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A CONGRUENCE THEOREM FOR CLOSED HYPERSURFACES IN RIEMANN SPACES

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Introduction

We consider two closed oriented surfaces F and $\bar{F}$ in Euclidean 3-space E^3 and a differentiable map $\Phi\colon F \to \bar{F}$ preserving the orientation. The word differentiable always means differentiable of class C^∞. Furthermore, we assume that the set of points on F where the lines $(p, \bar{p})$, $\bar{p} = \Phi(p)$, are tangent to F does not have inner points. Then the following theorems are known:

A) If all the lines $(p, \bar{p})$ are parallel and $H(p) = \bar{H}(\bar{p})$ (H and $\bar{H}$ are the mean curvatures of F and $\bar{F}$ respectively), then the surface $\bar{F}$ is obtained from F by a single translation, i.e., the distances $p\bar{p}$ are the same for all points p on F (H. Hopf and K. Voss [8]).

B) If all the lines $(p, \bar{p})$ go through a fixed point 0 (which does or does not lie on F or $\bar{F}$) and if $rH(p) = \bar{r}\bar{H}(\bar{p})$ (r and $\bar{r}$ are the distances of p and $\bar{p}$ from 0), then F is obtained from F by a homothety, in other words the ratio $\bar{r}/r$ is constant (A. Aeppli [1]).

In order to generalize these two theorems we consider the following case: Let R^{n+1} be an $(n+1)$-dimensional Riemann space, and $\Phi(p, s)$ be a one-parameter group of transformations of R^{n+1} into itself. Furthermore, let F^n and $\bar{F}^n$ be two n-dimensional hypersurfaces of R^{n+1} such that the points of $\bar{F}^n$ are given by the formula:

$$\bar{p} = \Phi(p, f(p)) \,, \qquad p \in F^n \,,$$

where $f(p)$ is a differentiable function of F^n. To generalize the condition for the mean curvatures, we have to introduce an additional family of hypersurfaces, one for every point of F^n, given by the formula:

$$\tilde{F}^n_p = \Phi(F^n, f(p)) \,.$$

Then the point $\bar{p} = \Phi(p, f(p))$ lies on the hypersurfaces $\bar{F}^n$ and $\tilde{F}^n_p$ and we define:

$$\bar{H}(\bar{p}) = \text{mean curvature of } \bar{F}^n \text{ at } \bar{p} \,,$$
$$\tilde{H}(\bar{p}) = \text{mean curvature of } \tilde{F}^n_p \text{ at } \bar{p} \,.$$

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We denote by S the set of points of $\bar{F}^n$ where the vector tangent to the orbit of $\Phi(p, s)$ through $\bar{p}$ lies in the tangent space of $\bar{F}^n$. For this general case, the following two theorems are known.

I) If $\bar{H}(\bar{p}) = \tilde{H}(\bar{p})$ for all $\bar{p} \in \bar{F}^n, \Phi(p, s)$ is a group of homothetic transformations, and the set S of the exceptional points is nowhere dense in $\bar{F}^n$, then F^n and $\bar{F}^n$ are congruent mod Φ; in other words, $f(p) = \text{const.}$ (Y. Katsurada [9]).

II) If $\bar{H}(\bar{p}) = \tilde{H}(\bar{p})$ for all $\bar{p} \in \bar{F}^n$, and the set S is empty, then F^n and $\bar{F}^n$ are congruent mod Φ (H. Hopf and Y. Katsurada [7]).

Theorem II is not a generalization of Theorem A, since in this case we always have exceptional points. However it suggests that Theorem I is true without the additional assumption of homotheticity.

Theorem I has been proved by Y. Katsurada by using the method of differential forms. For the proof of Theorem II the authors use the strong maximum principle of E. Hopf [5]. In [10], K. Voss gave a proof of Theorem A, using a generalized maximum principle. However, his proof worked only in the case where F and $\bar{F}$ are real analytic surfaces. Later, P. Hartman [3] gave a proof without using the assumption of analyticity, by generalizing the strong maximum principle for elliptic differential equations. In this paper we give a proof of the following theorem, which is a generalization of Theorem II since we may have exceptional points, but which is not a generalization of Theorem I since the assumption on the exceptional points is stronger than that in Theorem I.

