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ON THE ATIYAH-BOTT FORMULA FOR ISOLATED FIXED POINTS

DOMINGO TOLEDO

Introduction

The original Lefschetz fixed point theorem [11] states that if M is a compact manifold, $f\colon M \to M$ has isolated fixed points and $L(f)$ denotes the Lefschetz number of f, then

$$L(f) = \sum_p \deg_p (1 - f) ,$$

where the sum runs over the fixed points p of f. If f is a smooth map and the fixed point p is *simple* in the sense that $\det(1 - df_p) \neq 0$, then its local index has the infinitesimal description $\deg_p (1 - f) = \operatorname{sign}\det(1 - df_p)$. Atiyah and Bott have shown that Lefschetz theory also makes sense in the context of elliptic complexes. They show in [2] that if f induces a chain map of an elliptic complex E over M and the fixed points of f are simple, then

$$L(f, E) = \sum_p \nu(p) ,$$

where $\nu(p)$ are infinitesimal invariants of f at p. It is natural to ask whether their local index can be explained as a special case of a *cohomological* formula which always makes sense for isolated fixed points, as in the classical theorem.

The purpose of this paper is to present a general approach to fixed point theory which applied to isolated fixed points gives both the Atiyah-Bott formula and cohomological formulas. This method is based on a classical formula of de Rham [14, § 33] which expresses intersection numbers in Riemannian manifolds in terms of the Green kernel. It leads to an integral representation for the Lefschetz number from which the Atiyah-Bott theorem can be derived by some delicate but quite elementary analysis. Moreover, assuming that the Poincaré lemma holds, a cohomological expression for the index of an isolated fixed point can also be derived. For simple fixed points this reduces of course to the infinitesimal description.

The use of de Rham's formula was motivated by the intersection—theoretic proof of the classical theorem. An exposition of this proof in a form which suggests the steps to be taken in the elliptic context is included in the first

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section. The analysis which is carried out in detail in the next four sections is then outlined.

In § 6 the general theory is applied to the classical elliptic complexes. For the de Rham complex it just gives the classical formula. But for the Dolbeault complex the integral formula of § 5 is taken by Dolbeault's isomorphism into a sheaf-cohomological expression involving the Grothendieck residue. This formula was derived independently by Y. L. Tong, first by methods completely different from the ones given here [17]. Very recently he has discovered for the Dolbeault complex a similar intersection approach [18].

The intersection formula of de Rham was first brought to my attention by Professor J. Eells, and he conjectured to me that it may be useful in fixed point theory. I am also greatly indebted to him for much encouragement and advice while this work was in progress. I would also like to thank Professor G. R. Livesay for his encouragement to study these questions, and Professors C. J. Earle and R. S. Hamilton for several very helpful conversations.

1. Topological motivation

We begin by describing the intersection-theoretic proof of the classical theorem. This was actually Lefschetz's original approach [11]. The alternative proof based on traces at chain level, which is more familiar today, is due to Hopf [8]. For simplicity we assume that M is orientable. We use singular cohomology with real coefficients.

Let $\Gamma : M \to M \times M$ be the graph map defined by $\Gamma(x) = (fx, x)$. The fixed points of f are just the intersections of Γ with the diagonal Δ. To count these intersections one introduces the Thom class μ, namely, a generator for $H^n(M \times M, M \times M - \Delta) \approx \boldsymbol{R}$, where $n = \dim M$. A choice of Thom class is the same as a choice of orientation for M, and μ induces generators for the one-dimensional spaces $H^n(M)$, $H^n(M, M - \text{point})$. Consequently we can identify all these spaces with $\boldsymbol{R}$ in a compatible fashion.

The Lefschetz formula follows from the following observations.

(i) Poincaré duality and the Künneth formula identify

$$H^n(M \times M) \approx \sum_i H^iM \otimes H^iM^* = \sum_i \operatorname{Hom}(H^iM, H^iM) .$$

The basic fact discovered by Lefschetz is that if $\bar{\mu}$ denotes the image of μ in $H^n(M \times M)$, then $\bar{\mu} = \sum (-1)^i \operatorname{id}_H i$ under this identification.

(ii) If F denotes the fixed point set of f (F is so far arbitrary), we have the diagram:

$$\begin{array}{ccc} H^n(M, M - F) & \xleftarrow{\Gamma^*} & H^n(M \times M, M \times M - \Delta) \\ \big\downarrow j & & \big\downarrow j \\ H^n(M) & \xleftarrow{\Gamma^*} & H^n(M \times M) \end{array}$$

From (i) it is immediate that $(f \times 1)^*\bar{\mu} = \sum (-1)^i f_i^*$. Since $\Gamma^* = \Delta^*(f \times 1)^*$ and Δ^* gives the cup product, which is the Poincaré duality pairing, it follows that $\Gamma^*\bar{\mu} = L(f)$. Thus $L(f)$ can be interpreted as a cohomology class supported on F, namely, $j\Gamma^*\mu$.

(iii) Suppose the fixed points p are isolated. Take disjoint neighborhoods U_p of p, each homeomorphic to $\boldsymbol{R}^n$, and smaller neighborhoods V_p of p with $f(V_p) \subset U_p$. Then $H^n(V_p \times U_p, V_p \times U_p - \Delta)$ is generated by the local Thom class μ_p at p, which is the restriction of μ. Applying excision to the top part of the diagram and using Mayer-Vietoris to identify j with addition we get:

$$\begin{array}{ccc} \bigoplus_p H^n(V_p, V_p - 0) & \xleftarrow{\Gamma^*} & \bigoplus_p H^n(V_p \times U_p, V_p \times U_p - \Delta) \\ \Big\downarrow j & & \\ H^n(M) & & \end{array}$$

Thus $L(f) = \sum_p \nu(p)$ where $\nu(p) = \Gamma^*\mu_p$.

(iv) If $\delta\colon (V_p \times U_p, V_p \times U_p - \Delta) \to (U_p, U_p - 0)$ is the difference map $\delta(x, y) = y - x$, and $W_p \in H^n(U_p, U_p - 0)$ is the generator, then $\mu_p = \delta^* W_p$, because both classes have the same restriction to the fibre $p \times (U_p, U_p - p)$. Therefore $\nu(p) = \deg_p (1 - f)$.

From this point of view Lefschetz theory centers around the study of the Thom class. The fact that this class is supported on the diagonal (i.e., it acts as the δ-function on the diagonal) combined with its global interpretation as a map of cohomology gives at once the localization of the Lefschetz number on the fixed point set. The precise nature of this localization (the explicit formula for the local index) is then carried out by studying only the *local* Thom class.

Carrying out this procedure in the analytic context presents at once the difficulty of finding a class on $M \times M$ supported on the diagonal. One way of doing this is to use distributions. The various proofs of Atiyah and Bott [1], [2] use some procedure for approximating the Dirac measure on the diagonal by smooth sections. A very elegant proof along this line was given by Kotake [10]. He uses the heat equation to find such approximations, and actually avoids any use of distributions. All these methods give a "trace at chain level" which is very special to simple fixed points.

There is however another localization procedure due to de Rham [14, § 33] which for our purposes can be stated as follows:

Localize the Thom class by writing it as a smooth class on $M \times M$ which *cobounds off the diagonal*. We follows this procedure to find the analytic analogues of the steps just outlined. We work always in the context of smooth sections of vector bundles and do not need any theory of distributions.

In § 2 we review the main facts which we need on elliptic complexes, and carry out step (i) of the above outline. This is the only section in which we make essential use of pseudo-differential operators. These come in very heavily in the proof of Lemma 1, the basic estimates which we need.

§ 3 gives the basic localization theorem of the Lefschetz number (step (ii)). We also remark on coincidence theorems as they fall naturally under the general study of the Thom class. In § 4 we derive the Atiyah-Bott formula for simple fixed points from the general theory of the first two sections.

In § 5 we show how to construct local Thom classes under the assumption of sheaf-exactness and complete step (iii). For step (iv) (identification of the local Thom class) one needs more information on the local geometry of the complex in order to get useful formulas. This is done in § 6 for the classical complexes.

2. Elliptic complexes

Our notation is very close to that of Atiyah and Bott [2]. A good general reference for differential operators on manifolds is [12], and for pseudo-differential operators [13], [15].

M always denotes a compact smooth n-manifold. If E is a vector bundle over M, and U an open subset of M, then $\Gamma(E, U)$ denotes the space of C^∞-sections of E, and $\Gamma_c(E, U)$ those with compact support. We write $\Gamma(E, M)$ simply as $\Gamma(E)$. T^*M is the cotangent bundle of M.

Let $E_0, \cdots, E_N$ be complex vector bundles over M, and $D_i: \Gamma(E_i) \to \Gamma(E_{i+1})$ differential operators such that

(i) $D_{i+1}D_i = 0$,

(ii) for each $\xi \in T^*M$, $\xi \neq 0$, the symbol sequence

$$0 \longrightarrow E_0 \xrightarrow{\sigma(D_0, \xi)} E_1 \longrightarrow \cdots \xrightarrow{\sigma(D_{N-1}, \xi)} E_N \longrightarrow 0$$

is exact.

Then we say that $E = \{E_i, D_i\}$ is an *elliptic complex*. $H(E) = \{H^i(E)\}$ denotes its homology.

For simplicity of exposition, *we always assume that all the D_i have order one*. If the operators have arbitrary orders, then the Hodge theory of this section would have to be modified as in [10, §§ 4, 5]. The estimates of Lemma 1 would be somewhat different. Finally, $\sigma(D_i, \xi_j)$ in Lemma 4 would have to be replaced by suitable *differential* operators and subsequent formulas modified accordingly. All our arguments will then go through in this more general context.

