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EINSTEIN METRICS ON PRINCIPAL FIBRE BUNDLES

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Introduction

The following construction of Riemannian metrics on principal fibre bundles is well known. (See B. O'Neill [7] and A. Gray [1].) Let M be a manifold with Riemannian metric ds^2, and $\pi: P \to M$ be a principal fibre bundle over M with structure group G. Let γ be any connection form on P, and let $\langle\ ,\ \rangle$ denote a bi-invariant metric on G. Then $g \equiv \pi^* ds^2 + t^2\langle\gamma, \gamma\rangle$, for any $t > 0$, is a Riemannian metric on P with respect to which π becomes a Riemannian submersion. In [4], S. Kobayashi proved that if (M, ds^2) is a Kaehler-Einstein space of positive scalar curvature, then, for proper choices of P, γ and t, g becomes an Einstein metric. In this paper we generalize this result by showing that the above construction can be used to obtain examples of many homogeneous Einstein spaces.

In the first section we compute the Riemann and Ricci curvature tensors for the metric g. Although the Riemann curvature tensor has been computed in detail for such metrics by B. O'Neill [7] and A. Gray [1], we have repeated it here because it is necessary for our purposes to have an explicit expression for the Ricci tensor. Furthermore, we have carried out the computations in terms of forms on the bundle of frames, a technique considerably different from that used in [1] and [7], and one which can be used to great advantage in the applications considered in the second section.

In the second section we consider a class of principal bundles over homogeneous spaces, which are analogous to the example of the spheres $S^{4n+3} \subseteq Q^{n+1}$, $Q =$ quaternions, represented as the quotient spaces $Sp(n+1)/Sp(n)$. At any point $p \in S^{4n+3}$, the tangent space decomposes into the direct sum of a $4n$-dimensional subspace and a 3-dimensional subspace, each invariant under the linear isotropy representation of $Sp(n)$, and furthermore such that the linear isotropy action on the three dimensional subspace is trivial. Hence, by varying the scale of the metric on this three dimensional subspace, the Riemannian metric on S^{4n+3} is changed in such a way that it remains invariant under the action of $Sp(n+1)$. We show that in many such examples the scale on the trivial-action subspace can be chosen in such a way that the resulting metric is Einsteinian. For example, for S^{4n+3} one choice of scale just gives the

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constant curvature metric, but we show that there is precisely one other choice of scale which defines an Einstein metric of nonconstant sectional curvature on S^{4n+3}.

Some other examples contained in this class are listed in a table near the end of section two. Einstein metrics on the real Stieffel manifolds $SO(p+q)/SO(q)$ were found by Sagle in [8] using quite different methods.

Finally, this construction also gives some left-invariant (non-bi-invariant) Einstein metrics on most compact simple Lie groups. Those left-invariant Einstein metrics found in [3] are among the ones obtained here, but the construction presented here yields some additional examples.

1. We take as given a Riemannian manifold M with metric ds^2, a principal bundle $P(M, G)$ over M with structure group G, a bi-invariant Riemannian metric $\langle\ ,\ \rangle$ on G, and a connection form γ on P. Then $g = \pi^* ds^2 + t^2\langle\gamma, \gamma\rangle$ is a Riemannian metric on P, where $t > 0$ is an arbitrary (fixed) parameter, and $\pi\colon P \to M$ is the projection map. Explicitly, if $u \in P$, $X, Y \in P_u$, then $g(X, Y) = ds^2(\pi_* X, \pi_* Y) + t^2\langle\gamma(X), \gamma(Y)\rangle$. Recall that a connection form on P takes values in the Lie algebra of G.

The following properties of g are easily verified.

Proposition 1. i) *g is invariant under the right action of G on P,*

ii) *with respect to g on P and ds^2 on M, the projection map $\pi\colon P \to M$ is a Riemannian submersion.*

One would expect from the definition of g that the connection form, the curvature tensor and the Ricci tensor of g can be expressed in terms of the curvature tensors of ds^2 and γ. This is indeed the case, and we proceed now to determine the appropriate formulas.

The following convention for indices will be used: $1 \le i, j, k, m \le n = \dim M$ and $n + 1 \le a, b, c, d, f \le n + r$, where $r = \dim G$.

Let $X_{n+1}, \cdots, X_{n+r}$ be an orthonormal (o.n) frame field of left invariant vector fields on G with respect to $\langle\ ,\ \rangle$, i.e., an o.n frame in the Lie algebra $\mathfrak{g}$ of G. Set $[X_a, X_b] = \sum_c C^c_{ab} X_c$. Since $\langle\ ,\ \rangle$ is bi-invariant, the structure constants $\{C^c_{ab}\}$ are skew-symmetric in every pair of indices. Then $\gamma = \sum_a \gamma^a X_a$, where the γ^a are 1-forms on P.

From the right action of G on P, the vectors $\{X_a\}$ induce r fundamental vertical vector fields, (see [5, Vol. I, p. 51] for basic definitions), $X^*_{n+1}, \cdots, X^*_{n+r}$ on P which are linearly independent at every point of P. It is convenient to consider the set of *adapted* frames on P, denoted $A(P)$, which consists of all orthonormal frames for g on P of the form $X_1, \cdots, X_n, t^{-1}X^*_{n+1}, \cdots, t^{-1}X^*_{n+r}$. Namely, at each point of P, the vectors $X_1, \cdots, X_n$ can be any orthonormal set of n horizontal vectors, with respect to γ.

It is quite evident that $A(P)$ is a subbundle of the bundle of orthonormal frames over P, and has structure group $O(n)$.

Let $O(M)$ denote the principal $O(n)$-bundle of orthonormal frames on M with respect to ds^2. There is a natural map $\pi_1\colon A(P) \to O(M)$ defined by sending the adapted frame $X_1, \cdots, X_n, t^{-1}X^*_{n+1}, \cdots, t^{-1}X^*_{n+r}$ at the point $x \in P$ to the orthonormal frame $\pi_* X_1, \cdots, \pi_* X_n$ at the point $\pi(x) \in M$. Then $A(P)$ is a principal G-bundle over $O(M)$, if we use the right action of G on $A(P)$ given by: the element $z \in G$ sends the above frame at $x \in P$ to the frame $R_{z*}X_1, \cdots, R_{z*}X_n, t^{-1}X^*_{n+1}, \cdots, t^{-1}X^*_{n+r}$ at $xz \in P$, where R_z denotes the right action of G on P.

We now have the following commuting diagram:

$$
\begin{array}{ccccc}
 & A(P) & \xrightarrow{\pi_2} & P & \\
G & {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} & G \\
 & O(M) & \xrightarrow[\pi_3]{} & M &
\end{array}
\tag{1}
$$

where π_2 and π_3 denote the obvious projections.

The connection and curvature forms for g are defined on the bundle of orthonormal frames on P, aud hence they can be restricted to the subbundle $A(P)$. The above diagram indicates how the geometry of γ and ds^2 will enter into the calculations of these forms. We begin by finding the canonical form on $A(P)$. (See [5, Vol. I] for definitions.) Let θ denote the canonical form on $O(M)$. Thus θ is an $\boldsymbol{R}^n$-valued 1-form which, in terms of the standard basis $e_1, \cdots, e_n$ on $\boldsymbol{R}^n$, may be expressed as $\theta = \sum_i \theta^i e_i$, where the θ^i are ordinary 1-forms on $O(M)$.