Theorem. *Let $F^n, \bar{F}^n, \tilde{F}^n_p$ be closed oriented hypersurfaces in R^{n+1} as explained above, and assume all maps to be orientation-preserving. Furthermore let $\varphi(\bar{p}) = (w, \bar{n})$, where w is the vector tangent to the curve $\Phi(\bar{p}, s)$, $-\varepsilon < s < +\varepsilon$, at $\bar{p}$, and $\bar{n}$ is the normal vector of $\bar{F}^n$ at p. If* grad $\varphi \neq 0$ *whenever $\varphi = 0$ on $\bar{F}^n$, and $\bar{H}(\bar{p}) = \tilde{H}(\bar{p})$ for all $\bar{p} \in \bar{F}^n$, then the hypersurfaces F^n and $\bar{F}^n$ are congruent* mod Φ.

1. Variation of the mean curvature

Let F^n be a hypersurface in an $(n+1)$-dimensional Riemann space R^{n+1} given locally by the equations

$$x^i = x^i(u^\alpha) , \qquad i = 1, \cdots , n+1 ; \quad \alpha = 1, \cdots , n .$$

Then the tangent space to the surface is spanned by the n linearly independent vectors $t_\alpha = (\partial x^i / \partial u^\alpha) \partial / \partial x^i$. For the covariant derivative of the vector-field t_α in the direction of t_β in R^{n+1} we get

$$D_\beta t_\alpha = \nabla_{t_\beta} t_\alpha = \left(\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + \Gamma^i_{jk} \frac{\partial x^j}{\partial u^\alpha} \frac{\partial x^k}{\partial u^\beta} \right) \frac{\partial}{\partial x^i} ,$$

and for the second fundamental form

$$l_{\alpha\beta} = (D_\alpha t_\beta, n) = g_{il}\left(\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + \Gamma^i_{jk}\frac{\partial x^j}{\partial u^\alpha}\frac{\partial x^k}{\partial u^\beta}\right)n^l ,$$

where $n = n^i\partial/\partial x^i$ is the normal to the hypersurface, and g_{ij} is the metric tensor of the space R^{n+1}. The formula for the mean curvature of the hypersurface is

$$H = g^{\alpha\beta}l_{\alpha\beta}/n = l^\alpha_\alpha/n ,$$

where $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta} = g_{ij}(\partial x^i/\partial u^\alpha)\partial x^j/\partial u^\beta$.

Now let $\Phi(p, s)$ be a one-parameter group of transformations of R^{n+1}, and F^n and $\bar{F}^n$ be two hypersurfaces such that

$$\bar{F}^n = \{\Phi(p, f(p)), p \in F^n\}$$

as in the introduction. We introduce an additional family of hypersurfaces, depending on a point $p \in F^n$ and a parameter t, $0 \leq t \leq 1$, given by the equation

$$F^n(p, t) = \{\Phi(q, tf(q) + (1 - t)f(p)) \mid q \in F^n\} .$$

Since $\Phi(p, tf(p) + (1 - t)f(p)) = \Phi(p, f(p)) = \bar{p}$, the point $\bar{p}$ lies on all the hypersurfaces $F^n(p, t)$, p fixed and $0 \leq t \leq 1$. Furthermore we have, for $t = 1$,

$$F^n(p, 1) = \{\Phi(q, f(q)) \mid q \in F^n\} = \bar{F}^n ,$$

and, for $t = 0$,

$$F(p, 0) = \{\Phi(q, f(p)) \mid q \in F^n\} = \tilde{F}^n_p .$$

From these relations we get

$$\bar{H}(\bar{p}) - \tilde{H}(\bar{p}) = \int_0^1 \frac{dH(p, t)}{dt} dt ,$$

where $H(p, t)$ is the mean curvature of $F^n(p, t)$ at the point $\bar{p}$.