Let Ω denote the bundle of twisted n-forms (bundle of volumes) of M. If M is *oriented*, Ω can be identified with the bundle $\Lambda^n(T^*M)$ of usual n-forms. Let $E_i' = \operatorname{Hom}(E_i, \Omega) = \Omega \otimes E_i^*$. Then there exists a natural pairing

$$\langle\ ,\ \rangle \text{ (or tr)}: E_i \otimes E_i' \longrightarrow \Omega\ , \tag{2.1}$$

given by evaluation, or equivalently trace, which induces a pairing

$$\Gamma(E_i) \otimes \Gamma(E'_i) \longrightarrow C \tag{2.2}$$

defined by

$$s \otimes t \longrightarrow \int_M \langle s, t \rangle .$$

We can then define the transposed operators $D'_i : \Gamma(E'_{i+1}) \to \Gamma(E'_i)$ characterised by

$$\int_M \langle s, D'_i t \rangle = \int_M \langle Ds, t \rangle .$$

$\{E'_i, D'_i\}$ forms a complex E', also elliptic,called the *dual complex*. The pairing (2.2) induces a pairing:

$$H^i(E) \otimes H^i(E') \longrightarrow C . \tag{2.3}$$

Ellipticity implies that these homology groups are finite dimensional and that the pairing in homology is nonsingular. This follows easily from the Hodge theory outlined below.

Let $E_i \boxtimes E'_j$ denote the external tensor product over $M \times M$. There is a natural inclusion with dense image $\Gamma(E_i) \otimes \Gamma(E'_j) \to \Gamma(E_i \boxtimes E'_j)$. The operator $D_i : \Gamma(E_i) \to \Gamma(E_{i+1})$ determines an operator $\mathrm{D}_i(x)$ (or $D_i \boxtimes 1) : \Gamma(E_i \boxtimes E'_j) \to \Gamma(E_{i+1} \boxtimes E'_j)$ with the property that on $\Gamma(E_i) \otimes \Gamma(E'_j)$, $D_i(x)(s \otimes t) = (D_i s) \otimes t$. This determines $D_i(x)$ uniquely if it exists, and its existence need only be shown locally using coordinates. Similarly, we have $D'_j(y) : \Gamma(E_i \boxtimes E'_{j+1}) \to \Gamma(E_i \boxtimes E_j)$.

The bundles $E_i \boxtimes E'_j$ can be assembled into a complex $E \boxtimes E'$ over $M \times M$ by defining

$$(E \boxtimes E')_k = \sum_{i-j=k} E_i \boxtimes E'_j$$

with differentials

$$\hat{D}_k : \Gamma(E \boxtimes E')_k \longrightarrow \Gamma(E \boxtimes E')_{k+1}$$

given by

$$\hat{D}s(x, y) = D_i(x)s(x, y) + (-1)^{i-j} D'_{j-1}(y)s(x, y)$$

if $s(x, y) \in \Gamma(E_i \boxtimes E'_j)$.

The sections of $E \boxtimes E'$ can be thought of as kernels of integral operators $\Gamma(E) \to \Gamma(E)$ by the formula

$$s \longrightarrow \int_{y \in M} k(x, y)s(y) ,$$

where juxtaposition means the evaluation pairing (2.1). Thus $k \in \Gamma(E \boxtimes E')_k$ if and only if the associated integral operator maps $\Gamma(E_i)$ to $\Gamma(E_{i+k})$. For most of our work we need only a small portion of the complex $E \boxtimes E'$, namely, $\hat{D}_{-1}: \Gamma(E \boxtimes E')_{-1} \to \Gamma(E \boxtimes E')_0$, as we work mostly with operators of homogeneous degree zero and minus one. In § 5 we also need $\hat{D}_{-2}$.

Examples. (1) The most familiar elliptic complex is the de Rham complex $\Lambda = \Lambda(T^*M)$. If M is *oriented* and we identify Ω with Λ^n, then in the notation introduced above, $(\Lambda^i)' \approx \Lambda^{n-i}$. (If M is not oriented, $(\Lambda^i)'$ is the bundle of twisted $(n-i)$-forms.) The pairing $\Lambda^i \otimes \Lambda^{n-i} \to \Lambda^n$ is the usual wedge-product pairing. From Stokes' theorem we see that $(d_i)' = (-1)^{i-1}d_{n-i}$, and the induced pairing in homology is just the usual Poincaré duality pairing.

$\Lambda \boxtimes \Lambda'$ is the complex of double forms on $M \times M$ (in the sense of de Rham [14]) except that we use the grading from $-n$ to n instead of a the usual grading from 0 to $2n$. But taking note of the various sign conventions, d is seen to be the usual tensor product differential.

(2) Another elliptic complex of great geometric interest is the Dolbeault complex $\Lambda^{p,*}$ of differential forms of type (p, q) and a complex analytic manifold M. If $\dim_C M = n$, we always identify $\Omega \approx \Lambda^{n,n}$ using the natural orientation of M. Then $(\Lambda^{p,q})' \approx \Lambda^{n-p,n-q}$ again under the wedge product pairing, and $(\bar{\partial}p, q)' = (-1)^{p+q-1}\bar{\partial}_{n-p,n-q}$. The induced pairing in homology is the Serre duality pairing. $\Lambda^{p,*} \boxtimes (\Lambda^{p,*})'$ is again a complex of double forms on $M \times M$ with differential corresponding to $\bar{\partial}$ of the product complex structure on $M \times M$. Precisely, it is the subcomplex of $\Lambda^{n,*}(M \times M)$ of double forms of degree p in the holomorphic coordinates of the first factor and degree $n-p$ in the holomorphic coordinates of the second.

(3) For two other examples arising from Riemannian and Spin structures see [3]. These complexes are actually just a single elliptic operator.

Next, we need the techniques of Hodge theory. Choose Hermitian metrices $(\ ,\ ) = (\ ,\ )_i$ on E_i and a positive volume v on M. (Thus $v \in \Gamma(\Omega)$.) These then define conjugate linear bundle isomorphisms $*: E_i \to E_i'$ by $*e_x = (\ , e_x)v_x$. The operator $D_i^*: \Gamma(E_{i+1}) \to \Gamma(E_i)$ defined by $D_i^* = *^{-1}D_i'^*$ is easily seen to be the adjoint of D_i with respect to the L^2-inner product

$$\int_M (\ ,\ )v .$$

As usual we define the Laplacian $\Delta_i: \Gamma(E_i) \to \Gamma(E_i)$ by

$$\Delta_i = D_{i-1}D_{i-1}^* + D_i^*D_i .$$

This is a second order self-adjoint elliptic operator. (Ellipticity of the Δ_i is equivalent to ellipticity of the complex.)

Standard Hodge theory then gives the following facts:

(2.4) There exist pseudo-differential operators $G_i, H_i\colon \Gamma(E_i) \to \Gamma(E_i)$ such that

(a) H_i is L^2-orthogonal projection on the finite dimensional space $\ker \Delta_i = \ker D_i \cap \ker D_i^*$,

(b) $\Delta_i G_i = G_i \Delta_i = 1 - H_i$,

(c) H_i has a smooth kernel $h_i(x, y) \in \Gamma(E_i \boxtimes E_i')$ given explicity as follows: Let $\{h_{ij}\}$ be an orthonormal basis for $\ker \Delta_i$ Then

$$h_i(x, y) = \sum_j h_{ij}(x) \otimes *h_{ij}(y) ,$$

(d) $$(G_i s)(x) = \int_{y \in M} g_i(x, y) s(y) ,$$

where $g_i(x, y)$ is smooth for $x \neq y$. For each $x \in M$, $g_i(x, y) \in L^1(y)$ so that this integral always makes sense.

H_i is called the harmonic projection, and G_i the Green operator.

We sketch briefly the standard argument. One takes a cover $\{U_\alpha\}$ of M by coordinate charts, and trivializations of the E_i over U_α. Expressing the Δ_i in terms of these coordinates, the principal symbols $\sigma(\Delta_i, \xi)$ become matrices whose entries are homogeneous polynomials of degree 2 in ξ for each x. Define $q_\alpha^{-2}(x, \xi)$ on U_α, homogeneous of degree -2 in ξ, to be the inverse matrix.

Let θ be a smooth function vanishing near zero and identically one on $|\xi| \geq 1$. Then the operator $\Gamma_c(E_i, U_\alpha) \to \Gamma(E_i, U_\alpha)$ defined by

$$u \longrightarrow \left(\frac{1}{2\pi}\right)^n \int e^{ix\cdot\xi} \theta(\xi) q_\alpha^{-2}(x, \xi) \hat{u}(\xi) d\xi$$

gives a first approximation to an inverse for Δ_i on $\Gamma_c(E_i, U_\alpha)$.

Using the rule for composition of pseudo-differential operators, this approximation is improved by means of an iteration as in [13] or [15]. A sequence of of symbols $q_\alpha^{-\nu}(x, \xi)$, homogeneous degree $-\nu$ in ξ, is constructed, and these are assembled into a symbol $q_\alpha(x, \xi)$ which has for asymptotic expansion at infinity the formal sum of the homogeneous terms $q_\alpha^{-\nu}$. Let Q_α be the corresponding operator:

$$Q_\alpha u(x) = \frac{1}{(2\pi)}n \int e^{ix\cdot\xi} q_\alpha(x, \xi) \hat{u}(\xi) d\xi ,$$

and $\{\phi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$, and let $\psi_\alpha \in C_c^\infty(U_\alpha)$, $\psi_\alpha \equiv 1$ on $\operatorname{spt} \varphi_\alpha$. Then $Q = \sum \varphi_\alpha Q_\alpha \psi_\alpha$ is a pseudo-differential operator $\Gamma(E_i) \to \Gamma(E_i)$ which satisfies

$$\Delta_i Q = 1 - S_1 , \qquad Q\Delta_i = 1 - S_2 ,$$

where S_1, S_2 are smoothing operators, i.e.,

$$s_j u(x) = \int_{y \in M} s_j(x, y) u(y)$$

with $s_j \in \Gamma(E_i \boxtimes E_i')$, $j = 1, 2$.

