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MODELS OF THE RIEMANNIAN MANIFOLDS O_n^2 IN THE LORENTZIAN 4-SPACE

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1. Introduction

We denote by O_n^2 the 2-dimensional Riemannian manifold defined on the unit disk D^2: $u^2 + v^2 < 1$ in the uv-plane with the following metric:

$$(1.1) \quad ds^2 = (1 - u^2 - v^2)^{n-2}\{(1 - v^2)du^2 + 2uvdudv + (1 - u^2)dv^2\} ,$$

which is called the Otsuki manifold (of type number n) following W. Y. Hsiang and H. B. Lawson who treated it in [3] for any integer $n \geq 2$ and in particular for the case where $n = 2$. The second auther of this paper studied it about the angular periodicity of geodesics in [4], [5] and [6].

On the other hand, O_0^2 is the hyperbolic plane H^2 of curvature -1, and (1.1) is the metric described in the Cayley-Klein's model of H^2. O_1^2 is the hemisphere: $u^2 + v^2 + w^2 = 1$ and $w > 0$, and (1.1) is the metric described in the plane of the equator: $w = 0$ through the orthogonal projection.

As is well known, some part of H^2 but not whole plane can be represented as a surface of revolution in the Euclidean 3-space E^3. In the present paper, we shall show that O_n^2 $(n > 1)$ can be represented as a surface of revolution in E^3 for the part: $u^2 + v^2 \leq (2n - 1)/n^2$, and the whole space can be done as such a surface in the Lorentzian 4-space.

2. Preliminaries

Putting $u = r\cos\theta$, $v = r\sin\theta$, we can write (1.1) as

$$(2.1) \quad ds^2 = (1 - r^2)^{n-2}dr^2 + r^2(1 - r^2)^{n-1}d\theta^2 ,$$

which shows that the metric (1.1) is invariant under the group of rotations around the origin of D^2.

Putting $E = (1 - r^2)^{n-2}$ and $G = r^2(1 - r^2)^{n-1}$, from

$$K = -\frac{1}{\sqrt{EG}}\left\{\frac{\partial}{\partial r}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial r}\right)\right\} ,$$

we can obtain the Gaussian curvature K of O_n^2, namely,

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$$K = (2n - 1 - nr^2)(1 - r^2)^{-n} , \tag{2.2}$$

which leads immediately to

Proposition 1. *O_n^2 is of positive Gaussian curvature for $n \geq 1$, and of negative Gaussian curvature for $0 \leq n < \frac{1}{2}$.*

Next, we denote the length of curve $r = a$ by $l(a)$. Then

$$l(a) = 2\pi a(1 - a^2)^{\frac{1}{2}(n-1)} , \tag{2.3}$$

from which we can easily obtain

Proposition 2. *If $n > 1$, then $l(a)$ is maximal when $a = n^{-\frac{1}{2}}$, and $l(n^{-\frac{1}{2}}) = 2\pi(ne_{n-1})^{-\frac{1}{2}}$, where $e_{n-1} = [1 + 1/(n-1)]^{n-1}$.*

3. A representation of O_n^2 in E^3

In the following we suppose $n > 1$. In the Euclidean 3-space E^3 with canonical coordinates x, y, z, let us consider a smooth surface of revolution M^2 given by

$$p = (f(z) \cos \theta, f(z) \sin \theta, z) . \tag{3.1}$$

The induced Riemannian metric on M^2 from E^3 is

$$ds^2 = \{1 + (f'(z))^2\}dz^2 + (f(z))^2 d\theta^2 , \tag{3.2}$$

where z, θ are considered as local coordinates of M^2.