The variation of the mean curvature gives

$$dH(p, t)/dt = l_{\alpha\beta}\, dg^{\alpha\beta}/dt + g^{\alpha\beta}\, dl_{\alpha\beta}/dt ,$$

and by differentiating the relation $g^{\alpha\beta}g_{\gamma\beta} = \delta^\alpha_\gamma$ we get

$$dg^{\alpha\beta}/dt = -g^{\alpha\delta}g^{\beta\gamma}\, dg_{\gamma\delta}/dt = -g^{\alpha\delta}g^{\beta\gamma}\{(dt_\gamma/dt, t_\delta) + (t_\gamma, dt_\delta/dt)\} .$$

Furthermore, by taking the covariant derivative of the relations $(n, n) = 1$ and $(n, t_\alpha) = 0$, we obtain $(n, D_\alpha n) = 0$ and $(D_\alpha n, t_\beta) + (n, D_\alpha t_\beta) = 0$ or $D_\alpha n = \lambda^\beta_\alpha t_\beta$ with $\lambda^\beta_\alpha = -(n, D_\alpha t_\gamma)g^{\gamma\beta}$. Hence

$$D_\alpha n = -(n, D_\alpha t_\gamma)g^{\beta\gamma}t_\beta ,$$

$$\frac{dg^{\alpha\beta}}{dt} l_{\alpha\beta} = -g^{\alpha\delta}g^{\beta\gamma}\left\{\left(\frac{dt_\gamma}{dt}, t_\delta\right) + \left(t_\gamma, \frac{dt_\delta}{dt}\right)\right\} \qquad (D_\alpha t_\beta, n) = 2g^{\alpha\beta}(D_\alpha n, t_\beta) .$$

In order to compute the second term in the above expression for $dH(p, t)/dt$, differentiating the relations $(n, n) = 1$ and $(n, t_\alpha) = 0$ with respect to t we get $(dn/dt, n) = 0$ and $(dn/dt, t_\alpha) + (n, dt_\alpha/dt) = 0$, or $dn/dt = \lambda^\alpha t_\alpha$ with $\lambda^\alpha = -(dt_\beta/dt, n)g^{\alpha\beta}$. Hence

$$dn/dt = -(dt_\beta/dt, n)g^{\alpha\beta}t_\alpha ,$$

$$g^{\alpha\beta}\frac{dl_{\alpha\beta}}{dt} = g^{\alpha\beta}\frac{d}{dt}(D_\alpha t_\beta, n) = g^{\alpha\beta}\left(\frac{d}{dt}D_\alpha t_\beta, n\right) - g^{\alpha\beta}\Gamma^\delta_{\alpha\beta}\left(\frac{dt_\delta}{dt}, n\right) ,$$

where $\Gamma^\delta_{\alpha\beta} = (D_\alpha t_\beta, t_\gamma)g^{\gamma\delta}$. Finally we get the following formula for the variation of the mean curvature:

$$\frac{dH}{dt} = g^{\alpha\beta}\left(\frac{d}{dt}D_\alpha t_\beta, n\right) + 2g^{\alpha\beta}\left(D_\alpha n, \frac{dt_\beta}{dt}\right) - g^{\alpha\beta}\Gamma^\delta_{\alpha\beta}\left(\frac{dt_\delta}{dt}, n\right) .$$

Now using the definition of the hypersurfaces $F^n(p, t)$:

$$F^n(p, t) = \{\Phi(q, tf(q) + (1 - t)f(p)) \mid q \in F^n\} ,$$

or in local coordinates

$$x^i_p(u^\alpha, t) = \Phi^i(u^\alpha, tf(u^\alpha) + (1 - t)f(p)) ,$$

where $f(p)$ is independent of the u^α, we get

$$x^i_\alpha = \partial x^i_p/\partial u^\alpha = \partial\Phi^i/\partial u^\alpha + t(\partial\Phi^i/\partial s)\partial f/\partial u^\alpha ,$$

so that for the tangent vectors t_α of the hypersurface $F^n(p, t)$ at the point $\bar{p}$ we have

$$t_\alpha = (\partial\Phi^i/\partial u^\alpha|_{\bar{p}} + w^i t\, \partial f/\partial u^\alpha)\partial/\partial x^i ,$$

where $w^i = \partial\Phi(\bar{p}, s)/\partial s|_{s=0}$, and by differentiating with respect to t

$$dt_\alpha/dt = w\, \partial f/\partial u^\alpha , \qquad w = w^i\, \partial/\partial x^i .$$