This implies in particular that Δ_i is a Fredholm operator, and (a) follows trivially. If in addition we use the self-adjointness of Δ_i as in [14, § 31], it is not hard to construct (abstractly) the operator G_i satisfying (b). H_i is clearly given by (c).

Finally we note that

$$(G_i - Q)\Delta_i = H_i - S_2 ,$$

and applying Q on the right we get

$$G_i - Q = (G_i - Q)S_1 + (H_i - S_2)Q ,$$

which is a smoothing operator. Thus G_i is pseudo-differential and has the same symbol as Q. (d) then follows from standard facts on pseudo-differential operators but we omit the details now as they are given below for the operator $D_{i-1}^* G_i$.

The relevance of the Green operator to our problem is that it allows us to chain-retract $\Gamma(E)$ to $\ker \Delta$. Since $D_i \Delta_i = \Delta_{i+1} D_i$ and $\ker \Delta \subset \ker D$, if follows from (b) that $D_i G_i = G_{i+1} D_i$. Therefore, if we let $K_i = D_{i-1}^* G_i \colon \Gamma(E_i) \to \Gamma(E_{i-1})$, the first equation of (2.4b) can be rewritten as

$$D_{i-1} K_i + K_{i+1} D_i = 1 - H_i . \tag{2.5}$$

In other words, K_i is a chain homotopy between 1 and the projection on $\ker \Delta$. Since D acts trivially on $\ker \Delta$, this shows that

$$\ker \Delta_i \approx H^i(E) ,$$

where the isomorphism is induced by inclusion. This implies in particular that $H^i(E)$ is finite dimensional.

If we take on E' the Hermitian metrics induced by $*$ from the metrics on E, then L^2-adjoint of D_i' is $* D_i *^{-1}$. Since the Laplacian for E' is Δ_i', it is easy to see that the restriction of $*$ gives an isomorphism

$$* \colon \ker \Delta_i \xrightarrow{\approx} \ker \Delta_i' .$$

This implies at once that the duality pairing (2.3) is nonsingular. (This is only used in connection with coincidence theorems.)

All our arguments are based on the fact that the operator K_i of (2.5) is an integral operator with kernel smooth off the diagonal. We also need very precise estimates on the singularity of this kernel at the diagonal.

Lemma 1.
$$K_i s(x) = \int_{y \in M} k_i(x, y) s(y) ,$$

where $k_i(x, y) \in \Gamma(E_{i-1} \boxtimes E_i', M \times M - \Delta)$, *and near the diagonal the following estimates hold (expressed in local coordinates):*

(i) $|k_i(x, y)| \leq C|x - y|^{1-n}$,

(ii) $|(\partial/\partial x_j + \partial/\partial y_j) k_i(x, y)| \leq C|x - y|^{1-n}$, *or equivalently*

(ii′) $|k_i(x + z, y + z) - k_i(x, y)| \leq C|z||x - y|^{1-n}$.

Remark. Observe that (i) justifies writing K_i as an honest integral operator (rather than an operator with a distributional kernel), because $k_i(x, y) \in L^1(y)$ for each x. (ii′) implies that we have some control on the variation of k_i along directions parallel to the diagonal. If the D_i were *constant coefficient* operators, then K_i would be given by convolution with a function. Then $k_i(x, y)$ would depend only on $x - y$, and (ii′) would be trivial.

Proof of the lemma. This follows from the very thorough discussion of kernels of pseudo-differential operators given in [15], but we prefer to give a complete proof of the particular facts which we need.

Let U be our coordinate chart. Since K_i is the composition $D_{i-1}^* G_i$, it is pseudo-differential, given locally for $u \in \Gamma_c(E_i, U)$, by

$$K_i u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} p(x, \xi) \hat{u}(\xi) d\xi ,$$

where $p(x, \xi)$ is a symbol of order -1 which has an asymptotic expansion as a sum of homogeneous symbols. This expansion is obtained from that of G_i by applying the rule for composition of pseudo-differential operators. Since the matter is purely local, we can assume that u is supported in a relatively compact open set $V \subset U$, and that $p(x, \xi)$ is compactly supported in x.

Taking a function θ vanishing near zero and identically one for $|\xi| \geq 1$, we write

$$p(x, \xi) = \theta(\xi) p_1(x, \xi) + p_2(x, \xi) , \tag{2.6}$$

with $p_1(x, \xi)$ homogeneous of degree -1 in ξ and $p_2(x, \xi)$ a symbol of order -2, i.e.,

$$|(\partial/\partial x)^\alpha (\partial/\partial \xi)^\beta p_2(x, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{-2-|\beta|} . \tag{2.7}$$

The idea of the proof is to use separately the homogeneity of p_1 and the decay in high ξ-derivatives of p_2 to obtain the estimates.

We write $K_i = P_1 + P_2$ corresponding to the decomposition (2.6) of its symbol, and from now on we drop the irrelevant factor $(2\pi)^{-n}$. Then

$$P_2u(x) = \int e^{ix\cdot\xi}p_2(x, \xi)\hat{u}(\xi)d\xi = \int\int e^{i(x-y)\cdot\xi}p_2(x, \xi)u(y)dyd\xi \ .$$

Since the integrand is not absolutely convergent in ξ, we cannot interchange the order of integration. But we can take open cover of U given by $x_j - y_j \neq 0$ and a partition of unity $\{\varphi_j\}$ subordinate to this cover, and write

$$P_2u(x) = \sum_{j=1}^{n}\int\int e^{i(x-y)\cdot\xi}(x_j - y_j)^{n-1}\frac{\varphi_j(x-y)}{(x_j - y_j)^{n-1}}p_2(x, \xi)u(y)dyd\xi \ .$$

This we can integrate by parts $n - 1$ times, as the relevant y-integrands are in L^1 and the y-integrals are functions of ξ vanishing at infinity, so that we obtain

$$P_2u(x) = \sum_{j=1}^{n}\int\int e^{i(x-y)\cdot\xi}\frac{\varphi_j(x-y)}{(x_j - y_j)^{n-1}}(i\partial/\partial\xi_j)^{n-1}p_2(x, \xi)u(y)dyd\xi \ .$$

The integrand is then in L^1 with respect to y, and also with respect to ξ because of (2.7). Therefore we can interchange the order of integration and obtain that P_2 has an integrable kernel satisfying the estimate (i), namely,

$$\sum_{j=1}^{n}\frac{\varphi_j(x-y)}{(x_j - x_j)^{n-1}}\int e^{i(x-y)\cdot\xi}(i\partial/\partial\xi_j)^{n-1}p_2(x, \xi)d\xi \ .$$

As it stands this is only continuous for $x \neq y$. To see that it is actually smooth, note that we can integrate by parts k times for any $k > 0$ and rewrite the kernel as

$$\sum_{j=1}^{n}\frac{\varphi_j(x-y)}{(x_j - y_j)^{k+n-1}}\int e^{i(x-y)\cdot\xi}(i\partial/\partial\xi_j)^{k+n-1}p_2(x, \xi)d\xi \ .$$

But this can be differentiated under the integral sign k times.

To handle P_1, write $P_1 = P_1' + P_1''$ where

$$P_1'u(x) = \int_{|\xi|\leq 1} e^{ix\cdot\xi}\theta(\xi)p_1(x, \xi)\hat{u}(\xi)d\xi \ ,$$

$$P_1''u(x) = \int_{|\xi|\geq 1} e^{ix\cdot\xi}\theta(\xi)p_1(x, \xi)\hat{u}(\xi)d\xi \ .$$

P_1' clearly has the *smooth* kernel

$$\int_{|\xi|\leq 1} e^{i(x-y)\cdot\xi}\theta(\xi)p_1(x,\xi)d\xi \ ,$$

which can therefore be neglected.

Now, if Δ denotes the usual Laplacian in $\boldsymbol{R}^n$, then

$$P_1''u(x) = \int_{|\xi|\geq 1} e^{ix\cdot\xi}\theta(\xi)p_1(x,\xi)\,|\xi|^{-2r}(-\Delta)^r\hat{u}(\xi)d\xi \ .$$

For $2r > n-1$ we can write, after changing variables $y = x + z$,

$$P_1''u(x) = \int l(x, x+z)(-\Delta)_z^r u(x+z)dz \ , \tag{2.8}$$

where

$$l(x, x+z) = \int_{|\xi|\geq 1} e^{iz\cdot\xi}p_1(x,\xi)\,|\xi|^{-2r}d\xi \ .$$

Again repeated integrations by parts show that l is smooth for $z \neq 0$. Moreover, for $0 < t < 1$,

$$\begin{aligned} l(x, x+tz) &= \int_{|\zeta|\geq 1} e^{iz\cdot(t\xi)}p_1(x,\xi)\,|\xi|^{-2r}d\xi \\ &= \int_{|\eta|\geq 1} e^{iz\cdot\eta}p_1(x,\eta/t)\,|\eta/t|^{-2r}t^{-n}d\eta \\ &= t^{2r+1-n}l(x, x+tz) \ . \end{aligned}$$

But then $(\partial/\partial z)^\alpha l(x, x+z)$ is homogeneous in z of degree $2r+n-1-|\alpha|$ for $|z|$ small. Therefore we can integrate (2.8) by parts 2r times with L^1 kernel at each stage, and obtain

$$P_1''u(x) = \pm\int(-\Delta)_z^r l(x, x+z)u(x+z)dz \ .$$

The resulting kernel is then homogeneous of degree $1-n$ for $|z|$ small and hence satisfies (i).