Using the polar coordinates r, θ of R^2 regarded as an E^2, we consider a mapping from a neighborhood of the origin of R^2 to M^2: $O_n^2 \ni (r, \theta) \to (z, \theta) \in M^2$, given by

$$z = \varphi(r) . \tag{3.3}$$

Then from (2.1) and (3.2) it follows that this mapping is isometric if and only if the following equations are satisfied:

$$(1 - r^2)^{n-2} = \{1 + (f'(\varphi(r)))^2\}(\varphi'(r))^2 , \tag{3.4}$$

$$r^2(1 - r^2)^{n-1} = (f(\varphi(r)))^2 . \tag{3.5}$$

Since we may suppose $f \geq 0$, from (3.5) we get

$$f(\varphi(r)) = r(1 - r^2)^{\frac{1}{2}(n-1)} . \tag{3.6}$$

Differentiating (3.6), we have

$$f'(\varphi(r))\frac{d\varphi}{dr} = (1 - r^2)^{\frac{1}{2}(n-3)}(1 - nr^2) , \tag{3.7}$$

and substitution of this in (3.4) gives

$$(d\varphi/dr)^2 = r^2(1 - r^2)^{n-3}\lambda(r) ,$$

where

$$\lambda(r) = 2n - 1 - n^2r^2 . \tag{3.8}$$

Since we may suppose that $\varphi(r)$ is monotone increasing, we obtain

$$\varphi(r) = \int_0^r t(1 - t^2)^{\frac{1}{2}(n-3)}\sqrt{\lambda(t)}dt \qquad \text{for } 0 \leq r \leq \frac{\sqrt{2n-1}}{n} . \tag{3.9}$$

Now let

$$r = \psi(z) \tag{3.10}$$

be the inverse function of $\varphi(r)$. Then (3.6) implies

$$f(z) = \psi(z)\{1 - (\psi(z))^2\}^{\frac{1}{2}(n-1)} . \tag{3.11}$$

Finally, putting

$$\varphi(n^{-\frac{1}{2}}) = a , \qquad \varphi(\sqrt{2n-1}/n) = b , \tag{3.12}$$

we obtain

$$\begin{aligned} f(a) &= \frac{1}{\sqrt{n}}\left(1 - \frac{1}{n}\right)^{\frac{1}{2}(n-1)} = \frac{1}{\sqrt{n}e_{n-1}} , \\ f(b) &= \frac{\sqrt{2n-1}}{n}\left(1 - \frac{2n-1}{n^2}\right)^{\frac{1}{2}(n-1)} = \frac{\sqrt{2n-1}}{ne_{n-1}} , \\ &\lim_{n\to\infty} \frac{f(b)}{f(a)} = \sqrt{\frac{2}{e}} . \end{aligned} \tag{3.13}$$

Furthermore from (3.7), (3.8) and (3.9) it follows that

$$f'(z) = (1 - nr^2)r^{-1}(\lambda(r))^{-\frac{1}{2}} , \tag{3.14}$$

$$f'(0) = +\infty , \quad f'(a) = 0 , \quad f'(b) = -\infty . \tag{3.15}$$

Thus we have

Theorem 1. *O_n^2 can be represented as a surface of revolution*: $(f(z)\cos\theta, f(z)\sin\theta, z)$ *in E^3 for $0 \leq r \leq \sqrt{2n-1}/n$, where $z = \varphi(r)$ and $f(z)$ are given by* (3.9), (3.10) *and* (3.11).

Remark. The profile curve $\mathscr{C}$ of the surface of revolution in Theorem 1 is given by

(3.16) $$x = r(1 - r^2)^{\frac{1}{2}(n-1)} , \qquad z = \varphi(r) .$$

Let k_1 (= the curvature of $\mathscr{C}$) and k_2 be the principal curvatures of this surface. Then as is well known

$$k_1 = -f''(z)\{1 + (f'(z))^2\}^{-3/2} , \qquad k_2 = x^{-1}\{1 + (f'(z))^2\}^{-1/2} .$$

By using (3.14) and (3.16), we can easily obtain

(3.17) $$k_1 = \frac{2n - 1 - nr^2}{(1 - r^2)^{n/2}\sqrt{\lambda(r)}}, \qquad k_2 = \frac{\sqrt{\lambda(r)}}{(1 - r^2)^{n/2}} ,$$

from which follow

$$\lim_{z \to b} k_1 = +\infty , \qquad \lim_{z \to b} k_2 = 0 .$$

4. A surface theory in the Lorentzian 3-space

In this section, for our purpose we give a brief theory of surfaces in the Lorentzian 3-space.