Proposition 2. *Let φ denote the canonical form on $A(P)$. φ is an $\boldsymbol{R}^{n+r}$-valued form which we can decompose into $\varphi = (\varphi_1, \varphi_2)$ with respect to the decomposition $\boldsymbol{R}^{n+r} = \boldsymbol{R}^n \times \boldsymbol{R}^r$. Then $\varphi_1 = \pi_1^*\theta$ and $\varphi_2 = t\pi_2^*\gamma$, where in the latter expression we have identified the Lie algebra $\mathfrak{g}$ with $\boldsymbol{R}^r$ via the orthonormal basis $X_{n+1}, \cdots, X_{n+r}$ fixed above. In terms of components we have*

$$
\varphi = \sum_i (\pi_1^*\theta^i)e_i + t\sum_a (\pi_2^*\gamma^a)X_a \,. \tag{2}
$$

To avoid excessive notation we will frequently omit the π_1^* and π_2^* in (2).

Proof. The proof is an elementary application of the definitions of canonical forms and connection forms together with the commutativity of the diagram (1). We omit the details.

The curvature form Γ of γ can be expressed as $\Gamma = \sum_a \Gamma^a X_a$, where Γ^a is a horizontal 2-form on P (in the sense of [5, Vol. I, p. 75]), and $\pi_2^*\Gamma^a$ is a horizontal 2-form on $A(P)$, for $a = n + 1, \cdots, n + r$.

Lemma 1. *$\pi_2^*\Gamma^a = \sum_{i,j} H^a_{ij}(\pi_1^*\theta^i) \wedge (\pi_1^*\theta^j)$, where the H^a_{ij} are functions on $A(P)$ satisfying the relations $H^a_{ij} + H^a_{ji} = 0$. It is convenient to omit the π_2^* and π_1^*.*

Proof. Recall that at any point of a subbundle of the bundle of frames on a manifold, the kernel of the canonical form at that point consists precisely of the vertical vectors at that point, i.e., the vectors tangent to the fibre at that point. Thus, since $\pi_2^*\Gamma^a$ is horizontal for each a, it follows from (2) that $\pi_2^*\Gamma^a$ is at each point a linear combination of the forms I) $\pi_1^*\theta^i \wedge \pi_1^*\theta^j$, II) $\pi_1^*\theta^i \wedge \pi_2^*\gamma^c$, and III) $\pi_2^*\gamma^c \wedge \pi_2^*\gamma^b$ at that point, for all i, j, c and b. But, take $u \in A(P)$, $X, Y \in A(P)_u$, and suppose $(\pi_1^*\theta^i)(X) = 0$ for all i. Then $\pi_{1*}X$ is vertical on $O(M)$, i.e., $\pi_{3*}\pi_{1*}X = 0$. Since $\pi\,\pi_2 = \pi_3\pi_1$, it follows that $\pi_*\pi_{2*}X = 0$, i.e., that $\pi_{2*}X$ is vertical on P, and thus $\Gamma^a(\pi_{2*}X, \pi_{2*}Y) = 0$, since Γ^a is horizontal. But this implies that $\pi_2^*\Gamma^a$ must be a linear combination of only those forms of type I) above. Hence $\pi_2^*\Gamma^a = \sum_{i,j} H^a_{ij}\pi_1^*(\theta^i \wedge \theta^j)$, for some functions H^a_{ij} on $A(P)$, which we may assume satisfy $H^a_{ij} + H^a_{ji} = 0$.

Let ψ denote the Riemannian connection form on $A(P)$ for g. It is an $\mathcal{O}(n + r)$-valued 1-form on $A(P)$. Continuing to use our indexing conventions, the component forms of ψ will be denoted by ψ^i_j, ψ^i_a, ψ^a_b, with skew-symmetry in each pair of indices. Let $\omega = (\omega^i_j)$ denote the $\mathcal{O}(n)$-valued Riemannian connection form on $O(M)$ for ds^2.

Proposition 3.

i) $\psi^i_j = \pi_1^*\omega^i_j - t^2 \sum_a H^a_{ij}\pi_2^*\gamma^a$,

ii) $\psi^a_i = t \sum_j H^a_{ij}\pi_1^*\theta^j = -\psi^i_a$,

iii) $\psi^a_b = -\frac{1}{2} \sum_c C^a_{bc}\pi_2^*\gamma^c$.

Proof. Let β be the $\mathcal{O}(n + r)$-valued 1-form on $A(P)$ defined by i), ii) and iii). To show that $\beta = \psi$, it suffices to show that $\beta(A^*) = A$ for every fundamental vertical vector field A^* generated by $A \in \mathcal{O}(n) \subseteq \mathcal{O}(n + r)$, and that $d\varphi = -\beta \wedge \varphi$. For, having shown this, the usual uniqueness proof for the Riemannian connection shows that $\beta = \psi$, (cf. [5, Vol, I, p. 159]).

Using (2), it is an elementary exercise to show that $d\varphi = -\beta \wedge \varphi$. Let $A = (A^i_j) \in \mathcal{O}(n)$, let $u \in A(P)$, and let $\sigma_u : O(n) \to A(P)$ be defined by $\sigma_u(B) = uB$. Then $A^*_u = \sigma_{u*}A_e$, where e = identity of $O(n)$. But $\pi_1 \circ \sigma_u(B) = \pi_1(uB) = \pi_1(u)B = \sigma_{\pi_1(u)}(B)$, where $\sigma_{\pi_1(u)} : O(n) \to O(M)$ is defined in the same way. Thus $\pi_{1*}A^*_u = \pi_{1*}\sigma_{u*}A_e = \sigma_{\pi_1(u)*}A_e$ is a vertical vector to $O(M)$ at $\pi_1(u)$. Hence $(\pi_1^*\omega^i_j)A^*_u = \omega^i_j(\pi_{1*}A^*_u) = A^i_j$, since ω is a connection form on $O(M)$. Furthermore, $\pi_{2*}A^*_u = 0$ since A^*_u is vertical, and thus $(\pi_2^*\gamma^a)(A^*_u) = 0$. Hence, from i), $\beta^i_j(A^*_u) = A^i_j$; and from iii), $\beta^a_b(A^*_u) = 0$. Finally, from ii), $\beta^a_i(A^*_u) = 0$ since $\pi_{1*}A^*_u$ is vertical and so $\theta^j(\pi_{1*}A^*_u) = 0$. Hence $\beta(A^*_u) = A$, for every $u \in A(P)$, and any $A \in \mathcal{O}(n) \subseteq \mathcal{O}(n + r)$. q.e.d.

Let $\Omega = (\Omega^i_j)$ denote the Riemannian curvature form of ds^2, which is an $\mathcal{O}(n)$-valued 2-form on $O(M)$. Let Ψ denote the Riemannian curvature form of g on $A(P)$, which is an $\mathcal{O}(n + r)$-valued 2-form with components Ψ^i_j, Ψ^i_a and Ψ^a_b, all skew-symmetric in their indices. In the following proposition we again omit the π_1^*'s and π_2^*'s.