To prove (ii), observe that $[\partial/\partial x_j, K_i]$ is an operator of order -1, because both $\partial/\partial x_j, K_i$ and $K_i\cdot\partial/\partial x_j$ have principal symbol $P_1(x,\xi)\xi^j$. Hence our previous arguments applied to the operator $[\partial/\partial x_j, K_i]$ rather that K_i imply that

$$[\partial/\partial x_j, K_i]u(x) = \int \bar{k}_i(x,y)u(y)dy \ ,$$

where $\bar{k}_i$ satisfies (i).

Fix $x_0 \neq y_0$. Then for any u supported in a neighborhood of y_0 whose closure does not contain x_0, and for x close enough to x_0,

$$[\partial/\partial x_j, K_i]u(x) = \int (\partial/\partial x_j + \partial/\partial y_j)k_i(x, y)u(y)dy \ .$$

Since this holds for all such U, it follows that

$$\bar{k}_i(x, y) = (\partial/\partial x_j + \partial/\partial y_j)k_i(x, y) \qquad x \neq y \ ,$$

and this completes the proof of the lemma.

Lemma 2. *For $x \neq y$,*

$$D_{i-1}(x)k_i(x, y) + D'_i(y)k_{i+1}(x, y) = -h_i(x, y) \ .$$

Proof. (2.5) and Lemma 1 give

$$D_{i-1}\int_{y \in M} k_i(x, y)s(y) + \int_{y \in M} k_{i+1}(x, y)D_i s(y) = s(x) - \int_{y \in M} k_i(x, y)s(y) \ .$$

Again we can fix $x_0 \neq y_0$, and consider only sections s supported in a fixed neighborhood of y_0 whose closure does not contain x_0. Then for x away from this neighborhood we can rewrite this equation as

$$\int D_{i-1}(x)k_i(x, y)s(y) + \int D'_i(y)k_{i+1}(x, y)s(y) = -\int h_i(x, y)s(y) \ .$$

Since this holds for all such sections s, the lemma follows.

Remark. We emphasize that this and the following lemma give equations among elements of $\Gamma(E \boxtimes E', M \times M - \Delta)$. At no point are we interpreting the kernel k_i as distributions. The differential operators are only applied on $M \times M - \Delta$. Lemma 2 of course follows from a stronger statement about distributions, involving the δ-function on the diagonal, but we totally avoid this approach.

The next lemma is the basic localization statement for the Thom class in the de Rham formulation.

Lemma 3. *For $x \neq y$,*

$$\sum_{i=0}^{N} (-1)^i h_i(x, y) = -\hat{D}\left\{\sum_{i=1}^{N} (-1)^i k_i(x, y)\right\} \ .$$

Proof. $\qquad \hat{D}k_i(x, y) = D_{i-1}(x)k_i(x, y) - D'_{i-1}(y)k_i(x, y) \ .$

Therefore, by Lemma 2,

$$\hat{D}\left\{\sum_{i=1}^{N}(-1)^i k_i\right\} = \sum_{i=1}^{N}(-1)^i(D_{i-1}(x)k_i - D'_{i-1}(y)k_i)$$
$$= \sum_{i=0}^{N}(-1)^i(D_{i-1}(x)h_i + D'_i(y)k_{i+1} = -\sum_{i=0}^{N}(-1)^i h_i\ .$$

3. An integral formula for the Lefschetz number

Let $E = \{E_i, D_i\}$ be an elliptic complex over M, and $f\colon M \to M$ a smooth map. We say that f *lifts* E if there exist bundle maps

$$\varphi_i\colon f^*E_i \longrightarrow E_i$$

such that the map

$$f_i^\#\colon \Gamma(E_i) \longrightarrow \Gamma(E_i)$$

defined by $(f_i^\# s)(x) = \varphi_i s(fx)$ commutes with the differentials, i.e., $D_i f_i^\# = f_{i+1}^\# D_i$. Thus $f^\# = \{f_i^\#\}$ gives a *geometric endomorphism* of E in the sense of [2]. We call such lifts $f^\#$ simply *endomorphisms of* E, since we only consider lifts of this type.

$f^\#$ induces a map of $H(E)$ which we still denote by $f^\#$. Since $H(E)$ is finite dimensional, we can define the *Lefschetz number* of f

$$L(f, E) = \sum(-1)^i \operatorname{tr}(f^\#, H^i(E))\ .$$

Our first objective is to express this number in terms of the kernels k_i, h_i of § 2.

Remark. Note that for the de Rham complex Λ all smooth maps f have a natural lift with the φ_i given by the i'th exterior power of the differential of f. $f^\#$ is the usual pull-back of forms, and $L(f, \Lambda)$ is the usual Lefschetz number. The definitions for a general complex E are of course designed to abstract this situation. For the Dolbeault complex only *holomorphic* maps have a natural lift. Thus the existence of an endomorphism $f^\#$ will in general put strong restrictions on the maps f for which Lefschetz theory can be formulated.

If $f\colon M \to M$ lifts to E, then $f \times 1\colon M \times M \to M \times M$ has a natural lift to $E \boxtimes E'$

$$(f \times 1)^\#\colon \Gamma(E \boxtimes E') \longrightarrow \Gamma(E \boxtimes E')$$

defined by

$$(f \times 1)_{ij}^\# s(x, y) = \varphi_i(x) s(fx, y)$$

for $s \in \Gamma(E_i \boxtimes E'_j)$. It is clear that $\hat{D}(f \times 1)^\# = (f \times 1)^\# \hat{D}$.

Applying $(f \times 1)^{\#}$ to the expression of Lemma 3, and using the fact that it is an endomorphism, we get

$$(3.1) \qquad \sum (-1)^i (f \times 1)^{\#} h_i = -\hat{D}\{\sum (-1)^i (f \times 1)^{\#} k_i\}$$

for $fx \neq y$. Note that even though $\sum (-1)^i (f \times 1)^{\#} k_i$ is defined only for $fx \neq y$, $\hat{D}\{\sum (-1)^i (f \times 1)^{\#} k_i\}$ extends smoothly to all of $M \times M$, namely, by the left hand side of (3.1).

Now if $s \in \Gamma((E \boxtimes E')_0, U)$ where U is an open subset of $M \times M$, restricting to the diagonal we obtain a section over an open subset of M

$$\Delta^* s \in \Gamma((E \otimes E')_0, \Delta^{-1} U) .$$

To this section we can apply the pairing (2.1) and obtain

$$\operatorname{tr} \Delta^* s \in \Gamma(\Omega, \Delta^{-1} U) .$$

Applying this operation to (3.1) we get

$$(3.2) \qquad \sum (-1)^i \operatorname{tr} \Delta^* (f \times 1)^{\#} h_i = -\operatorname{tr} \Delta^* \hat{D}\{\sum (-1)^i (f \times 1)^{\#} k_i\} .$$

Observe again that the alternating sum is an element of $\Gamma(\Omega, M)$, even though $\operatorname{tr} \Delta^* \hat{D}(f \times 1)^{\#} k_i \in \Gamma(\Omega, M - F)$ for each i, where F denotes the fixed point set.

Proposition 1.

$$L(f, E) = -\int_M \operatorname{tr} \Delta^* \{\sum (-1)^i \hat{D}(f \times 1)^{\#} k_i\} .$$

Proof. Recall that from (2.4c),

$$h_i(x, y) = \sum h_{ij}(x) \otimes *h_{ij}(y) ,$$

$$\int_M \operatorname{tr} \Delta^* (f \times 1)^{\#} h_i = \sum_j \int_M \langle f_i^{\#} h_{ij}, *h_{ij} \rangle = \operatorname{tr} H_i f_i^{\#} .$$

But under the isomorphism $\ker \Delta_i \xrightarrow{\approx} H^i(E)$ induced by the inclusion $\ker \Delta_i \to \Gamma(E)$, $H_i f_i^{\#}$ clearly corresponds to the induced map in homology. Therefore

$$L(f, E) = \sum (-1)^i \int_M \operatorname{tr} \Delta^* (f \times 1)^{\#} h_i ,$$

and the proposition follows from (3.2).

If E is the de Rham complex, then $k_i(x, y)$ is a double form on $M \times M$, and the operation $\operatorname{tr} \Delta^*$ is the usual restriction of an n-form on $M \times M$ to an n-form on M via the diagonal map (the cup product of algebraic topology). If we use $\sharp$ to denote the usual pull-back of forms, then

$$\operatorname{tr} \Delta^* \hat{D}(f \times 1)^* k_i = \Delta^* d(f \times 1)^* k_i = d\Delta^*(f \times 1)^* k_i ,$$

where the last equality makes sense only over $M - F$. In other words, for the de Rham complex the integrand in Proposition 1 is *exact* on $M - F$. The same argument holds for the Dolbeault complex.

For a general elliptic complex there is no natural $(n-1)$-form defined on $M - F$ whose exterior derivative is our integrand restricted to $M - F$. But forms with this property can always be constructed as follows.