Let R^3 denote the Cartesian product $R \times R \times R$ where R is the set of real numbers. On R^3 with the canonical coordinates x_1, x_2, x_3, the Euclidean 3-space E^3 and the Lorentzian 3-space L^3 are defined by the metrices

$$E^3 \colon ds^2 = dx_1^2 + dx_2^2 + dx_3^2 , \qquad L^3 \colon ds^2 = dx_1^2 + dx_2^2 - dx_3^2 ,$$

respectively. We denote the inner products, in E^3 and L^3, of any two vectors $X = \sum X_i \partial/\partial x_i$ and $Y = \sum Y_i \partial/\partial x_i$ by

(4.1) $$(X, Y) = X_1Y_1 + X_2Y_2 + X_3Y_3 ,$$

(4.2) $$\langle X, Y \rangle = X_1Y_1 + X_2Y_2 - X_3Y_3 ,$$

respectively, denote the symmetry of E^3 with respect to the x_1x_2-plane by φ, and extend φ to vectors as follows:

(4.3) $$\varphi(X) = X_1\partial/\partial x_1 + X_2\partial/\partial x_2 - X_3\partial/\partial x_3 .$$

Then we have

(4.4) $$\langle X, Y \rangle = (X, \varphi(Y)) = (\varphi(X), Y) .$$

Let $X \wedge Y$ be the outer product of X and Y in E^3, that is,

$$X \wedge Y = (X_2Y_3 - X_3Y_2)\frac{\partial}{\partial x_1} + (X_3Y_1 - X_1Y_3)\frac{\partial}{\partial x_2} + (X_1Y_2 - X_2Y_1)\frac{\partial}{\partial x_3} ,$$

and let $\{X, Y\}$ denote the space spanned by X and Y. Then we obtain easily

Lemma 1. *$\varphi(X \wedge Y) \in \{X, Y\}$ if and only if $X \wedge Y$ is a null vector of L^3.*

Now let M be a surface in R^3, and M_x the tangent space at $x \in M$. Let N_x and $\tilde{N}_x$ be the normal tangent spaces of M_x in E^3 and L^3, and denote the normal bundles of M in E^3 and L^3 by $N(M)$ and $\tilde{N}(M)$, respectively. By virtue of (4.4), we have immediately

Lemma 2. *$\tilde{N}_x = \varphi(N_x)$.*

A point of $x \in M$ is said to be regular if $\tilde{N}_x$ is linearly independent of M_x. For any tangent vector fields $X, Y \in$ the set $\Gamma(T(M))$ of smooth cross sections of the tangent bundle $T(M)$ of M, we have

$$d_X Y = \nabla_X Y + T_X Y \, , \tag{4.5}$$

where $d_X Y$ is the ordinary derivative of Y with respect to X in R^3, $\nabla_X Y \in \Gamma(T(M))$, and $T_X Y \in \Gamma(N(M))$.

Supposing every point of M is regular in L^3, we have the following formula with respect to L^3 analogous to (4.5):

$$d_X Y = \tilde{\nabla}_X Y + \tilde{T}_X Y \, , \quad \tilde{\nabla}_X Y \in \Gamma(T(M)) \, , \quad \tilde{T}_X Y \in \Gamma(\tilde{N}(M)) \, . \tag{4.6}$$

Let (x, e_1, e_2, e_3) be an orthonormal frame of E^3 at $x \in M$ such that $e_3 \in N_X$. Then

$$T_X Y = A(X, Y) e_3 \, , \tag{4.7}$$

where $A(X, Y)$ is the 2nd fundamental form of M in E^3.

Proposition 3. *For any $X, Y \in \Gamma(T(M))$ at any regular point of M in L^3, we have*

$$\tilde{\nabla}_X Y = \nabla_X Y - \frac{A(X, Y)}{\langle e_3, e_3 \rangle} \operatorname{Proj} \varphi(e_3) \, , \tag{4.8}$$

$$\tilde{T}_X Y = \frac{A(X, Y)}{\langle e_3, e_3 \rangle} \varphi(e_3) \, , \tag{4.9}$$

$$\operatorname{Proj} \varphi(e_3) = \langle e_1, e_3 \rangle e_1 + \langle e_2, e_3 \rangle e_2 \, . \tag{4.10}$$

Proof. At a regular point, we easily obtain

$$e_3 = -\operatorname{Proj} \varphi(e_3) / \langle e_3, e_3 \rangle + \varphi(e_3) / \langle e_3, e_3 \rangle \, . \tag{4.11}$$

Substitution of (4.11) in (4.5) gives

$$d_X Y = \nabla_X Y + A(X, Y) \{ -\operatorname{Proj} \varphi(e_3) + \varphi(e_3) \} / \langle e_3, e_3 \rangle \, ,$$

which implies (4.8) and (4.9). q.e.d.