Proposition 4. *The Riemannian curvature form of g on A(P) is given by*:

i) $\Psi^i_j = \Omega^i_j - t^2 \sum_{a,k,m} (H^a_{ij}H^a_{km} + H^a_{ik}H^a_{jm})\theta^k \wedge \theta^m$
$\qquad - t \sum_{a,m} H^a_{ij;m}\theta^m \wedge t\gamma^a$
$\qquad + \sum_{a,b} (t^2 \sum_k H^a_{ik}H^b_{kj} \quad - \frac{1}{2} \sum_c H^c_{ij}C^c_{ab})t\gamma^a \wedge t\gamma^b$,

ii) $\Psi^i_a = t \sum_{j,m} H^a_{ij;m}\theta^j \wedge \theta^m + \sum_{j,b} (-t^2 \sum_k H^b_{ik}H^a_{kj} + \frac{1}{2} \sum_c H^c_{ij}C^a_{cb})\theta^j \wedge t\gamma^b$,

iii) $\Psi^a_b = \sum_{i,j} (t^2 \sum_k H^a_{ik} H^b_{kj} + \frac{1}{2} \sum_c H^c_{ij} C^a_{cb}) \theta^i \wedge \theta^j$
$- \frac{1}{4} t^{-2} \sum_{c,d,f} C^a_{fc} C^c_{bd} t\gamma^d \wedge t\gamma^f$,

where

iv) $\sum_m H^a_{ij;m} \theta^m \equiv dH^a_{ij} + \sum_k (H^a_{jk} \omega^k_i - H^a_{ik} \omega^k_j) - \sum_{b,c} H^b_{ij} C^a_{bc} \gamma^c$, $H^a_{ij;m} = -H^a_{ji;m}$, *and* $H^a_{ij;m} + H^a_{mi;j} + H^a_{jm;i} = 0$. *The content of* iv) *is that the right hand side is a* 1*-form contained in the span of* $\theta^1, \cdots \theta^n$.

Proof. The curvature forms are defined by the equations:

$$\Psi^i_j = d\psi^i_j + \sum_k \psi^i_k \wedge \psi^k_j + \sum_a \psi^i_a \wedge \psi^a_j ,$$

$$\Psi^i_a = -\Psi^a_i = d\psi^i_a + \sum_k \psi^i_k \wedge \psi^k_a + \sum_b \psi^i_b \wedge \psi^b_a ,$$

$$\Psi^a_b = d\psi^a_b + \sum_i \psi^a_i \wedge \psi^i_b + \sum_c \psi^a_c \wedge \psi^c_b .$$

Thus, using iv) and Proposition 3, it is a straightforward, but lengthy, calculation to derive i), ii) and iii).

To prove iv), one takes the exterior derivative on $A(P)$ of the two expressions

1) $\Gamma^a = d\gamma^a + \frac{1}{2} \sum_{b,c} C^a_{bc} \gamma^b \wedge \gamma^c$, 2) $\Gamma^a = \sum_{i,j} H^a_{ij} \theta^i \wedge \theta^j$,

and then equates the right hand sides. The rest is an elementary exercise in the algebra of exterior differential forms together with an application of the Jacobi identity of $\mathfrak{g}$.

On $O(M)$ we have $\Omega^i_j = \frac{1}{2} \sum_{k,m} K_{ijkm} \theta^k \wedge \theta^m$, where the K_{ijkm} are functions on $O(M)$ satisfying, among other identities, $K_{ijkm} = -K_{jikm} = -K_{ijmk}$, (cf. [5, Vol. I, p. 145]).

Similarly, on $A(P)$ we have

$$(3) \quad \Psi^A_B = \frac{1}{2} \sum_{k,m} R_{ABkm} \theta^k \wedge \theta^m + \sum_{k,a} R_{ABka} \theta^k \wedge t\gamma^a + \frac{1}{2} \sum_{a,b} R_{ABab} t\gamma^a \wedge t\gamma^b ,$$

where the R_{ABCD} are functions on $A(P)$ satisfying, among other identities, $R_{ABCD} = -R_{BACD} = -R_{ABDC}$, $1 \leq A, B, C, D \leq n + r$.

Skew-symmetrizing the coefficients of Ψ^A_B in Proposition 4 and comparing them to those in (3), we obtain the components of the Riemann curvature tensor of g expressed in terms of the curvatures of ds^2 and γ.

Proposition 5.

i) $R_{ijkm} = K_{ijkm} - t^2 \sum_a (2H^a_{ij} H^a_{km} + H^a_{ik} H^a_{jm} - H^a_{im} H^a_{jk})$,

ii) $R_{ijma} = -tH^a_{ij;m}$,

iii) $R_{ijab} = t^2 \sum_k (H^a_{ik} H^b_{kj} - H^b_{ik} H^a_{kj}) - \sum_c H^c_{ij} C^c_{ab}$,

iv) $R_{iajb} = -t^2 \sum_k H^b_{ik} H^a_{kj} + \frac{1}{2} \sum_c H^c_{ij} C^a_{cb}$,

v) $R_{abic} = 0$,

vi) $R_{abcd} = \frac{1}{4} t^{-2} \sum_f C^a_{bf} C^f_{cd}$.

All the components of the curvature tensor can be obtained from these using the identities $R_{ABCD} = -R_{BACD} = -R_{ABDC}$, $R_{ABCD} = R_{CDAB}$, *and* $R_{ABCD} + R_{ADBC} + R_{ACDB} = 0$.

The components of the Ricci tensor of ds^2 on $O(M)$ and g on $A(P)$, respectively, are given by $K_{ij} = \sum_k K_{kikj}$ and $R_{AB} = \sum_{s=1}^{n+r} R_{sAsB}$.

Proposition 6.

i) $R_{jk} = K_{jk} + 2t^2 \sum_{i,a} H^a_{ji} H^a_{ik}$,

ii) $R_{ja} = -t \sum_i H^a_{ij;i}$,

iii) $R_{ab} = -t^2 \sum_{i,k} H^a_{ik} H^b_{ki} - \frac{1}{4} t^{-2} \cdot B(X_a, X_b)$,

where B denotes the Killing form of $\mathfrak{g}$.

Proof. Recall that $B(X_a, X_b) = \sum_{c,d} C^c_{ad} C^d_{bc}$. With this, everything follows directly from Proposition 5.

2. The computations and formulas of section one provide a useful tool for studying a certain class of homogeneous spaces, many of which provide new examples of Einstein spaces, which are analogous to the spheres $S^{4n+3} = Sp(n+1)/Sp(n)$ described in the introduction.

Let K be a compact connected Lie group with Lie algebra $\mathfrak{k}$, and let H be a closed connected subgroup with Lie algebra $\mathfrak{h}$. Suppose that H is locally the direct product of two closed normal subgroups H_1 and H_2, with $\dim H_1 \geq 1$. This means that $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, (direct sum of ideals), where $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are the Lie algebras of H_1 and H_2, respectively, and $\mathfrak{h}_1 \neq 0$.

Let b denote a positive definite bi-invariant metric on K, with respect to which $\mathfrak{h}_1$ is orthogonal to $\mathfrak{h}_2$. If we let $\mathfrak{m}$ denote the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{k}$, then $\mathfrak{m}$ is ad(H)-invariant. For any $t > 0$, the inner product $g = b|\mathfrak{m} \times \mathfrak{m} + t^2 b|\mathfrak{h}_1 \times \mathfrak{h}_1$ is ad(H_2)-invariant, (note that the adjoint action of H_2 on $\mathfrak{h}_1$ is trivial), and hence induces a K-invariant metric, also denoted g, on K/H_2.

It is our aim to show that for certain choices involved, the parameter t can be chosen so that g becomes an Einstein metric. To do this, we place these spaces into the context of section one. For the rest of this section let $P = K/H_2$, $M = K/H$, $G = H_1$, and let $\pi: K/H_2 \to K/H$ be the natural projection, i.e., $\pi(kH_2) = kH$, for $k \in K$. Then P is a principal fibre bundle over M with projection π and structure group G. In fact, H_1 acts on K/H_2 on the right by: $R_h(kH_2) = khH_2$, for $h \in H_1$, $k \in K$. This action is well defined since elements of H_1 commute with elements of H_2. The local triviality of P over M is well-known.