Choose one-forms $\xi_1, \cdots, \xi_m$ and vector fields $X_1, \cdots, X_m$ on M such that any one-form $\xi \in \Gamma(T^*M)$ can be written

$$\xi = \sum_{j=1}^{m} \xi(X_j)\xi_j . \tag{3.3}$$

Their existence can be shown using either local coordinates or general algebraic facts on finitely generated projective modules. If $s \in \Gamma(E_{i-1})$ and $t \in \Gamma(E'_i)$, then by the very definition of D' we know that $\langle Ds, t\rangle - \langle s, D't\rangle \in \Gamma(\Omega)$ is exact. But choosing ξ_j, X_j satisfying (3.3) we can write explicitly

$$\langle Ds, t\rangle - \langle s, D't\rangle = d \sum_j X_j \rfloor \langle \sigma(D_{i-1}, \xi_j)s, t\rangle , \tag{3.4}$$

where $\rfloor$ denotes interior product.

This is a standard fact which can be proved as follows: choose connections ∇ for E_{i-1} and $\bar{\nabla}$ for E'_{i-1} such that for all vector fields X

$$L_X\langle s, s'\rangle = \langle \nabla_X s, s'\rangle + \langle s, \bar{\nabla}_X s'\rangle ,$$

where L_X is the Lie derivative. (If ∇ is a metric connection for E_{i-1}, then $\bar{\nabla}$ defined by $\bar{\nabla}_X = *\nabla_X *^{-1} + \operatorname{div}_X v$ satisfies our requirement.) Then

$$Ds = \sum_j \sigma(D_{i-1}, \xi_j)\nabla_{X_j} s + Bs ,$$

$$D't = -\sum_j \bar{\nabla}_{X_j}(\sigma(D_{i-1}, \xi_j)'t) + B't$$

for some bundle map $B\colon E_{i-1} \to E_i$. Thus

$$\begin{aligned} &d\{X_j \rfloor \langle \sigma(D_{i-1}, \xi_j)s, t\rangle\} \\ &\quad = d\{X_j \rfloor \langle s, \sigma(D_{i-1}, \xi_j)'t\rangle\} = L_{X_j}\langle s, \sigma(D_{i-1}, \xi_j)'t\rangle \\ &\quad = \langle \sigma(D_{i-1}, \xi_j)\nabla_{X_j} s, t\rangle + \langle s, \bar{\nabla}_{X_j}\sigma(D_{i-1}, \xi_j)'t\rangle , \end{aligned}$$

and this gives (3.4).

Lemma 4. *Let ξ_j, X_j satisfy* (3.3), *and let $u \in \Gamma(E_{i-1} \boxtimes E'_i, U)$ where U is open in $M \times M$. Then over $\Delta^{-1}U$*

$$\operatorname{tr} \Delta^* \hat{D}u = d \sum_j X_j \rfloor \operatorname{tr} \sigma(d_{i-1}, \xi_j)u . \tag{3.5}$$

Proof. If $U = M \times M$ and $u = s \otimes t$, then (3.5) is just (3.4). By continuity, since $\Gamma(E_{i-1}) \otimes \Gamma(E_i')$ is dense in $\Gamma(E_{i-1} \boxtimes E_i')$, (3.5) must hold for any u defined on all of $M \times M$. If u is defined only on an open subset U, for any $(x_0, y_0) \in U$ we can find a global section v which agrees with u in a neighborhood of (x_0, y_0). (3.5) then holds for v and therefore for u in a neighborhood of (x_0, y_0), and hence is valid everywhere on U.

Applying Lemma 4 to each term of Proposition 1 we obtain the following theorem.

Theorem 1. *There exists a (twisted) $(n-1)$-form $\mu \in \Gamma((T^*M)', M - F)$ such that $d\mu$ extends to a global form $\lambda \in \Gamma(\Omega)$ and*

$$L(f, E) = -\int_M \lambda \ .$$

Moreover, for any choice of vector fields $X_1, \cdots, X_m$ and one-forms $\xi_1, \cdots, \xi_m$ satisfying (3.3) *an explicit form with this property is $\mu = \sum (-1)^i \mu_i$ where*

$$\mu_i = \sum X_j \rfloor \operatorname{tr} \Delta^* \sigma(D_{i-1}, \xi_j)(f \times 1)^\# k_i \ .$$

This gives at once the most elementary result of Lefschetz theory.

Corollary 1. *If f has no fixed points, then $L(f, E) = 0$.*

If the fixed point set is not empty, the theorem shows that $L(f, E)$ depends only on the behavior of the map f and the endomorphism $f^\#$ near the fixed point set F. If F has measure zero in M, then one can attempt to compute explicitly this dependence by taking a fundamental system of neighborhoods $\{U\}$ of F, which are manifolds with boundary, and by writing

$$L(f, E) = -\int_{M-F} \lambda = -\lim \int_{M-U} d\mu = \lim \int_{\partial U} \mu \ .$$

The limit is taken over a suitable sequence of such neighborhoods.

In this way one can define a local index for each isolated component of F. For quite simple reasons this local index will be independent of the various choices made. But we do not discuss these matters further because it seems hopeless to obtain interesting results in this generality. To derive *computable* formulas for the local index one needs a good deal of information on the singularities of μ on F. The simplest case in which this information is available is when F consists of *isolated* points, and this is discussed in detail in the remaining sections.

The strarting point is the following corollary of Theorem 1. We have written $\Delta^*(f \times 1)^\# k_i$ explicitly as $\varphi_{i-1}(x) k_i(fx, x)$, where φ_{i-1} is applied on the first variable.

Corollary 2. *Suppose the fixed points are isolated, and choose disjoint Euclidean neighborhoods U_p centered at each fixed point p. Then*

$$L(f, E) = \sum_{p \in F} \nu(p) ,$$

where

$$\nu(p) = \lim_{\varepsilon \to 0} \sum (-1)^i \int_{S_\varepsilon(p)} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \lrcorner \operatorname{tr} (\sigma(D_{i-1}, dx^j)\varphi_{i-1}(x)k_i(fx, x)) ,$$

and $S_\varepsilon(p)$ denotes the sphere of radius ε about p.

Remark. $\nu(p)$ is independent of the local coordinates used in its definition; this follows trivially from the fact that their sum is independent of all choices. We emphasize that the individual integrals in the definition of $\nu(p)$ do not converge as $\varepsilon \to 0$; it is only the alternating sum which converges. Taking limits over spheres in the Euclidean structure is totally irrelevant in the convergence of the alternating sum. We would use the boundaries of any other sequence of manifolds exhausting $U_p - p$.

Proof of the corollary. Take neighborhoods V_p of each p with $\overline{V}_p \subset U_p$, and vector fields $X_1, \cdots, X_n$, one-forms $\xi_1, \cdots, \xi_n$ on M such that

$$X_j = \begin{cases} \partial/\partial x_j & \text{in a neighborhood of } \overline{V}_p , \\ 0 & \text{on } M - U_p , \end{cases}$$

$$\xi_j = \begin{cases} dx^j & \text{in a neighborhood of } \overline{V}_p , \\ 0 & \text{on } M - U_p . \end{cases}$$

These can be completed to a dual basis $\{\xi_j, X_j\}$, $j = 1, \cdots, m$, in the sense of (3.3) with $\xi_j = X_j = 0$ on each V_p if $j > n$. Then

$$L(f, E) = \lim_{\varepsilon \to 0} \sum_p \int_{S_\varepsilon(p)} \mu ,$$

where the limit exists independently for each p because λ is smooth on all of M. But for ε small enough μ coincides with the expression in the corollary, where we have written the map $(f \times 1)^\#$ explicitly in terms of the φ_i.

Remarks on coincidence theorems. Observe that everything we have done so far also works in the following more general context. Let $\overline{M}$, M be two manifolds (always compact and smooth), and $\overline{E}$, E elliptic complexes over $\overline{M}$, M respectively. Suppose two maps $f, g \colon \overline{M} \to M$ and bundle maps

$$\varphi_i \colon f^*E_i \longrightarrow \overline{E}_i ,$$
$$\psi_i \colon g^*E'_i \longrightarrow \overline{E}'_i$$

be such that the maps

$$f_i^\# \colon \Gamma(E_i) \longrightarrow \Gamma(\overline{E}_i) ,$$
$$g_i^\# \colon \Gamma(E'_i) \longrightarrow \Gamma(\overline{E}'_i)$$

defined by $(f_i^\# s)(x) = \varphi_i s(fx)$, $(g_i^\# t)(x) = \psi_i t(gx)$ commute with the differentials. In other words, f and g induce chain maps $\Gamma(E) \to \Gamma(\bar{E})$ and $\Gamma(E') \to \Gamma(\bar{E}')$ respectively. Then we obtain a chain map

$$(f \times g)^\#: \Gamma(E \boxtimes E') \longrightarrow \Gamma(\bar{E} \times \bar{E}') .$$

If k_i, h_i denote again the kernels for the complex E, and $\hat{\bar{D}}$ is the differential of $\Gamma(\bar{E} \boxtimes \bar{E}')$, then we have the analogue of (3.2):

$$\sum (-1)^i \operatorname{tr} \Delta^*(f \times g)^\# h_i = -\operatorname{tr} \Delta^* \hat{\bar{D}} \{\sum (-1)^i (f \times g)^\# k_i\} . \tag{3.6}$$

Since

$$\sum (-1)^i \int_M \operatorname{tr} \Delta^*(f \times g)^\# h_i = \sum (-1)^i \int_M \sum_j \langle f_i^\# h_{ij}, g_i^\# * h_{ij} \rangle ,$$

using the usual identification of the harmonic sections with the homology of the complex we see that this number is just the *Lefschetz coincidence number* of f and g:

$$L(f, g; E, \bar{E}) = \sum (-1)^i \operatorname{tr} ((g_i^\#)^t f_i^\#, H^i(E)) .$$

Here

$$(g_i^\#)^t: H^i(\bar{E}) \longrightarrow H^i(E)$$

is the transpose of

$$g_i^\#: H^i(E') \longrightarrow H^i(\bar{E}')$$

under the duality pairing (2, 3) in homology. Thus taking $M = \bar{M}$, $E = \bar{E}$ and $g = 1$ we can reduce this to $L(f, E)$.