Now let us consider a surface of revolution around the x_3-axis in L^3 given by

$$p = (x \cos \theta, x \sin \theta, f(x)) . \tag{4.12}$$

Take the orthonormal frame (p, e_1, e_2, e_3) of E^3 given by

$$\begin{aligned} e_1 &= (1 + f'^2)^{-\frac{1}{2}} (\cos \theta, \sin \theta, f') , \\ e_2 &= (-\sin \theta, \cos \theta, 0) = \varphi(e_2) , \\ e_3 &= (1 + f'^2)^{-\frac{1}{2}}(-f' \cos \theta, -f' \sin \theta, 1) , \\ \varphi(e_3) &= (1 + f'^2)^{-\frac{1}{2}}(-f' \cos \theta, -f' \sin \theta, -1) , \end{aligned}$$

from which we obtain

$$\langle e_3, e_3 \rangle = -1/\mu = -\langle e_1, e_1 \rangle , \qquad \langle e_2, e_2 \rangle = 1 , \tag{4.13}$$

where

$$\mu = (1 + f'^2)/(1 - f'^2) . \tag{4.14}$$

so that $(e_1, e_2, \varphi(e_3))$ is an orthogonal basis of L^3.

In the following, we consider the case where

$$|f'(x)| < 1 . \tag{4.15}$$

Then putting

$$\tilde{e}_1 = \sqrt{\mu}\, e_1 , \quad \tilde{e}_2 = e_2 , \quad \tilde{e}_2 = \sqrt{\mu}\, \varphi(e_3) , \tag{4.16}$$

we see that $(p, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ is an orthonormal frame of L^3 in the following sense:

$$\begin{aligned} \langle \tilde{e}_1, \tilde{e}_1 \rangle &= \langle \tilde{e}_2, \tilde{e}_2 \rangle = -\langle \tilde{e}_3, \tilde{e}_3 \rangle = 1 , \\ \langle \tilde{e}_1, \tilde{e}_3 \rangle &= \langle \tilde{e}_2, \tilde{e}_3 \rangle = \langle \tilde{e}_1, \tilde{e}_2 \rangle = 0 . \end{aligned}$$

Proposition 4. *For a surface M of revolution around the x_3-axis in L^3 with the profile curve $x_3 = f(x_1)$ such that $|f'(x_1)| < 1$, its principal curvatures $\tilde{k}_1$ and $\tilde{k}_2$ satisfy the following equations:*

$$\tilde{k}_1 = -\mu^{3/2} k_1 , \qquad \tilde{k}_2 = -\mu^{1/2} k_2 , \tag{4.17}$$

where k_1 and k_2 are the principal curvatures of M considered as a surface in E^3.

Proof. Let us compute the principal curvatures $\tilde{k}_1$ and $\tilde{k}_2$ of the surface M in L^3 by means of the frame $(p, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ stated above. Define the 2nd fundamental form $\tilde{A}(X, Y)$ of M in L^3 by

$$\tilde{T}_X Y = \tilde{A}(X, Y) \tilde{e}_3 , \qquad X, Y \in \Gamma(T(M)) . \tag{4.18}$$

From (4.9), (4.13), (4.16) and (4.18), it follows that

$$\tilde{A}(X, Y) = -\sqrt{\mu}\, A(X, Y) . \tag{4.19}$$

Putting

$$X = X_1 e_1 + X_2 e_2 = \tilde{X}_1 \tilde{e}_1 + \tilde{X}_2 \tilde{e}_2 , \qquad Y = Y_1 e_1 + Y_2 e_2 = \tilde{Y}_1 \tilde{e}_1 + \tilde{Y}_2 \tilde{e}_2 ,$$

we have

$$\tilde{X}_1 = \mu^{-\frac{1}{2}} X_1 , \quad \tilde{X}_2 = X_2 , \quad \tilde{Y}_1 = \mu^{-\frac{1}{2}} Y_1 , \quad \tilde{Y}_2 = Y_2 .$$