Observe that $\mathfrak{m}$ is an ad(H) invariant subspace of $\mathfrak{k}$, and that $\mathfrak{h}_1 + \mathfrak{m}$ is an ad(H_2)-invariant subspace of $\mathfrak{k}$. We identify the tangent space at $e \cdot H$ of M with $\mathfrak{m}$ via the natural projection $\pi_3: K \to K/H$, and identify the tangent space at $e \cdot H_2$ of P with $\mathfrak{h}_1 + \mathfrak{m}$ via the natural projection $\pi_2: K \to K/H_2$. Then the metric $ds^2 = b|\mathfrak{m} \times \mathfrak{m}$ makes M into a naturally reductive homogeneous space, (cf. [5, Vol. II, p. 202] for definitions). Taking g on P and ds^2 on M makes $\pi: P \to M$ a Riemannian submersion. It is not difficult to see that, if the adjoint representation of H on $\mathfrak{m}$ is faithful, which we henceforth assume, then g is a naturally reductive metric on P if and only if $t = 1$.

We explain how, in the present context, diagram (1) can be replaced with the commutative diagram:

$$(*)\qquad \begin{tikzcd} K \arrow[r, "\pi_2"] \arrow[dr, "\pi_3"'] & K/H_2 \arrow[d, "\pi"] \\ & K/H \end{tikzcd}$$

Lemma. *If $Y \in \mathfrak{h}_1$, then $\pi_{2*}Y$ is a well-defined vector field on P, and equals the fundamental vertical vector field Y^* on P generated by Y.*

Proof. Take $u = xH_2 \in P$, where $x \in K$, and let $\sigma\colon H_1 \to P$ be defined by $\sigma(h) = uh = xhH_2$. Then $Y_u^* = \sigma_* Y_e$. Let L_k denote left action of $k \in K$ on either K or K/H_2. Then $\pi_2 \circ L_k = L_k \circ \pi_2$ for every $k \in K$, and $\sigma(h) = \pi_2(xh) = \pi_2 \circ L_x(h)$ for every $h \in H_1$, i.e., $\sigma = (\pi_2 \circ L_x)\,|\,H_1$. Hence $\pi_{2*}Y_x = \pi_{2*}L_{x*}Y_e = \sigma_* Y_e = Y_u^*$ for every $Y \in \mathfrak{h}_1$. Therefore $\pi_{2*}Y = Y^*$ proving the lemma.

Corollary. *If $k \in K$ and $Y \in \mathfrak{h}_1$, then $L_{k*}Y^* = Y^*$.*

Choose and fix an orthonormal basis $X_1, \cdots, X_{n+r}$ on $\mathfrak{m} + \mathfrak{h}_1$ with respect to b such that the first n vectors are in $\mathfrak{m}$ and the remaining r vectors are in $\mathfrak{h}_1$, where $n = \dim \mathfrak{m}$ and $r = \dim \mathfrak{h}_1$. Let $A(P)$ be the set of adapted frames on P for g with respect to $t^{-1}X_{n+1}^*, \cdots, t^{-1}X_{n+r}^*$. Namely, the adapted frames at $p \in P$ are all orthonormal frames at p in which the last r vectors are $t^{-1}X_{n+1}^*, \cdots, t^{-1}X_{n+r}^*$ at p, (cf. section one).

By the corollary, L_{k*} sends adapted frames to adapted frames for every $k \in K$. Thus setting $u_0 = \pi_{2*}(X_1, \cdots, X_n, t^{-1}X_{n+1}, \cdots, t^{-1}X_{n+r})_e$, an adapted frame at $\pi_2(e)$, then $L_{k*}u_0$ is an adapted frame at $\pi_2(k)$, for any $k \in K$, and therefore we have a map $K \to A(P)$ defined by $k \mapsto L_{k*}u_0$. It is not difficult to verify that this map is a bundle monomorphism of the principal bundle $K(P, H_2)$ into $A(P)(P, O(n))$, i.e., K is a subbundle of $A(P)$.

K is also a subbundle of $O(M)$, the bundle of orthonormal frames on M with respect to ds^2. In fact, letting v_0 denote the fixed orthonormal frame $\pi_{3*}(X_1, \cdots, X_n)_e$ at $\pi_3(e)$, then $L_{k*}v_0$ is an orthonormal frame at $\pi_3(k)$ and the map $K \to O(M)$ given by $k \mapsto L_{k*}v_0$ is a bundle monomorphism of $K(M, H)$ into $O(M)(M, O(n))$.

The map $\pi_1\colon A(P) \to O(M)$, as defined in (1) of section one, when restricted to $K \subseteq A(P)$ is just the identity map onto $K \subseteq O(M)$. Hence restricting to K, diagram (1) becomes the commutative diagram $(*)$. Thus we can restrict the canonical forms and connection forms on $A(P)$ and $O(M)$ to K, obtaining, as we shall see, left invariant forms on K.

To apply the results of section one, we must express g in terms of some connection on $P(M, H_1)$. Recall that a connection γ on P is K-invariant if $L_k^*\gamma = \gamma$ for every $k \in K$.

Proposition 7. *If γ is a K-invariant connection on $P(M, H_1)$, then $\pi_2^*\gamma$ is a left invariant $\mathfrak{h}_1$-valued 1-form on K satisfying:*

i) $\pi_2^*\gamma(X) = 0$ *for every* $X \in \mathfrak{h}_2$,

ii) $\pi_2^*\gamma(X) = X$ *for every* $X \in \mathfrak{h}_1$,
iii) $\pi_2^*\gamma(\mathrm{ad}(h)X) = \pi_2^*\gamma(X)$ *for every* $X \in \mathfrak{k}$ *and* $h \in H_2$,
iv) $\pi_2^*\gamma(\mathrm{ad}(h)X) = \mathrm{ad}(h)(\pi_2^*\gamma(X))$ *for every* $X \in \mathfrak{k}$ *and* $h \in H_1$.

Conversely, if $\tilde{\gamma}$ *is a left invariant* $\mathfrak{h}_1$*-valued* 1*-form on* K *satisfying* i)-iv), *then there is a unique* K*-invariant connection* γ *on* K/H_2 *such that* $\pi_2^*\gamma = \tilde{\gamma}$.

We omit the proof, (cf. [6] or [5, Vol. II, Chapter 10]).

The left invariant $\mathfrak{h}_1$-valued 1-form $\tilde{\gamma}$ on K, defined by $\tilde{\gamma}(X) =$ the $\mathfrak{h}_1$-component of X with respect to the decomposition $\mathfrak{k} = \mathfrak{h}_1 + \mathfrak{h}_2 + \mathfrak{m}$, satisfies the properties i)-iv) in Proposition 7, and hence induces a K-invariant connection γ on K/H_2 such that $\pi_2^*\gamma = \tilde{\gamma}$. We call this connection the canonical connection on $P(M, H_1)$. Using this connection on P, we have

$$g = \pi^* ds^2 + t^2 b(\gamma, \gamma) \ . \tag{4}$$

Proposition 8. *Let* γ *be the canonical connection on* $P(M, H_1)$, *and let* Γ *denote its curvature form. Then* $\pi_2^*\Gamma$ *is a left-invariant* $\mathfrak{h}_1$*-valued* 2*-form on* K. *Letting* $Z_{\mathfrak{h}_1}$ *denote the* $\mathfrak{h}_1$*-component of* $Z \in \mathfrak{k}$, *we have*

$$\pi_2^*\Gamma(X, Y) = \begin{cases} 0, & \text{if } X \text{ or } Y \text{ is in } \mathfrak{h} \ , \\ -\frac{1}{2}[X, Y]_{\mathfrak{h}_1}, & \text{if } X \text{ and } Y \text{ are in } \mathfrak{m} \ . \end{cases}$$

Proof. This proposition follows easily from the equation $\Gamma = d\gamma - \frac{1}{2}[\gamma, \gamma]$ combined with the properties i)-iv) of Proposition 7 satisfied by $\pi_2^*\gamma$.