Let $C = \{x \in \bar{M}: fx = gx\}$, the set of *coincidence points* of f and g. The same formal reasoning we used to prove Theorem 1 gives

Theorem 1′. *There exists a (twisted) $(n-1)$-form $\mu \in \Gamma((T^*\bar{M})', \bar{M} - C)$ such that $d\mu$ extends to a global form λ on $\bar{M}$ and*

$$L(f, g; E, \bar{E}) = -\int_M \lambda .$$

For any choice of vector fields and one-forms on $\bar{M}$ satisfying (3, 3), an explicit form with this property is $\mu = \sum (-1)^i \mu_i$ where

$$\mu_i = \sum_{j=1}^m X_j \rfloor \operatorname{tr} \Delta^* \sigma(\bar{D}_{i-1}, \xi_j)(f \times g)^\# k_i .$$

Corollary 1′. *If f and g have no coincidences, then $L(f, g; E, \bar{E}) = 0$.*

Corollary 2′. *Suppose the coincidence points are isolated, and choose disjoint Euclidean neighborhoods U_p centered at each coincidence point p. Then*

$$L(f, g; E, \bar{E}) = \sum_{p \in C} \nu(p) ,$$

where

$$\nu(p) = \lim_{\varepsilon \to 0} \sum (-1)^i \int_{S_\varepsilon(p)} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \lrcorner \operatorname{tr} (\sigma(\bar{D}_{i-1}, dx^j)(\varphi_{i-1} \otimes \psi_i)(x) k_i(fx, gx) .$$

If M and $\bar{M}$ are manifolds of the same dimension, and E and $\bar{E}$ are their respective de Rham complexes, then our hypothesis hold for any smooth maps f and g. $L(f, g; \Lambda)$ is the classical Lefschetz coincidence number. For complex analytic manifolds M and $\bar{M}$ of the same dimension and holomorphic f, g we can also define $L(f, g; \Lambda^{p,*})$.

The theorems of the next two sections have obvious coincidence analogues. But we do not mention coincidence theorems again until we discuss the Dolbeault complex.

4. Simple fixed points

Suppose that p is a *simple* fixed point of f in the sense that $\det(1 - df_p) \neq 0$. Geometrically this means that the graph of f and the diagonal intersect *transversely* at $(p, p) \in M \times M$. Then the local index $\nu(p)$ can be computed explicitly by going to the limit in the expression of Corollary 2. The result is the formula of Atiyah and Bott [2]. Since this formula involves the values of the φ_i *at* p, the Dirac measure at p (evaluation at p) must be relevant to the formula. But this evaluation functional must be hidden in the singularity of the k_i, and the idea of our proof of this case is to find the Dirac measure explicitly in terms of k_i.

We work in a coordinate patch U_p centered at p and with fixed trivializations of the E_i over U_p. We write the kernels k_i, h_i and the symbols $\sigma(D_i, \xi)$ simply as matrices with respect to these trivializations. (Strictly speaking, we should write $k_i(x, y) \otimes dy$, $h_i(x, y) \otimes dy$ where dy is the volume element given by the local coordinates.)

We write dy^J for $dy^1 \cdots \widehat{dy^j} \cdots dy^n$, and $\sigma_i(\xi)$ for $\sigma(D_i, \xi)$.

Lemma 5. *Let* $s \in \Gamma_c(E_i, U_p)$. *Then*

$$s(0) = -\lim_{\varepsilon \to 0} \int_{\partial U_\varepsilon} \sum (-1)^{j-1} \{\sigma_{i-1}(dx^j) k_i(0, y) + k_{i+1}(0, y) \sigma_i(dy^j)\} s(y) dy^J ,$$

where $U_\varepsilon \subset U_p - 0$ is any increasing family of manifolds with boundary whose union is $U_p - 0$.

Proof. For $s \in \Gamma_c(E_i, U_p)$ we know that

$$
\tag{4.1}
\begin{aligned}
D_{i-1}\int k_i(x, y)s(y)\,dy + \int k_{i+1}(x, y)D_i s(y)dy \\
= s(x) - \int h_i(x, y)s(y)dy \,.
\end{aligned}
$$

Write $D_{i-1} = \sum_j \sigma_{i-1}(dx^j)\partial/\partial x^j + B_{i-1}$ where $B_{i-1}\colon E_{i-1}|U_p \to E_i|U_p$ is a bundle map. To prove the lemma it is enough to assume that $\operatorname{spt} s$ is small enough so that $x + y \in U_p$ whenever $x, y \in \operatorname{spt} s$. Then we can interchange the order of differentiation and integration in the first term of (4.1) by making the substitution $y = x + z$:

$$
\begin{aligned}
\frac{\partial}{\partial x^j}\int k_i(x, y)s(y)\,dy &= \frac{\partial}{\partial x^j}\int k_i(x, x+z)s(x+z)dz \\
&= \int\left(\frac{\partial}{\partial x^j} + \frac{\partial}{\partial y^j}\right)\{k_i(x, y)s(y)\}dy \,.
\end{aligned}
$$

Differentiation under the integral sign is now justified because by Lemma 1 (ii) the integrand is in L^1. Therefore (4.1) becomes

$$
\tag{4.2}
\begin{aligned}
&\int_U \Big[D_{i-1}(x)k_i(x, y)s(y) \\
&\qquad + \sum_j \sigma_{i-1}(dx^j)(\partial/\partial y^j)\{k_i(x, y)s(y)\} + k_{i+1}(x, y)D_i s(y)\Big]dy \\
&= s(x) - \int h_i(x, y)s(y)dy \,.
\end{aligned}
$$

Since the integrand is in L^1 (note that the first two terms are not integrable, but their sum is), setting $x = 0$ we can write (4.2) as the limit of the integrals over U_ε. But over U_ε we can use the following three equations; the first is Lemma 2, and the second follows from Lemma 4:

$$
\begin{aligned}
&D_{i-1}(x)k_i(0, y) + D_i'(y)k_i(0, y) = -h_i(0, y) \,, \\
&k_{i+1}(0, y)D_i s(y) - D_i'(y)k_i(0, y)s(y) \\
&\qquad = d_y\{\textstyle\sum (-1)^{j-1}k_{i+1}(0, y)\sigma_{i-1}(dx^j)s(y)dy^J\} \,, \\
&\sum_j \sigma_{i-1}(dx^j)(\partial/\partial x^j)\{k_i(0, y)s(y)\}dy \\
&\qquad = d_y\{\textstyle\sum (-1)^{j-1}\sigma_{i-1}(dx^j)k_i(0, y)s(y)dy^J\} \,.
\end{aligned}
$$

Combining these with (4.2) we see that the terms involving $D_{i-1}(x)$ and $h_i(0, y)$ are cancelled with each other, and Stokes' theorem gives the lemma.

Theorem 2 (*Atiyah-Bott formula*). *Suppose that p is a simple fixed point. Then*

$$\nu(p) = \sum_{i=0}^{n} (-1)^i \operatorname{tr} \varphi_i(p)/|\det(1 - df_p)| \, .$$

Proof. Applying $\varphi_i(0)$ to both sides of Lemma 5 and noting that since $\sigma_{i-1}(dx^j)_0 = [D_{i-1}, x^j](0)$ and $\varphi_i(0)\sigma_{i-1}(dx^j)_0 = \sigma_{i-1}(df^j)_0\varphi_{i-1}(0)$, it follows that

$$(4.3) \qquad \begin{aligned}(\varphi_i s)(0) = -\lim_{\varepsilon \to 0} \int_{\partial U_\varepsilon} \sum (-1)^{j-1}\{&\sigma_{i-1}(df^j)_0\varphi_{i-1}(0)k_i(0, y) \\ &+ \varphi_i(0)k_{i+1}(0, y)\sigma(dy^j)\}s(y)dy^J \, .\end{aligned}$$

Therefore

$$(4.4) \qquad \begin{aligned}\operatorname{tr} \varphi_i(0) = -\lim_{\varepsilon \to 0} \int_{\partial U_\varepsilon} \sum (-1)^{j-1} \operatorname{tr} \{&\sigma_{i-1}(df^j)_0\varphi_{i-1}(0)k_i(0, y) \\ &+ \phi_i(0)k_{i+1}(0, y)\sigma_i(dy^j)\}dy^J \, .\end{aligned}$$

Here tr means the trace of the matrix in brackets, even though the map it represents is not an endomorphism. (4.4) can be deduced readily from (4.3) by taking $s_1, \cdots, s_k \in \Gamma_c(E_i, U_p)$, which agree near 0 with the basis of the given trivialization of E_i, and by computing $\sum_l \langle \varphi_i s_l(0), s_l(0) \rangle$ according to (4.3).