Thus by noticing that $A(X, Y) = k_1 X_1 Y_1 + k_2 X_2 Y_2$, $\tilde{A}(X, Y) = \tilde{k}_1 \tilde{X}_1 \tilde{Y}_1 + \tilde{k}_2 \tilde{X}_2 \tilde{Y}_2$, from (4.19) we can easily obtain (4.17).

Proposition 5. *Let M be a surface in L^3 such that every point is regular. With respect to an orthonormal frame $(p, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ of M in L^3, we have*

$$\tilde{R}_{1212} = \tilde{A}_{11} \tilde{A}_{22} - \tilde{A}_{12} \tilde{A}_{12} , \tag{4.20}$$

where $\tilde{A}_{\alpha\beta} = \tilde{A}(\tilde{e}_\alpha, \tilde{e}_\beta)$.

Proof. For any $X, Y, Z \in \Gamma(T(M))$, we have

$$d_X Y = \tilde{\nabla}_X Y + \tilde{A}(X, Y) \tilde{e}_3 , \quad \tilde{R}(X, Y)Z := \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z ,$$

where $\tilde{R}$ is the curvature tensor of M in L^3. From the above first equation follow immediately

$$d_X d_Y Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z + \tilde{A}(Y, Z) d_X \tilde{e}_3 \pmod{\tilde{e}_3} , \qquad d_X \tilde{e}_3 \in \Gamma(T(M)) .$$

Substitution of these equations in the identity $d_X d_Y Z - d_Y d_X Z - d_{[X,Y]} Z = 0$ gives

$$\tilde{R}(X, Y)Z = \tilde{A}(X, Z) d_Y \tilde{e}_3 - \tilde{A}(Y, Z) d_X \tilde{e}_3 . \tag{4.21}$$

On the other hand, we have

$$\begin{aligned} \langle d_{\tilde{e}_\alpha} \tilde{e}_3, \tilde{e}_\beta \rangle &= -\langle \tilde{e}_3, d_{\tilde{e}_\alpha} \tilde{e}_\beta \rangle = -\langle \tilde{e}_3, \tilde{T}_{\tilde{e}_\alpha} \tilde{e}_\beta \rangle \\ &= -\tilde{A}(\tilde{e}_\alpha, \tilde{e}_\beta) \langle \tilde{e}_3, \tilde{e}_3 \rangle = \tilde{A}_{\alpha\beta} . \end{aligned}$$

Hence we can easily obtain (4.10) from $\tilde{R}_{1212} := \langle \tilde{R}(\tilde{e}_1, \tilde{e}_2) \tilde{e}_1, \tilde{e}_2 \rangle$.

Using Proposition 5 for the surface in Proposition 4, we obtain

$$\tilde{K} = -\tilde{R}_{1212} = -\tilde{A}_{11} \tilde{A}_{22} = -\tilde{k}_1 \tilde{k}_2 = -\mu k_1 k_2 ,$$

where $\tilde{K}$ is the Gaussian curvature of M.

Supposing the curve $x_3 = f(x_1)$ as is shown in Fig. 1, i.e.,

$$-1 < f'(x_1) < 0 , \qquad f''(x_1) > 0 , \tag{4.22}$$

we have

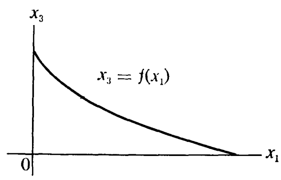


Fig. 1

$$k_1 = f''(1 + f'^2)^{-3/2}, \qquad k_2 = f'(1 + f'^2)^{-1/2}/x_1,$$

and therefore

$$\tilde{K} = -f'f''(1 - f'^2)^{-2}/x_1. \tag{4.23}$$

5. A representation of O_n^2 in L^4

We showed in § 3 that the subdomain of O_n^2 $(0 \leq r \leq \sqrt{2n-1}/n)$ is represented as a surface of revolution in E^3, but we could not extend it over $r = \sqrt{2n-1}/n$. In this section, we shall do it in the Lorentzian 4-space L^4 $(\supset E^3)$ defined by the metric:

$$ds^2 = dx^2 + dy^2 + dz^2 - dw^2 \tag{5.1}$$

on R^4 with the canonical coordinates x, y, z, w as a surface of revolution around the zw-plane.