Furthermore

$$D_\alpha t_\beta = (\partial x^i_\alpha/\partial u^\beta + \Gamma^i_{jk}x^j_\alpha x^k_\beta)\partial/\partial x^i ,$$

so

$$\frac{d}{dt}D_\alpha t_\beta = \left(\frac{d}{dt}\frac{\partial x^i_\alpha}{\partial u^\beta} + \Gamma^i_{jk}\frac{\partial x^j_\alpha}{\partial t}x^k_\beta + \Gamma^i_{jk}x^j_\alpha\frac{\partial x^k_\beta}{\partial t}\right)\frac{\partial}{\partial x^i} .$$

For the derivative of x^i_α we get

$$\frac{\partial x^i_\alpha}{\partial u^\beta} = \frac{\partial^2 \Phi^i}{\partial u^\alpha \partial u^\beta} + t\frac{\partial^2 \Phi^i}{\partial u^\alpha \partial s}\frac{\partial f}{\partial u^\beta} + t\frac{\partial^2 \Phi^i}{\partial u^\beta \partial s}\frac{\partial f}{\partial u^\alpha}$$

$$+ t^2 \frac{\partial^2 \Phi^i}{\partial s^2}\frac{\partial f}{\partial u^\alpha}\frac{\partial f}{\partial u^\beta} + t\frac{\partial \Phi^i}{\partial s}\frac{\partial^2 f}{\partial u^\alpha \partial u^\beta} .$$

Since $\partial^2\Phi^i/\partial u^\alpha \partial u^\beta, \partial^2\Phi^i/\partial u^\alpha \partial s, \partial^2\Phi^i/\partial s^2, \partial\Phi^i/\partial s$ do not depend on t when considered only at the point $\bar{p}$ on the hypersurfaces $F^n(p, t)$, we get for the derivative of the above expression with respect to t:

$$\frac{d}{dt}\frac{\partial x^i_\alpha}{\partial u^\beta} = \frac{\partial w^i}{\partial u^\alpha}\frac{\partial f}{\partial u^\beta} + \frac{\partial w^i}{\partial u^\beta}\frac{\partial f}{\partial u^\alpha} + 2t\frac{\partial w^i}{\partial s}\frac{\partial f}{\partial u^\alpha}\frac{\partial f}{\partial u^\beta} + w^i\frac{\partial^2 f}{\partial u^\alpha \partial u^\beta} ,$$

so

$$\frac{d}{dt}D_\alpha t_\beta = w\frac{\partial^2 f}{\partial u^\alpha \partial u^\beta} + D_\beta w\frac{\partial f}{\partial u^\alpha} + D_\alpha w\frac{\partial f}{\partial u_\beta} + 2t\frac{\partial w}{\partial s}\frac{\partial f}{\partial u^\alpha}\frac{\partial f}{\partial u^\beta} .$$

Therefore the formula for the variation of the mean curvature in our case is the following:

$$\frac{dH}{dt} = (w, n)g^{\alpha\beta}\frac{\partial^2 f}{\partial u^\alpha \partial u^\beta} + 2g^{\alpha\beta}\frac{\partial(w, n)}{\partial u^\alpha}\frac{\partial f}{\partial u^\beta} + 2t\left(\frac{\partial w}{\partial s}, n\right)g^{\alpha\beta}\frac{\partial f}{\partial u^\alpha}\frac{\partial f}{\partial u^\beta}$$

$$- g^{\alpha\delta}\Gamma^\beta{}_{\alpha\delta}(w, n)\frac{\partial f}{\partial u^\beta} .$$

2. A lemma on partial differential equations

For the proof of our main theorem we need a generalization of the strong maximum principle for elliptic partial differential equations. We consider a linear differential expression of the form

$$L(f) = \sum_{\alpha,\beta=1}^{n} A_{\alpha\beta}(x)\frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} + \sum_{\alpha=1}^{n} B_\alpha(x)\frac{\partial f}{\partial x^\alpha} ,$$

where $A_{\alpha\beta}(x)$ and $B_\alpha(x)$ are differentiable functions, together with a differentiable function $\varphi(x)$ in a normal domain G of the n-dimensional number space $\boldsymbol{R}^n$. We assume that $\varphi(x)$ and $L(f)$ have the following properties:

a) $\operatorname{grad}\varphi(x) \neq 0$ whenever $\varphi(x) = 0$,

b) $\sum_{\alpha,\beta=1}^{n} A_{\alpha\beta}(x)\lambda^\alpha\lambda^\beta$ is positive definite for every x with $\varphi(x) > 0$, negative definite for every x with $\varphi(x) < 0$, and identically 0 for every x with $\varphi(x) = 0$.