By assumption $1 - f$ is a diffeomorphism in a neighborhood of 0. Choose $U_\varepsilon = \{y : |(1 - f)^{-1}y| \geq \varepsilon\}$. Then $\operatorname{vol}(\partial U_\varepsilon) \sim \text{const.}\, \varepsilon^{n-1}$ and, since by Lemma 1 (i) $|k_i(0, y)| = 0(|y|^{1-n})$, we can replace $\sigma_i(dy^j)_y$ by $\sigma_i(dy^j)_0$ in (4.4) because the difference of the two integrals goes to zero. Making this modification in (4.4), taking alternating sums, rearranging terms and using $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, we get

$$\begin{aligned}&\sum (-1)^i \operatorname{tr} \varphi_i(0) \\ &\quad = \lim_{\varepsilon \to 0} \int \sum (-1)^{j-1} \operatorname{tr} (\sigma_{i-1}((1 - f)^* dy^j)_0 \varphi_{i-1}(0) k_i(0, y)) dy^J \, .\end{aligned}$$

The integral is over $|(1 - f)^{-1}y| = \varepsilon$. Now change variables $y = x - fx$ to get

$$(4.5) \qquad \begin{aligned}&\sum (-1)^i \operatorname{tr} \varphi_i(0) \\ &\quad = \pm \lim_{\varepsilon \to 0} \int_{|x| = \varepsilon} \sum (-1)^{j-1} \operatorname{tr} (\sigma_{i-1}((1 - f)^* dy^j)_0 \\ &\qquad\qquad \cdot \varphi_i(0) k_i(0, x - fx))(1 - f)^* dy^J \, ,\end{aligned}$$

where $\pm =$ sign $\det(1 - df_0)$.

Now, by Lemma 1 (ii′),

$$|k_i(0, x - fx) - k_i(fx, x)| \leq C\,|fx|\,|x - fx|^{1-n} \, .$$

Since $|fx| \leq c_1 |x|$ and $|x - fx| \geq c_2 |x|$, we get

$$|k_i(0, x - fx) - k_i(fx, x)| \leq C^1 |x|^{2-n} .$$

Therefore the integral of this difference goes to zero in the limit, and (4.5) is the same as

$$(4.6) \qquad \pm \lim \int_{|x|=\varepsilon} \sum (-1)^{j-1} \operatorname{tr} (\sigma_{i-1}((1-f)^* dy^j) \varphi_{i-1}(0) k_i(fx, x))(1-f)^* dy^J .$$

If we let (a_{jk}) be the matrix of $1 - f^*$ in the local coordinates, then

$$(1 - f)^* dy^J = \sum_j b_{jl}(x) dx^J ,$$

where (b_{jl}) is the matrix of minors of (a_{jk}), and hence

$$(4.7) \qquad \sum_j (-1)^{j+l} b_{jl}(x) a_{jk}(x) = \delta_{lk} \det (1 - f^*)(x) .$$

The integrand in (4.6) can be written as

$$\sum_j (-1)^{j-1} \operatorname{tr} \left(\sum_k a_{jk}(0) \sigma(dx^k)_0 k_i(fx, x) \right) \sum_l b_{jl}(x) dx^J .$$

Replacing $b_{jl}(x)$ by $b_{jl}(0)$ by the usual argument that any smooth factor can be replaced by its value at the origin because $k_i(fx, x) = 0(|x|^{1-n})$, (4.7) gives that the integral of

$$\pm \det (1 - f^*) \sum_l (-1)^{l-1} \operatorname{tr} (\sigma(dx^l)_0 \varphi_{i-1}(0) k_i(fx, x)) dx^J$$

has the same limit as (4.6). Therefore replacing now $\sigma(dx^l)_0 \varphi_{i-1}(0)$ by $\sigma(dx^l)_x$ $\varphi_{i-1}(x)$ we get

$$\begin{aligned} &\sum (-1)^i \operatorname{tr} \varphi_i(0) \\ &\quad = |\det (1 - df_0)| \lim_{\varepsilon \to 0} \int_{|x|=\varepsilon} \sum (-1)^{l-1} \operatorname{tr} (\sigma(dx^l)_x \varphi_{i-1}(x) k_i(fx, x) dx^J \\ &\quad = |\det (1 - df_0)| \, \nu(p) , \end{aligned}$$

and the proof is complete.

Remark. Note the reason for the change of variable in the proof of this theorem (and the presence of the "twisting factor" $|\det (1 - df_p)|$ in the formula for $\nu(p)$). The differential equation satisfied by $k_i(x, y)$ gives information on this kernel along the fibres $x =$ constant. This is the content of Lemma 5. But $\nu(p)$ involves the singularity of k_i along the graph of f, or equivalently, the singularity of $(f \times 1)^\# k_i$ along Δ. The proof relates these two singularities via the change of variable $y = x - fx$ and the estimate (ii′) of Lemma 1.

5. Isolated fixed points

Suppose that the fixed points are *isolated* but *not necessarily simple*. In this case it is hopeless to carry an explicit computation of $\nu(p)$ as in Theorem 2. First of all, the change of variable $y = x - fx$ is no longer available. But more significantly from our point of view, in the absence of the estimate $|x - fx| \geq c\,|x|$ the estimates of Lemma 1 are totally useless in studying the limits. For this reason we want to avoid taking limits in the expression of Corollary 2. This can be done if the integrand in that expression can be replaced by a *closed* form. The topological argument of § 1 suggests that this is possible provided we can replace k_i by a *cohomology class* in the product—the local Thom class.

To do this we need a local cohomological condition on E, the Poincaré lemma in the sense of sheaf exactness. Precisely, we say that E is *sheaf exact in dimension* i if given any $x \in M$ and $s \in \Gamma(E_i, U)$, $D_i s = 0$ where U is a neighborhood of x, there exists a possibly smaller neighborhood V of x and $t \in \Gamma(E_{i-1}, V)$ such that $s\,|\,V = D_{i-1}t$. If E is sheaf exact in dimension i for all $i > 0$ we say simply that E is *sheaf exact*. Thus the local cohomology of E is just the germs of solutions of $D_0 s = 0$. Of course since $E', E \boxtimes E'$ have not been graded in increasing order starting at 0, these definitions have to be modified in the obvious way for these complexes.

We observe the following simple fact:

Lemma 6. *If $E = \{E_i, D_i\}_{i=0}^N$ is an elliptic complex, then E is sheaf exact in dimension N.*

Proof. Since the integrability condition $D_N s = 0$ is vacuous, this just says that any $s \in \Gamma(E_N)$ is locally exact. Fix a neighborhood U of $x \in M$ and let $\Delta_N = D_{N-1}D_{N-1}^*$ be the Laplacian of § 2. Since the null space of Δ_N is finite dimensional, there is a possibly smaller neighborhood V of x such that there are no solutions of $\Delta_N s = 0$ supported in V.

Take a neighborhood W of x with $\overline{W}$ compact in V and a real valued function ρ identically zero near $\overline{W}$ and identically one near $M - V$. Then $\Delta_N + \rho$ is still elliptic and self-adjoint, and by the positivity of Δ_N it is easy to see that $\ker \Delta_N + \rho = 0$. Thus there exists a Green operator

$$G_\rho \colon \Gamma(E_N) \longrightarrow \Gamma(E_N)$$

for $\Delta_N + \rho$ which is bijective. Since $\Delta_N + \rho = D_{N-1}D_{N-1}^*$ near $\overline{W}$, if we take a function ψ identically one on $\overline{W}$ and with $\operatorname{spt}\psi \cap \operatorname{spt}\rho = \emptyset$, it follows that if $s \in \Gamma(E_N, U)$, then $D_{N-1}(D_{N-1}^* G_\rho \psi s) = s$ on W.

Remarks. (1) For the complex E' the lemma says that if $s \in \Gamma(E_0', U)$, then $s\,|\,V = D_0' t$ for some $t \in \Gamma(E_1', V)$ and $V \subset U$. This is the form in which we will apply the lemma.

(2) Sheaf exactness of E in dimensions $< N$ is a very hard problem. The proof of the lemma does not work in this case because the integrability condi-

tion $D_i s = 0$ is present. For this reason the *local* problem over U cannot be reduced to the *global* theory of elliptic equations over closed compact manifolds. The standard approach to this problem is to reduce it to a boundary value problem. For more information on these questions see [16] and the references given there.

(3) The classical elliptic complexes are well known to be sheaf exact, in fact locally exact.

From now on p denotes an isolated fixed point of f, U_p a sufficiently small Euclidean neighborhood centered at p, and V_p a smaller neighborhood of p with $fV_p \subset U_p$. If

$$q_i(x, y) \in \Gamma(E_{i-1} \boxtimes E'_i, U_p \times U_p - \Delta) , \qquad i = 1, \cdots, N ,$$

we write

$$q(x, y) = \sum (-1)^i q_i(x, y) .$$

Let

$$\mu(q_i) = \sum_{j=1}^{n} (\partial/\partial x^j) \lrcorner \operatorname{tr} (\sigma_{i-1}(dx^j)\varphi_{i-1}(fx, x)) ,$$

where φ_i are the bundle maps defining $f_i^\#$. Then $\mu(q_i)$ and $\mu(q) = \sum (-1)^i \mu(q_i)$ are $(n-1)$-forms on $V_p - p$. Observe that if $q_i(x, y)$ is smooth on $U_p \times U_p$, then $\mu(q_i)$ is a smooth form on all of V_p.