We need to determine the components of $\pi_2^*\Gamma$ with respect to the coframe on $\mathfrak{m} + \mathfrak{h}_1$ dual to the above fixed frame $X_1, \cdots, X_{n+r}$. Denote this dual coframe by $\theta^1, \cdots, \theta^n, \gamma^{n+1}, \cdots, \gamma^{n+r}$, which all are left invariant 1-forms on K. We adopt the same index conventions as were used in the first section. Namely, $1 \le i, j, k, m \le n$ and $n + 1 \le a, b, c, d \le n + r$. The structure constants C^A_{BC} are defined by $C^A_{BC} = b(X_A, [X_B, X_C])$ for $1 \le A, B, C \le n + r$, and are skew-symmetric in every pair of indices since b is bi-invariant.

If γ is the canonical connection on $P(M, H_1)$, then $\pi_2^*\gamma = \sum_a X_a \gamma^a$. If Γ is its curvature form, then $\pi_2^*\Gamma = -\frac{1}{2} \sum_{a,i,j} (C^a_{ij}\theta^i \wedge \theta^j)X_a$ by Proposition 8. Hence, using the notation of Proposition 6, we now have

$$H^a_{ij} = -\tfrac{1}{2} C^a_{ij} \ . \tag{5}$$

At this point we must digress briefly and consider the Riemannian geometry of the naturally reductive homogeneous space $M \equiv (K/H, ds^2)$. It was observed above that $K \subseteq O(M)$. The canonical form of $O(M)$ when restricted to K is just the $\boldsymbol{R}^n$-valued left-invariant 1-form $\sum_i \theta^i e_i$, where $e_1, \cdots, e_n$ is the standard orthonormal basis on $\boldsymbol{R}^n$, and $\theta^1, \cdots, \theta^n$ are the same as above. The restriction of the Riemannian connection form ω to K is the $\mathcal{O}(n)$-valued left-invariant 1-form given by

$$\omega(X) = \begin{cases} \frac{1}{2}\,\mathrm{ad}_{\mathfrak{m}}(X), & \text{if } X \in \mathfrak{m}\ , \\ \mathrm{ad}_{\mathfrak{m}}(X), & \text{if } X \in \mathfrak{h}\ , \end{cases}$$

where $\mathrm{ad}_{\mathfrak{m}}(X)$ is the endomorphism of $\mathfrak{m}$ given by $\mathrm{ad}_{\mathfrak{m}}(X)Y = [X, Y]_{\mathfrak{m}}$ for $Y \in \mathfrak{m}$, (cf. [6] for details). These endomorphisms are skew-symmetric with respect to the inner product b on $\mathfrak{m}$, and thus the matrix of $\omega(X)$ with respect to $X_1, \cdots, X_n$ is in $\mathcal{O}(n)$. In this sense ω is $\mathcal{O}(n)$-valued.

If we let $\theta^{n+r+1}, \cdots, \theta^{n+r+r_2}$ be an orthonormal coframe on $\mathfrak{h}_2$ with respect to b, then ω is represented by an $n \times n$ skew-symmetric matrix (ω^i_j) of left-invariant 1-forms on K given by

$$(6) \qquad \omega^i_j = \frac{1}{2} \sum_k C^i_{kj}\theta^k + \sum_a C^i_{aj}\gamma^a + \sum_{t=n+r+1}^{n+r+r_2} C^i_{tj}\theta^t\ ,$$

where the C's are the structure constants of $\mathfrak{k}$ with respect to the coframe $\theta^1, \cdots, \theta^n, \gamma^{n+1}, \cdots, \gamma^{n+r}, \theta^{n+r+1}, \cdots, \theta^{n+r+r_2}$, and are skew-symmetric in every pair of indices since b is bi-invariant. Furthermore, the relations $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}, [\mathfrak{h}_t, \mathfrak{h}_t] \subseteq \mathfrak{h}_t, t = 1, 2$, and $[\mathfrak{h}_1, \mathfrak{h}_2] = 0$ imply that:

$$(7) \qquad \begin{array}{ll} 0 = C^b_{ai} = C^t_{ai} = C^s_{ti}, & 0 = C^i_{ab} = C^t_{ab} = C^i_{ts} = C^a_{ts}, \quad \text{and} \\ 0 = C^i_{at} = C^b_{at} = C^s_{at}, & \text{where} \quad n + r + 1 \leq s, t \leq n + r + r_2\ . \end{array}$$

The curvature form of ω is the $\mathfrak{h}$-valued left-invariant 2-form on K given by $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$. Thus we have

$$\Omega(X, Y) = \begin{cases} 0, & \text{if either } X \text{ or } Y \text{ is in } \mathfrak{h}\ , \\ -\frac{1}{4}\,\mathrm{ad}_{\mathfrak{m}}([X, Y]_{\mathfrak{m}}) - \frac{1}{2}\,\mathrm{ad}_{\mathfrak{m}}([X, Y]_{\mathfrak{h}}) + \frac{1}{8}[\mathrm{ad}_{\mathfrak{m}}(X), \mathrm{ad}_{\mathfrak{m}}(Y)] & \\ & \text{if } X, Y \in \mathfrak{m}\ . \end{cases}$$

Setting $\Omega = (\Omega^i_j)$, a skew-symmetric matrix of left-invariant 2-forms on K, we have

$$(8) \qquad \Omega^i_j = \sum_{m,p=1}^{n} \Big\{ \frac{1}{8} \sum_k (C^i_{pk}C^k_{mj} - C^i_{mk}C^k_{pj}) - \frac{1}{4} \sum_k C^k_{pm}C^i_{kj} - \frac{1}{2} \sum_{t=n+1}^{n+r+r_2} C^i_{tj}C^t_{pm} \Big\} \theta^p \wedge \theta^m\ .$$

put

$$(9) \qquad \Omega^i_j = \frac{1}{2} \sum_{k,m} K^i_{jkm}\theta^k \wedge \theta^m\ ,$$

where the K^i_{jkm} are constants satisfying $K^i_{jkm} + K^i_{jmk} = 0$.