Using the complex coordinate $\eta = u + iv$ on D^2, we can write the metric (1.1) of O_n^2 as

$$ds^2 = \tfrac{1}{4}(1 - \eta\bar{\eta})^{n-2}\{\bar{\eta}^2 d\eta^2 + 2(2 - \eta\bar{\eta})d\eta d\bar{\eta} + \eta^2 d\bar{\eta}^2\}. \tag{5.2}$$

Putting $\xi = x + iy$ and $\zeta = z + iw$, by Theorem 1 we can write the representation of O_n^2 $(0 \leq r \leq \sqrt{2n-1}/n)$ in $E^3 \subset L^4$ as

$$\xi = \eta(1 - \eta\bar{\eta})^{\frac{1}{2}(n-1)}, \qquad \zeta = \int_0^r t(1 - t^2)^{\frac{1}{2}(n-3)}\sqrt{\lambda(t)}\,dt, \tag{5.3}$$

where E^3 is considered as a hypersurface of L^4 defined by $w = 0$.

Noticing the expressions of the righthand side of (5.3), we define a mapping

$$O_n^2\,(\sqrt{2n-1}/n \leq r < 1) \to L^3 \subset L^4$$

given by

$$\xi = \eta(1 - \eta\bar{\eta})^{\frac{1}{2}(n-1)}, \qquad \zeta = b + i\int_{\sqrt{2n-1}/n}^r t(1 - t^2)^{\frac{1}{2}(n-3)}\sqrt{-\lambda(t)}\,dt, \tag{5.4}$$

where L^3 is given by $z = b$ ((3.12)) in L^4.

Theorem 2. *The mapping* (5.4) *is an isometric imbedding of* O_n^2 $(\sqrt{2n-1}/n \leq r < 1)$ *into* L^3.

Proof. From (5.3) an elementary calculation gives

$$d\xi d\bar{\xi} + d\zeta d\zeta = \tfrac{1}{4}(1-\eta\bar{\eta})^{n-2}\{\bar{\eta}^2 d\eta^2 + 2(2-\eta\bar{\eta})d\eta d\bar{\eta} + \eta^2 d\bar{\eta}^2\} .$$

Since in L^4, (5.1) can be written as $ds^2 = \mathrm{Re}\,(d\xi d\bar{\xi} + d\zeta d\zeta)$, from (5.2) it thus follows that (5.4) is an isometric immersion of O_n^2 $(\sqrt{2n-1}/n \leq r < 1)$ in L^4. We can easily see that (5.4) is one-to-one. q.e.d.

Now, the first equation of (5.4) shows that the image of the mapping (5.4) is a surface of revolution in L^4 around the zw-plane. The profile curve of the surface in L^3 is given by

$$x = r(1-r^2)^{\frac{1}{2}(n-1)} , \qquad w = \int_{\sqrt{2n-1}/n}^{r} t(1-t^2)^{\frac{1}{2}(n-3)}\sqrt{-\lambda(t)}dt . \tag{5.5}$$

Differentiating (5.5) we obtain

$$\frac{dw}{dx} = \frac{r\sqrt{-\lambda(r)}}{1-nr^2} , \tag{5.6}$$

$$\frac{d^2w}{dx^2} = -\frac{2n-1-nr^2}{(1-nr^2)^3\sqrt{-(1-r^2)^{n-3}\lambda(r)}} . \tag{5.7}$$