Then we prove the following

Lemma. *Let $f(x)$ be a solution of $L(f) = 0$, and x_0 be a point in G such* *that* $f(x) \leq f(x_0)$ *for all* x *in* G. *If either* $\varphi(x_0) \neq 0$ *or* $\varphi(x_0) = 0$ *and* $\sum_{\alpha=1}^{n} B_\alpha(x_0)(\partial\varphi/\partial x^\alpha)(x_0) > 0$, *then* $f(x) \equiv f(x_0)$ *in a neighborhood of* x_0.

This lemma is a special case of a theorem proved in the paper [4] by Hartman and Sacksteder. However, since we use only a simple case, we give a sketch of the proof.

Proof. The case $\varphi(x_0) \neq 0$ follows directly from the strong maximum principle of E. Hopf [5]. Therefore we may assume $\varphi(x_0) = 0$, and $\sum_{\alpha=1}^{n} B_\alpha (\partial\varphi/\partial x^\alpha)(x_0) > 0$. The proof for this case is a modification of the proof of E. Hopf's second lemma [6]. Since by assumption $\operatorname{grad} \varphi \neq 0$ whenever $\varphi = 0$, the set of points where $\varphi(x) = 0$ is a differentiable curve through x_0. Therefore there exists an open ball K_1 in G such that its boundary has exactly the point x_0 in common with the curve $\varphi(x) = 0$ and that $\varphi(x) > 0$ in $\bar{K}_1 - x_0$. We choose its center as the origin of the coordinate system, and set $r = |x|$, $r_0 = |x_0|$. We may assume $f(x_0) = 0$ and $f(x) \geq 0$ in K_1. By the strong maximum principle this implies either $f(x) > 0$ in $\bar{K}_1 - x_0$ and $f(x_0) = 0$ or $f(x) \equiv 0$ in $\bar{K}_1$. We show that $f(x) > 0$ in $\bar{K}_1 - x_0$ leads to a contradiction. We consider the auxiliary function $h(x) = e^{-r^2} - e^{r_0^2}$, which has the properties: $h(x) > 0$ for $|x| < r_0$, $h(x) = 0$ for $|x| = r_0$, and

$$L(h)(x_0) = \sum_{\alpha=1}^{n} B_\alpha \frac{\partial h}{\partial x^\alpha}(x_0) = -2e^{-r^2} \sum_{\alpha=1}^{n} B_\alpha x_0^\alpha = c \sum_{\alpha=1}^{n} B_\alpha \frac{\partial \varphi}{\partial x^\alpha}(x_0) , \quad c > 0 ,$$

since the vector $x_0 = (x_0^1, \cdots, x_0^n)$ is a negative multiple of $\operatorname{grad} \varphi$. Therefore $L(h)(x_0) > 0$, and hence $L(h) > 0$ in the closure of a ball K_2 with center x_0. Now we consider the function $g(x) = f(x) - \varepsilon h(x)$ in the domain $K = K_1 \cap K_2$. Then $g \geq 0$ on $S_1 \cap \bar{K}_2$, where $S_1 =$ boundary of K_1, and $g(x_0) = 0$. Furthermore, by choosing $\varepsilon > 0$ sufficiently small, we also have $g \geq 0$ on $S_2 \cap \bar{K}_1$, since $f > 0$ there.

Since $L(f) = 0$, and $L(h) > 0$ in K, we have $L(g) < 0$ in K, and therefore $g \geq 0$ in K by the strong maximum principle. Hence $(dg/dn)(x_0) \leq 0$, where dg/dn is the derivative in the direction of the outer normal of K. But then

$$\frac{df}{dn}(x_0) = \frac{dg}{dn}(x_0) + \varepsilon \frac{dh}{dn}(x_0) < 0 ,$$

since $(dh/dn)(x_0) < 0$. This contradicts the fact that $\operatorname{grad} f(x_0) = 0$.