Comparison of (8) with (9) yields a formula for the Riemann curvature tensor for ds^2:

$$K^i_{jpm} = \tfrac{1}{4} \sum_k (C^i_{pk} C^k_{mj} - C^i_{mk} C^k_{pj}) - \tfrac{1}{2} \sum_k C^k_{pm} C^i_{kj} - \sum_{t=n+1}^{n+r+r_2} C^i_{tj} C^t_{pm} .$$

From this follows the Ricci tensor of ds^2:

$$K_{jm} \equiv \sum_i K^i_{jim} = -\tfrac{1}{4} \sum_{i,k} C^k_{ji} C^i_{mk} - \sum_i \sum_{t=n+1}^{n+r+r_2} C^i_{jt} C^t_{mi} . \tag{10}$$

The Ricci tensor involves the Killing form F of $\mathfrak{k}$ and a certain symmetric bilinear form T on $\mathfrak{m}$. To see this, observe that

$$\begin{aligned} F(X_j, X_m) &= \text{Trace ad}\,(X_j)\,\text{ad}\,(X_m) \\ &= \sum_{s,t=1}^{n+r+r_2} C^t_{js} C^s_{mt} = \sum_{i,k} C^i_{jk} C^k_{mi} + 2 \sum_i \sum_{t=n+1}^{n+r+r_2} C^i_{jt} C^t_{mi} . \end{aligned} \tag{11}$$

Combining (10) and (11) gives

$$K_{jm} = -\tfrac{1}{2} F(X_j, X_m) + \tfrac{1}{4} \sum_{i,k} C^i_{jk} C^k_{mi} .$$

Let T be the symmetric bilinear form on $\mathfrak{m}$ defined by

$$T(X, Y) = \sum_{i=1}^{n} b([X, X_i]_{\mathfrak{m}}, [X_i, Y]_{\mathfrak{m}}) . \tag{12}$$

Then T is negative semi-definite since $T(X, X) = -\sum_i \|[X, X_i]_{\mathfrak{m}}\|^2 \leq 0$ for any $X \in \mathfrak{m}$. Notice that $T(X_j, X_m) = \sum_{i,k} C^k_{ji} C^k_{im} = \sum_{i,k} C^k_{ji} C^i_{mk}$.

The Ricci tensor of ds^2 on M is K-invariant, as it can be regarded as the symmetric bilinear form S on $\mathfrak{m}$ satisfying $S(X_i, X_j) = K_{ij}$, (which are given by (10)). Then

$$S = -\tfrac{1}{2} F|_{\mathfrak{m} \times \mathfrak{m}} + \tfrac{1}{4} T . \tag{12a}$$

We must make one more computation before we can apply Proposition 6 to the present situation. Namely, we must find the covariant derivatives $H^a_{ij;k}$ of the components of Γ given in (5).

Lemma.

$$H^a_{ij;m} = \tfrac{1}{4} \sum_k C^a_{mk} C^k_{ij} , \tag{13}$$

$$\sum_i H^a_{ij;i} = \tfrac{1}{4} F(X_a, X_j) . \tag{14}$$

Proof. Combining iv) of Proposition 4 with (5) and (6) gives:

$$\begin{aligned} &\sum_m H^a_{ij;m} \theta^m \\ &= \sum_k \Big\{ -\tfrac{1}{2} C^a_{jk} \Big(\tfrac{1}{2} \sum_m C^k_{mi} \theta^m + \sum_b C^k_{bi} \gamma^b + \sum_{t=n+r+1}^{n+r+r_2} C^k_{ti} \theta^t \Big) \\ &\quad + \tfrac{1}{2} C^a_{ik} \Big(\tfrac{1}{2} \sum_m C^k_{mj} \theta^m + \sum_b C^k_{bj} \gamma^b + \sum_{t=n+r+1}^{n+r+r_2} C^k_{tj} \theta^t \Big) \Big\} + \tfrac{1}{2} \sum_{b,c} C^b_{ij} C^a_{bc} \gamma^c . \end{aligned}$$

Applying the Jacobi identity, together with (7), one finds that the coefficients of the γ^b and the θ^t are zero, as we know they must be from Proposition 4, while the coefficients of the θ^m yield (13). It is a simple computation to obtain (14) from (13), (11) and (7).

Proposition 10. *Let γ be the canonical connection on $P(M, H_1)$, and g be the corresponding metric on P given by* (4). *Retain the notation and indexing conventions used above. Let R_{uv}, $1 \leq u, v \leq n + r$, denote the components of the Ricci tensor of g with respect to the orthonormal frame $X_1, \cdots, X_n, t^{-1}X_{n+1}, \cdots, t^{-1}X_{n+r}$. Then*

$$R_{jk} = K_{jk} + \tfrac{1}{2}t^2 \sum_{i,a} C^a_{ji} C^a_{ik} , \tag{15}$$

$$R_{ja} = -\tfrac{1}{4}tF(X_j, X_a) , \tag{16}$$

$$R_{ab} = -\tfrac{1}{4}t^2 F(X_a, X_b) + \tfrac{1}{4}(t^2 - t^{-2})F_1(X_a, X_b) , \tag{17}$$

where F_1 denotes the Killing form of $\mathfrak{h}_1$.

Proof. (15) follows from Proposition 6, i) together with (5). (16) follows from Proposition 6, ii) together with (14). From Proposition 6, iii) combined with (5) we have $R_{ab} = -\frac{1}{4}t^2 \sum_{i,k} C^a_{ik} C^b_{ki} - \frac{1}{4}t^{-2}F_1(X_a, X_b)$. But, using (7), it is easy to see that $\sum_{i,k} C^a_{ik} C^b_{ki} = F(X_a, X_b) - F_1(X_a, X_b)$. Substituting this into the preceding formula for R_{ab} gives (17).

It will be convenient to have basis free expressions for (15), $\cdots$, (17). Let S (respectively S_g) denote the Ricci tensor of ds^2 on M (resp. of g on P), which is a symmetric bilinear form on $\mathfrak{m}$ (resp. on $\mathfrak{h}_1 + \mathfrak{m}$). Let α_1 (resp. α_2) denote the symmetric bilinear form on $\mathfrak{m}$ defined by $\alpha_1(X, Y) = \sum_i b([X, X_i]_{\mathfrak{h}_1}, [X_i, Y]_{\mathfrak{h}_1})$ (resp. by $\alpha_2(X, Y) = \sum_i b([X, X_i]_{\mathfrak{h}_2}, [X_i, Y]_{\mathfrak{h}_2})$). It is easy to see that α_1 and α_2 are negative semi-definite and ad (H)-invariant. These forms are related as follows.

$$F | \mathfrak{m} \times \mathfrak{m} = T + 2\alpha_1 + 2\alpha_2 , \tag{18}$$

where $F | \mathfrak{m} \times \mathfrak{m}$ denotes the restriction of F to $\mathfrak{m} \times \mathfrak{m}$, and T was defined by (12).

Then, since $R_{jk} = S_g(X_j, X_k)$, $R_{ja} = S_g(X_j, t^{-1}X_a)$, $R_{ab} = S_g(t^{-1}X_a, t^{-1}X_b)$ and $K_{jk} = S(X_j, X_k)$, we have

$$S_g | \mathfrak{m} \times \mathfrak{m} = S + \tfrac{1}{2}t^2\alpha_1 , \tag{19}$$

$$t^{-1}S_g | m \times \mathfrak{h}_1 = -\tfrac{1}{4}tF | \mathfrak{m} \times \mathfrak{h}_1 , \tag{20}$$

$$t^{-2}S_g | \mathfrak{h}_1 \times \mathfrak{h}_1 = -\tfrac{1}{4}t^2 F | \mathfrak{h}_1 \times \mathfrak{h}_1 + \tfrac{1}{4}(t^2 - t^{-2})F_1 . \tag{21}$$

It is our intention now to investigate conditions under which g can be an Einstein metric, that is, when S_g will be a constant multiple of g. For this purpose, the following conditions seem to be quite natural.

Condition 1. $b = -F$.

Condition 2. ds^2 is an Einstein metric, say $S = kds^2$, for some $k \in \boldsymbol{R}$.