Since $n > 1$ and $1 - nr^2 < 0$ for $\sqrt{2n-1}/n < r$, (5.6) and (5.7) imply

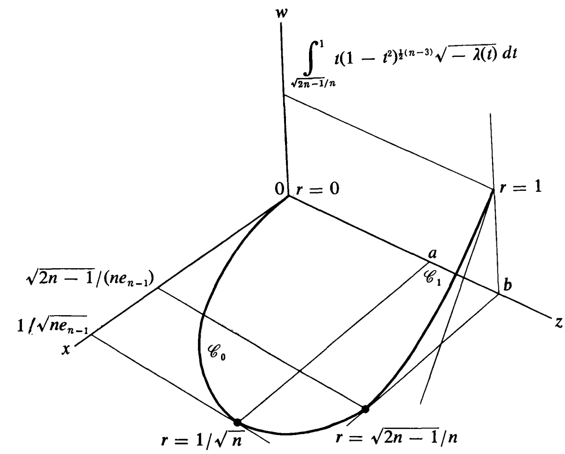
$$\begin{aligned} &\left.\frac{dw}{dx}\right|_{r=\sqrt{2n-1}/n} = 0 , \qquad \left.\frac{dw}{dx}\right|_{r=1} = -1 , \\ &\frac{d^2w}{dx^2} > 0 , \qquad -1 < \frac{dw}{dx} < 0 \qquad \text{for } \frac{\sqrt{2n-1}}{n} < r < 1 . \end{aligned} \tag{5.8}$$

The last inequality shows that the profile curve satisfies the condition in Proposition 4. By means of (5.5), (5.6), (5,7) and (2.2), and using $w(x)$ for $f(x_1)$ in (4.23) we can easily see that in L^3 the Gaussian curvature $\tilde{K}$ of the surface of revolution is equal to the Gaussian curvature K of O_n^2.

Thus putting (5.3) and (5.4) together we get an isometric imbedding of O_n^2 into L^4, the image of which is a surface of revolution around the zw-plane with the profile curve $\mathscr{C} = \mathscr{C}_0 \cup \mathscr{C}_1$ where $\mathscr{C}_0$ and $\mathscr{C}_1$ are given by

$$\mathscr{C}_0 : \begin{cases} x = r(1-r^2)^{\frac{1}{2}(n-1)} , \\ z = \varphi(r) = \displaystyle\int_0^r t(1-t^2)^{\frac{1}{2}(n-3)}\sqrt{\lambda(t)}dt , \\ w = 0 , \qquad (0 \leq r \leq \sqrt{2n-1}/n) , \end{cases} \tag{5.9}$$

$$(5.10)\quad \mathscr{C}_1 : \begin{cases} x = r(1-r^2)^{\frac{1}{2}(n-1)} , \qquad z = b , \\ w = \displaystyle\int_{\sqrt{2n-1}/n}^{r} t(1-t^2)^{\frac{1}{2}(n-3)}\sqrt{-\lambda(t)}dt , \qquad \left(\frac{\sqrt{2n-1}}{n} \le r < 1\right) . \end{cases}$$



Fig, 2

Proposition 6. *The profile curve $\mathscr{C} = \mathscr{C}_0 \cup \mathscr{C}_1$ is C^1 and not C^2. The sub-arcs $\mathscr{C}_0$ and $\mathscr{C}_1$ are C^ω.*

Proof. We have

$$\frac{dx}{dr} = (1-nr^2)(1-r^2)^{\frac{1}{2}(n-3)} ,$$

$$\frac{d^2x}{dr^2} = -(n-1)r(3-nr^2)(1-r^2)^{\frac{1}{2}(n-5)} .$$

For $\mathscr{C}_0$, we have

$$\frac{dz}{dr} = r(1-r^2)^{\frac{1}{2}(n-3)}\sqrt{\lambda(r)} , \qquad \frac{d^2z}{dr^2} = \frac{(1-r^2)^{\frac{1}{2}(n-5)}P(r)}{\sqrt{\lambda(r)}} ,$$

where

$$(5.11)\qquad P(r) = 2n-1-(4n^2-5n+2)r^2 + n^2(n-1)r^4 .$$

Since $P(r)|_{r=\sqrt{2n-1}/n} = -(2n-1)(n-1)^2/n^2 < 0$,

we get

$$(5.12)\qquad \frac{dz}{dr} \to +0 , \qquad \frac{d^2z}{dr^2} \to -\infty \qquad \text{as } r \to \frac{\sqrt{2n-1}}{n} - 0 .$$