Proof of the Theorem. By using the formula for the variation of the mean curvature and the relation

$$\bar{H}(\bar{p}) - \tilde{H}(\bar{p}) = \int_0^1 \frac{dH(p,t)}{dt} dt = 0 ,$$

we get the following differential equation for the function f:

$$\sum_{\alpha,\beta=1}^{n} A_{\alpha\beta}\frac{\partial^2 f}{\partial u^\alpha \partial u^\beta} + \sum_{\beta=1}^{n} B_\beta \frac{\partial f}{\partial u^\beta} = 0 ,$$

where

$$A_{\alpha\beta} = \int_0^1 (w, n(t)) g^{\alpha\beta}(t) dt ,$$

$$B_\beta = \int_0^1 \left\{ 2g^{\alpha\beta}\frac{\partial(w,n)}{\partial u^\alpha} + 2t\left(\frac{\partial w}{\partial s}, n\right) g^{\alpha\beta}\frac{\partial f}{\partial u^\alpha} - g^{\alpha\beta}\Gamma^\beta{}_{\alpha\delta}(w,n)\right\} dt .$$

From the relation

$$(w, n(t)) dA(t) = (w, \bar{n}) d\bar{A}$$

(proved in [2]), where $dA(t)$ is the volume element of the hypersurface $F^n(p, t)$ at p, it follows that if $(w, \bar{n}) \neq 0$, then $(w, n(t)) \neq 0$ for all t, $0 \leq t \leq 1$, and that if $(w, \bar{n}) = 0$, then $(w, n(t)) = 0$ for all t. Therefore by setting $\varphi(\bar{p}) = (w, \bar{n})$, $\sum_{\alpha,\beta=1}^{n} A_{\alpha\beta}\lambda^\alpha\lambda^\beta$ is positive definite if $\varphi > 0$, negative definite if $\varphi < 0$, and identically 0 if $\varphi = 0$, since $g^{\alpha\beta}(t)$ is positive definite for every t.

Now let $\bar{p}_0$ be a maximum point of f, so that $f(\bar{p}) \leq f(\bar{p}_0)$ for all $\bar{p}$ in $\bar{F}^n$. Such a point exists, since $\bar{F}^n$ is supposed to be compact. Then either $\varphi(\bar{p}_0) \neq 0$, or $\varphi(\bar{p}_0) = 0$; the latter implies that $(w, n(t)) = 0$ for all t, and

$$B_\beta = 2\int_0^1 \left(g^{\alpha\beta}\frac{\partial(w,n)}{\partial u^\alpha}\right) dt .$$

Since $(w, n(t)) \neq 0$, if $(w, \bar{n}) \neq 0$, then the set of points $\bar{p}$ on $\bar{F}^n$ where $(w, n(t)) = 0$ is the same as the set where $(w, \bar{n}) = 0$. Furthermore, if $(w, \bar{n}) > 0$, then $(w, n(t)) > 0$, and

$$\operatorname{grad}(w, n(t)) = c(t) \operatorname{grad}(w, \bar{n}) ,$$

with $c(t) \geq 0$. Thus

$$\sum_{\beta=1}^{n} \frac{\partial\varphi}{\partial u^\beta}(\bar{p}_0) = 2\int_0^1 g^{\alpha\beta}\frac{\partial(w, n(t))}{\partial u^\alpha}\frac{\partial(w,\bar{n})}{\partial u^\beta} dt$$
$$= 2\int_0^1 c(t) g^{\alpha\beta}\frac{\partial(w,\bar{n})}{\partial u^\alpha}\frac{\partial(w,\bar{n})}{\partial u^\beta} dt > 0 ,$$

since $g^{\alpha\beta}$ is positive definite and $c(t) \geq 0, c(1) = 1$. Therefore by our lemma, $f(\bar{p}) = f(\bar{p}_0)$ in a neighborhood of $\bar{p}_0$; in other words, the set $U_1 = \{\bar{p} \in \bar{F}^n \mid f(\bar{p}) = f(\bar{p}_0)\}$ is open in $\bar{F}^n$. This implies that $\bar{F}^n = U_1 \cup U_2$, where $U_2 = \{\bar{p} \in \bar{F}^n \mid f(\bar{p}) < f(\bar{p}_0)\}$, so that $\bar{F}^n$ is the disjoint union of two open sets. Since $\bar{F}^n$ is connected, it follows that $U_1 = \bar{F}^n$, i.e., $f(\bar{p}) = \text{const.}$ on $\bar{F}^n$. Hence the theorem is proved.

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