If we assume Conditions 1 and 2, then (19) and (20) simplify somewhat to:

$$S_g \,|\, \mathfrak{m} \times \mathfrak{m} = -kF \,|\, \mathfrak{m} \times \mathfrak{m} + \tfrac{1}{2} t^2 \alpha_1 \,, \tag{22}$$

$$S_g \,|\, \mathfrak{m} \times \mathfrak{h}_1 = 0, \text{ since } \mathfrak{m} = \mathfrak{h}^{\perp} \text{ with respect to } b = -F. \tag{23}$$

It is clear from (21) and (22) that if g is an Einstein metric, then α_1 must be a multiple of $F \,|\, \mathfrak{m} \times \mathfrak{m}$, and F_1 must be a multiple of $F \,|\, \mathfrak{h}_1 \times \mathfrak{h}_1$.

Condition 3. $F_1 = cF \,|\, \mathfrak{h}_1 \times \mathfrak{h}_1$ and $\alpha_1 = aF \,|\, \mathfrak{m} \times \mathfrak{m}$, where $a, c \in \boldsymbol{R}$.

In fact, $a \geq 0$ and $c \geq 0$ since F, F_1 and α_1 are all negative semi-definite.

Proposition 11. *Suppose Conditions* 1, 2 *and* 3 *are satisfied* (*by the spaces, metrics, etc. we have been working with in* § 2), *with k, a and c being the constants used above. Then*

i) $S_g \,|\, \mathfrak{m} \times \mathfrak{m} = (\frac{1}{2} n^{-1} t^2 r(1 - c) - k) F \,|\, \mathfrak{m} \times \mathfrak{m}$,

ii) $S_g \,|\, \mathfrak{m} \times \mathfrak{h}_1 = 0$,

iii) $S_g \,|\, \mathfrak{h}_1 \times \mathfrak{h}_1 = \frac{1}{4}(-t^2 + c(t^2 - t^{-2})) t^2 F \,|\, \mathfrak{h}_1 \times \mathfrak{h}_1$,

and g is an Einstein metric if and only if

$$(2r/n + 1)(1 - c)t^4 - 4kt^2 + c = 0 \,, \tag{25}$$

where $n = \dim \mathfrak{m}$ *and* $r = \dim \mathfrak{h}_1$ *as before.*

Proof. i), ii) and iii) follow directly from (21), (22), (23) and Condition 3.

Now g is an Einstein metric if and only if S_g is a constant multiple of g. But, from the definition of g and Condition 1, $g \,|\, \mathfrak{m} \times \mathfrak{m} = -F \,|\, \mathfrak{m} \times \mathfrak{m}$, $g \,|\, \mathfrak{h}_1 \times \mathfrak{h}_1 = -t^2 F \,|\, \mathfrak{h}_1 \times \mathfrak{h}_1$, and $\mathfrak{m}$ is orthogonal to $\mathfrak{h}_1$ with respect to g or F. Thus, from i), ii) and iii), g is an Einstein metric if and only if $-(\frac{1}{2} n^{-1} t^2 r(1 - c) - k) = -\frac{1}{4}(-t^2 + c(t^2 - t^{-2}))$. Rewriting this equation gives (25).

(25) is quadratic in t^2. In order to know anything about possible solutions of (25), we need to have some estimates on the size of c and k, and their relationship to r and n.

Lemma. *Retaining the hypotheses and notation of Proposition* 11, *we have*

$$\text{i) } r(1 - c) = an, \text{ and } 0 \leq c < 1 \,; \qquad \text{ii) } \tfrac{1}{4} \leq k \leq \tfrac{1}{2} \,. \tag{26}$$

Proof. i) $F(X_b, X_b) = \sum_{i,j} C^i_{bj} C^j_{bi} + \sum_{c,d} C^c_{bd} C^d_{bc}$, so that $\sum_b F(X_b, X_b) = \sum_i \alpha_1(X_i, X_i) + \sum_b F_1(X_b, X_b)$, i.e., $-r = -an - rc$ by Conditions 1 and 3. Thus $r(1 - c) = an$, which implies that $c \leq 1$. We observed above that $c \geq 0$. Furthermore, from the above expression for $F(X_b, X_b)$, we have $-1 = -\sum_i b([X_b, X_j], [X_b, X_j]) - c$. Hence $c = 1$ if and only if $[X_a, X_j] = 0$ for all a and j, i.e., if and only if $[\mathfrak{h}_1, \mathfrak{m}] = 0$. But this last condition implies

$\mathfrak{h}_1$ is a nontrivial ideal in $\mathfrak{k}$, which contradicts our assumption that K act essentially on M. Hence $c \neq 1$.

From (12a) and Condition 2 we get $kn = \sum_i S(X_i, X_i) = \frac{1}{2}n + \frac{1}{4}\sum_i T(X_i, X_i)$. By (18) and negative semi-definitness of T, we have $0 \geq \sum_i T(X_i, X_i) \geq -n$. Hence $\frac{1}{4} \leq k \leq \frac{1}{2}$.

Without making further assumptions it is difficult to determine the solutions of (25). Thus we consider the special cases: 1) where M is an irreducible Riemannian symmetric space, and 2) where $H_2 = \{e\}$.

Proposition 12. *Suppose that M is a compact irreducible Riemannian symmetric space, that $b = -F$, and that $\mathfrak{h}_2 \neq 0$. Suppose also that $F_1 = cF \mid \mathfrak{h}_1 \times \mathfrak{h}_1$ for some constant $c \geq 0$. Then Conditions* 1, 2 *and* 3 *are satisfied and, by Proposition* 11, *g is an Einstein metric if and only if t satisfies* (25).

i) *When $\mathfrak{h}_1$ is nonabelian,* (25) *has two distinct positive solutions, neither of which equals* 1.

ii) *When $\mathfrak{h}_1$ is abelian,* (25) *has just one positive solution, which equals* 1 *only when $n = 2$ and $r = 1$.*

Remark. The condition $F_1 = cF \mid \mathfrak{h}_1 \times \mathfrak{h}_1$ is certainly satisfied whenever $\mathfrak{h}_1$ is simple or abelian.

Proof. The invariant metric on an irreducible Riemannian symmetric space is unique up to a constant multiple. We may assume it is $ds^2 = -F \mid \mathfrak{m} \times \mathfrak{m}$. The irreducibility implies that ds^2 is an Einstein metric. Furthermore, M symmetric means $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$, which implies that $T = 0$, and thus $S = -\frac{1}{2}F$ by (12a), and $k = \frac{1}{2}$.

The irreducibility of M means that the adjoint representation of H on $\mathfrak{m}$ is irreducible. Thus the ad (H)-invariant bilinear forms α_1 and α_2 on $\mathfrak{m}$ are each multiples of ds^2, and Condition 3 is satisfied since $F_1 = cF \mid \mathfrak{h}_1 \times \mathfrak{h}_1$ by assumption.

Hence Conditions 1, 2 and 3 are satisfied.

Consider $D = 4k^2 - (2r/n + 1)(1 - c)c$, the discriminant of (25). Now

$$\begin{aligned} (27) \qquad -1 &= F(X_i, X_i) = 2\alpha_1(X_i, X_i) + 2\alpha_2(X_i, X_i) && \text{by (18)} \\ &= -2a + 2\alpha_2(X_i, X_i) && \text{by Condition 3.} \end{aligned}$$

Since α_2 is negative semidefinite, $2a \leq 1$, and therefore $2n^{-1}r(1 - c) \leq 1$, by (26). Hence $D = 1 - (2n^{-1}r(1 - c) + 1 - c)c \geq 1 - (2 - c)c = (1 - c)^2 > 0$. On the other hand, from (26) it is clear that $D \leq 1$, and $D = 1$ if and only if $c = 0$, i.e., $F_1 = 0$, i.e., $\mathfrak{h}_1$ is abelian. Hence, if $\mathfrak{h}_1$ is not abelian, then $0 < D < 1$, and (25) has two distinct positive solutions.

