \qquad \Delta(0) = \frac{d\Delta}{d\theta}(0) = 0 \ .$$

Therefore

$$\Delta(\theta) = \int_0^\theta \phi(t) \sin(\theta - t) dt \ , \tag{6}$$

which implies that $\Delta \geq 0$ for $-\pi \leq \theta \leq \pi$. In addition, $\Delta$ is periodic with period $2\pi$ so $\Delta \geq 0$ for all $\theta$. Therefore $p \leq p'$ on $\Gamma$, and the proof is complete.

## 3. Applications and remarks

The simplest situation where the theorem applies is when $II_x \geq cI \geq II_{x'}$, that is, the smallest principal curvature of $M$ at $x$ is larger than all principal curvatures of $M'$ at $x'$. This special case was obtained by Koutroufiotis [2]. A further specialization occurs if $M$ (resp. $M'$) is a sphere, in which case we need an upper (resp. lower) bound on the principal curvatures of $M'$ (resp. $M$). As a typical application of these ideas we have the following characterization of spheres.

**Corollary.** *The sphere is the only ovaloid whose principal radii of curvature are less than or equal to half its width at each point.*

Recall that the width of an ovaloid $M$ at a point $x$ is the distance between the two tangent planes to $M$ which are orthogonal to $n(x)$.

*Proof.* Let $r$ be the largest principal radius of curvature of $M$, and choose $x \in M$ so that $r$ is a principal radius at $x$. Let $M'$ be the sphere of radius $r$ internally tangent to $M$ at $x$. The theorem asserts that $M$ is inside $M'$, so the width at $x$ is less than or equal to $2r$. To complete the proof we must show that if equality holds, then $M = M'$. If equality holds then the point opposite $x$ on $M'$ must also lie on $M$. If $\Delta, \Gamma, p, \phi$ have the same meaning as in the proof of the theorem, then we have $\Delta(-\pi) = \Delta(\pi) = 0$. Equation (6) then implies that $\phi \equiv 0$ so $\Delta \equiv 0$. Therefore $p = p'$ on each $\Gamma$, so $p = p'$ everywhere and $M$ must coincide with $M'$. q.e.d.

It is also true that an ovaloid whose principal radii of curvature are always at least half the width must be a sphere. We prove a stronger result. For any $x \in M$ the normal line at $x$ intersects $M$ in exactly two points. The distance between these two points is called the *diameter at x*. The diameter is less than or equal to the width.

**Corollary.** *The sphere is the only ovaloid whose principal radii of curvature are greater than or equal to half the diameter at all points.*

*Proof.* One shows that the sphere whose radius is equal to the minimum radius of curvature of the ovaloid and is internally tangent at a point where the minimum is achieved lies inside $M$. The argument is completed exactly as for the preceding corollary.

## References

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