Next, for $\mathscr{C}_1$ we have

$$\frac{dw}{dr} = r(1-r^2)^{\frac{1}{2}(n-3)}\sqrt{-\lambda(r)}\,, \qquad \frac{d^2w}{dr^2} = -\frac{(1-r^2)^{\frac{1}{2}(n-5)}P(r)}{\sqrt{-\lambda(r)}}\,,$$

so that

$$(5.13)\qquad \frac{dw}{dr} \to +0\,, \quad \frac{d^2w}{dr^2} \to +\infty \qquad \text{as } r \to \frac{\sqrt{2n-1}}{n} + 0\,.$$

These relations imply the proposition.

In conclusion, we obtain

Theorem 3. *The surface of revolition in L^4 around the zw-plane with the profile curve $\mathscr{C} = \mathscr{C}_0 \cup \mathscr{C}_1$ given by* (5.9) *and* (5.10) *is a C^1-model of O_n^2 and the parts corresponding to $\mathscr{C}_0$ and $\mathscr{C}_1$ are analytic models of O_n^2 $(0 \le r \le \sqrt{2n-1}/n)$ and O_n^2 $(\sqrt{2n-1}/n \le r \le 1)$, respectively.*

Examples. I) When $n = 2$, $\mathscr{C}_0$ and $\mathscr{C}_1$ are given by

$$\mathscr{C}_0 : \begin{cases} x = r\sqrt{1-r^2}\,, \\ z = \dfrac{1}{2}\{\sqrt{3} - \sqrt{1-r^2}\sqrt{3-4r^2}\} + \dfrac{1}{4}\log\dfrac{2\sqrt{1-r^2}+\sqrt{3-4r^2}}{2+\sqrt{3}}\,, \\ w = 0\,, \qquad \text{for } 0 \le r \le \frac{1}{2}\sqrt{3}\,. \end{cases}$$

and

$$a = \frac{1}{2}\left(\sqrt{3} - \frac{1}{2}\right) - \frac{1}{4}\log\frac{2+\sqrt{3}}{1+\sqrt{2}}\,,$$

$$b = \tfrac{1}{2}\sqrt{3} - \tfrac{1}{4}\log(2+\sqrt{3})\,;$$

$$\mathscr{C}_1 : \begin{cases} x = r\sqrt{1-r^2}\,, \qquad z = b\,, \\ w = \frac{1}{8}\pi - \frac{1}{2}\sqrt{1-r^2}\sqrt{4r^2-3} - \frac{1}{4}\sin^{-1} 2\sqrt{1-r^2}\,, \\ \qquad\qquad\qquad\qquad \text{for } \frac{1}{2}\sqrt{3} \le r < 1\,. \end{cases}$$

II) When $n = 3$, $\mathscr{C}_0$ and $\mathscr{C}_1$ are given by

$$\mathscr{C}_0 : \quad x = r(1-r^2)\,, \quad z = \frac{1}{27}\{5\sqrt{5} - (5-9r^2)^{3/2}\}\,, \quad w = 0\,,$$
$$\text{for } 0 \le r \le \sqrt{5}/3\,,$$

or

$$x = \left\{\frac{4}{9} + \left(\frac{5\sqrt{5}}{27} - z\right)^{2/3}\right\}\left\{\frac{5}{9} - \left(\frac{5\sqrt{5}}{27} - z\right)^{2/3}\right\}^{1/2}\,,$$
$$\text{for } 0 \le z \le 5\sqrt{5}/27\,,$$

and

$$a = \frac{1}{27}(5\sqrt{5} - 2\sqrt{2}) , \qquad b = \frac{5\sqrt{5}}{27} ;$$

$\mathscr{C}_1$: $x = r(1 - r^2)$, $z = b$, $w = (r^2 - 5/9)^{3/2}$, for $\sqrt{5}/3 \leq r < 1$,

or

$$x = (4/9 - w^{2/3})(5/9 + w^{2/3})^{1/2} , \qquad \text{for } 0 \leq w < 8/27 .$$

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