Furthermore, substituting $t^2 = 1$ into (25) and using (26), it can be seen that 1 is a solution if and only if $2a - 1 = 0$. But $2a - 1 = 0$ if and only if $\alpha_2 = 0$ by (27), and this means $[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}_2} = (0)$ by the definition of α_2. Finally, $[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}_2} = (0)$ if and only if $\mathfrak{h}_2 = (0)$, since $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$ for a symmetric space of noneuclidean type. This completes the proof of i).

To prove ii), observe that when $c = 0$, (25) becomes $t^2((2r/n + 1)t^2 - 2) = 0$, which has only one positive solution. This solution equals 1 if and only if $n = 2r = 2$, since $r = 1$ if $\mathfrak{h}_1$ is abelian.

The irreducible Riemannian globally symmetric spaces satisfying the hypotheses of Proposition 12, and for which neither $\mathfrak{h}_1$ nor $\mathfrak{h}_2$ is trivial, are contained in the list given in Helgason [2, p. 354]. In the following table we list them together with the possible choices of $\mathfrak{h}_1$ and $\mathfrak{h}_2$. In each case take H_t to be the analytic subgroup of H corresponding to $\mathfrak{h}_t$, $t = 1, 2$. Then K/H_2 has at least one K-invariant Einstein metric.

K	H or $\mathfrak{h}$	$\mathfrak{h}_1$	$\mathfrak{h}_2$	Conditions
$SU(4)$	$SO(4)$	$SO(3)$	$SO(3)$	
$SU(p+q)$	$S(U_p \times U_q)$	$SU(p)$	$\boldsymbol{R} \times SU(q)$	$p \geq 2,\ q \geq 1$
		$\boldsymbol{R}$	$SU(p) \times SU(q)$	$p \geq 2,\ q \geq 1$
$SU(2p)$	$S(U_p \times U_p)$	$SU(p) \times SU(p)$	$\boldsymbol{R}$	$p \geq 2$
$SO(p+q)$	$SO(p) \times SO(q)$	$SO(p)$	$SO(q)$	$p \geq 2,\ q \geq 2$
		$SO(3)$	$SO(3) \times SO(q)$	$p = 4,\ q \geq 2$
$SO(2n)$	$U(n)$	$SU(n)$	$\boldsymbol{R}$	$n \geq 3$
		$\boldsymbol{R}$	$SU(n)$	
$Sp(n)$	$U(n)$	$SU(n)$	$\boldsymbol{R}$	$n \geq 1$
		$\boldsymbol{R}$	$SU(n)$	
$Sp(p+q)$	$Sp(p) \times Sp(q)$	$Sp(p)$	$Sp(q)$	$p \geq 1,\ q \geq 1$
E_6	$SU(6) \times SU(2)$	$SU(6)$	$SU(2)$	
		$SU(2)$	$SU(6)$	
E_6	$SO(10) \times \boldsymbol{R}$	$SO(10)$	$\boldsymbol{R}$	
		$\boldsymbol{R}$	$SO(10)$	
E_7	$SO(12) \times SU(2)$	$SO(12)$	$SU(2)$	
		$SU(2)$	$SO(12)$	
E_7	$e_6 \times \boldsymbol{R}$	e_6	$\boldsymbol{R}$	
		$\boldsymbol{R}$	e_6	
E_8	$e_7 \times SU(2)$	e_7	$SU(2)$	
		$SU(2)$	e_7	
F_4	$Sp(3) \times SU(2)$	$Sp(3)$	$SU(2)$	
		$SU(2)$	$Sp(3)$	
G_2	$SU(2) \times SU(2)$	$SU(2)$	$SU(2)$	Two cases because of the nonequivalent imbedding of each factor $SU(2)$ in $\mathfrak{g}_2$.

Example. In the introduction we cited the example of $S^{4p+3} = Sp(p+1)/Sp(p)$. In this example, $K = Sp(p+1)$, $H = Sp(p) \times Sp(1)$, $H_2 = Sp(p)$ and $H_1 = Sp(1)$. Thus $r = 3$, $n = 4p$, $k = \frac{1}{2}$, $c = 2/(p+2)$, and the solutions of (25) are $t_1^2 = 2$ and $t_2^2 = 2/(2p+3)$. For the first solution, the metric g is the standard metric of constant sectional curvature on S^{4p+3}. However, for the second solution, g is an Einstein metric of nonconstant sectional curvature on S^{4p+3}.

Proposition 13. *Retain the hypotheses and notation of Proposition* 11. *If, furthermore,* $H_2 = \{1\}$, *then* $t^2 = 1$ *is a solution to* (25) *and the other solution is nonnegative, equaling* 0 *if and only if* $\mathfrak{h}_1$ *is abelian, and equaling* 1 *if and only if* $c = \frac{1}{2}(1 + r/(n+r))$.

Proof. In this case, $P = K$ and g is a left-invariant metric on K. If $t = 1$, then $g = -F$, a bi-invariant metric which is always an Einstein metric on K. Thus $t^2 = 1$ is a solution of (25). It follows that the other solution is $c/[(2r/n + 1)(1-c)]$, which is nonnegative, equals 0 if and only if $c = 0$, and equals 1 if and only if $C = \frac{1}{2}(1 + r/(n+r))$.

Under the assumptions of Proposition 13 and the assumption that M be symmetric, the Einstein metrics coming from solutions of (25) are just the left-invariant Einstein metrics on K found in [3]. But now this construction works for a wider class of spaces. For example, the nonsymmetric isotropy irreducible spaces listed in Wolf [9, pp. 107–110] are all naturally reductive Einstein spaces, and Condition 3 is satisfied at least whenever $\mathfrak{h}$ is simple. Thus they satisfy the hypotheses of Proposition 13 whenever $\mathfrak{h}$ is simple.

For example, let H be any compact simple Lie group of dimension r. Fix an orthonormal basis $X_1, \cdots, X_r$ on its Lie algebra $\mathfrak{h}$ with respect to the Killing form F_1 of $\mathfrak{h}$. Then the adjoint representation defines a locally faithful representation ad: $H \to SO(r)$. Taking K to be $SO(r)$, the space $K/\mathrm{ad}(H)$ is, according to Wolf [9], isotropy irreducible and nonsymmetric, and is a naturally reductive Einstein space. The Killing form F of K is given by $F(A,B) = (r-2)\mathrm{Tr}\,AB$. If $X \in \mathfrak{h}$, then $F_1(X,X) = \mathrm{Tr}(\mathrm{ad}(X)\,\mathrm{ad}(X)) = (r-2)^{-1}F(\mathrm{ad}(X), \mathrm{ad}(X))$. Hence $c = 1/(r-2)$ in this case. Finally we have $n + r = \dim K = \frac{1}{2}r(r-1)$. By Proposition 13, one of the solutions of (25) will be unequal to 1 if and only if $c \neq \frac{1}{2}(1 + r/(n+r))$. But $c = \frac{1}{2}(1+r/(n+r))$ in this case if and only if $r = 3$, i.e., if and only if $H = K = SO(3)$. Hence wo do obtain some left-invariant Einstein metrics on $SO(r)$, (for certain r), which were not found in [3].

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