Applying Lemma 4 to each $(f \times 1)^\# q_i$ and taking alternating sums it follows that on $V'_p - p$

$$d\mu(q) = \operatorname{tr} \Delta^*(f \times 1)^\# q . \tag{5.1}$$

Proposition 2. *Assume E is sheaf exact. If U_p is sufficiently small, then there exist $k_i^{\mathrm{loc}} \in \Gamma(E_{i-1} \boxtimes E'_i, U_p \times U_p - \Delta)$ such that*

(i) *$\mu(k^{\mathrm{loc}})$ is closed in $V_p - p$,*

(ii) $\displaystyle\int_{S_\varepsilon} \mu(k^{\mathrm{loc}}) = \nu(p)$.

Proof. By shrinking U_p sufficiently, and by sheaf exactness of E and Lemma 6 applied to E' we may assume that there exist

$$s_{ij} \in \Gamma(E_i, U_p) , \qquad t_{1,j} \in \Gamma(E'_1, U_p) ,$$

such that

$$Ds_{i-1,j} = h_{ij} , \qquad D' t_{1,j} = * h_{0j} .$$

$\{h_{ij}\}$ denotes as usual an orthonormal base for the space of harmonic sections of E over M, as in (2.4).

We can restrict the operators K_i, H_i on M to operators

$$K_i\colon \Gamma_c(E_i, U_p) \longrightarrow \Gamma(E_{i-1}, U_p) ,$$
$$H_i\colon \Gamma_c(E_i, U_p) \longrightarrow \Gamma(E_i, U_p) .$$

Then (2.5) gives

$$D_{i-1}K_i + K_{i+1}D_i = 1 - H_i \qquad \text{on } \Gamma_c(E_i, U_p) .$$

Define operators $L_i\colon \Gamma_c(E_i, U_p) \to \Gamma(E_{i-1}, U_p)$ by

$$L_1 = \sum_j h_{0,j} \int_U \langle\ ,\, t_{ij}\rangle + \sum_j s_{0,j} \int_U \langle\ ,\, * h_{1,k}\rangle ,$$
$$L_i = \sum_j s_{i-1,j} \int_U \langle\ ,\, * h_{0j}\rangle \qquad \text{for } i > 1 .$$

Since

$$H_i u(x) = \sum_j h_{ij} \int_U \langle u, * h_{ij}\rangle ,$$

it is immediate that on $\Gamma_c(E_i, U_p)$

$$D_{i-1}L_i + L_{i+1}D_i = H_i .$$

Hence, if we define

$$K_i^{\mathrm{loc}} = K_i - L_i ,$$

it follows that on $\Gamma_c(E_i, U_p)$

(5.2) $$D_{i-1}K_i^{\mathrm{loc}} + K_{i+1}^{\mathrm{loc}}D_i = 1 .$$

Note that this identity holds only on *compactly supported* sections.

Now L_i clearly has *smooth kernel* $l_i(x, y)$ on $U_p \times U_p$, hence K_i^{loc} has kernel $k_i^{\mathrm{loc}}(x, y) = k_i(x, y) - l_i(x, y)$ which is smooth for $x \neq y$ and satisfies the estimates of Lemma 1. Standard reasoning as in the proof of Lemma 2 gives

$$D_{i-1}(x)k_i^{\mathrm{loc}}(x, y) + D_i'(y)k_{i+1}^{\mathrm{loc}}(x, y) = 0 \qquad \text{for } x \neq y .$$

Hence taking alternating sums we get, as in Lemma 3,

$$\hat{D}k^{\mathrm{loc}}(x, y) = 0 \qquad \text{for } x \neq y .$$

Therefore, by (5.1),

$$d\mu(k^{\mathrm{loc}}) = 0 \qquad \text{in } V_p - p .$$

This proves (i), and shows that

$$\int_{S_\varepsilon} \mu(k^{\mathrm{loc}})$$

is independent of ε. To see that its value is $\nu(p)$, write

$$\int_{S_\varepsilon} \mu(k^{\mathrm{loc}}) = \int_{S_\varepsilon} \mu(l) + \int_{S_\varepsilon} \mu(k) .$$

The first term on the right hand side converges to zero because $\mu(l)$ is smooth on U_p, the second converges to $\nu(p)$ by definition of $\nu(p)$, and this completes the proof.

Thus the kernel $k^{\mathrm{loc}}(x, y) = \sum (-1)^i k_i^{\mathrm{loc}}(x, y)$ can be thought of as a representative of the "local Thom class" of E. But the formulation of the proposition is still far from satisfactory. The operator K^{loc} is defined in terms of the *global* operator K, whose construction involves the whole manifold M. Both for aesthetic and practical reasons we would like to have a purely *local* description of $\nu(p)$. More precisely, we would like to know that given any locally defined operators

$$K_i^{\mathrm{loc}} \colon \Gamma_c(E_i, U_p) \longrightarrow \Gamma(E_{i-1}, U_p)$$

satisfying

$$D_{i-1}K_i^{\mathrm{loc}} + K_{i+1}^{\mathrm{loc}}D_i = 1 \qquad \text{on } \Gamma_c(E_i, U_p) \tag{5.2}$$

with kernels $k_i^{\mathrm{loc}}(x, y)$, then the cohomology class of $\mu(k^{\mathrm{loc}})$ always gives the local index $\nu(p)$.

Observe that if E consists of just one operator $D_0 \colon \Gamma(E_0) \to \Gamma(E_1)$, then the uniqueness of the local index is trivial. In this case, (5.2) determines the *symbol* of K^{loc} and hence determines K^{loc} up to smoothing operators. But smooth kernels do not affect the local index, the fact which we used at the end of the proof of Proposition 2.

If the complex consists of more than one operator, then (5.2) no longer determines the symbol. K^{loc} is determined only up to operators A satisfying

$$D_{i-1}A_i + A_{i+1}D_i = 0 \qquad \text{on } \Gamma_c(E_i, U_p) . \tag{5.3}$$

We proceed to study the kernels of operators satisfying (5.3).

First we enlarge the class of kernels which we consider. So far we have been working with operators Q with integral representation

$$Qu(x) = \int_y q(x, y)u(y) , \tag{5.4}$$

where $q(x, y)$ is smooth for $x \neq y$ and $q(x, y) \in L^1(y)$ for each x. K_i and K_i^{loc} are of this type because of Lemma 1 (i). The proof was based on the fact that K_i is pseudo-differential of order -1, and that its top order symbol is homogeneous in ξ. For estimate (ii) we needed also the next term in the homogeneous expansion of the symbol. These estimates were essential in the study of simple fixed points because we needed delicate information on the kernel near the diagonal.

But in the cohomological approach of this section this delicate information is irrelevant. Also, even though operators K_i^{loc} satisfying these estimates are the most likely to turn up in practice, (5.3) need not even determine the *order* of K_i^{loc}. For these reasons it is unnatural to consider only this restrictive class of operators. The following condition is just what we need.

Let $Q: \Gamma_c(E_i, U_p) \to \Gamma(E_{i+k}, U_p)$. We say that $q(x, y) \in \Gamma(E_{i+k} \boxtimes E_i', U_p \times U_p - \Delta)$ is the *kernel of* Q if (5.4) holds for x *outside the support of* u. (Strictly speaking we should say that q is the kernel of Q outside the diagonal.) It is clear that if Q has a kernel, it is unique. The following facts are well known [2, § 4].

Lemma 7. *Let $Q: \Gamma_c(E_i, U_p) \to \Gamma(E_{i+k}, U_p)$ be pseudo-differential. Then*

(i) *Q has kernel $q(x, y) \in \Gamma(E_{i+k} \boxtimes E_i', U_p \times U_p - \Delta)$,*

(ii) *DQ has kernel $D(x)q(x, y)$, QD has kernel $D'(y)q(x, y)$.*

Proof. (i) follows by the same argument as in the beginning of the proof of Lemma 1. One integrates by parts by taking enough ξ derivatives of the symbol (depending on the order of Q) until the ξ integrand is covergent. Since $x \notin \operatorname{spt} u$, the y integrand is smooth and there is no problem in interchanging the order of integration.

(ii) follows by the same argument which we used in Lemma 2.

Proposition 3. *Let $A_i: \Gamma_c(E_i, U_p) \to \Gamma(E_{i-1}, U_p)$ be pseudo-differential operators satisfying*

$$D_{i-1}A_i + A_{i+1}D_i = 0 \qquad \text{on } \Gamma_c(E_i, U_p)\ , \tag{5.3}$$

and $a_i(x, y)$ be the kernel of A_i. Then there exist

$$b(x, y) \in \Gamma((E \boxtimes E')_{-2}, U_p \times U_p - \Delta)\ ,$$
$$c(x, y) \in \Gamma((E \boxtimes E')_{-1}, U_p \times U_p)\ ,$$

such that

$$\hat{D}_{-2}b(x, y) + c(x, y) = a(x, y) \quad \text{for } x \neq y\ , \quad \hat{D}_{-1}c(x, y) = 0\ .$$

Proof. We write simply K_i for the operator K_i^{loc} constructed in Proposition 2. Take a locally finite cover $\{U_\alpha\}$ of U_p by relatively compact open sets and a partition of unity $\{\psi_\alpha\}$ subordinate to $\{U_\alpha\}$. Then

$$K_i = \sum_{\alpha,\beta} \psi_\alpha K_i \psi_\beta ,$$

and if we let

$$\bar{K}_i = \sum_{\psi_\alpha\psi_\beta \not\equiv 0} \psi_\alpha K_i \psi_\beta , \qquad \bar{S}_i = \sum_{\psi_\alpha\psi_\beta \equiv 0} \psi_\alpha K_i \psi_\beta ,$$

then $K_i = \bar{K}_i + \bar{S}_i$ where $\bar{S}_i$ is smoothing and $\bar{K}_i \colon \Gamma_c(E_i, U_p) \to \Gamma_c(E_{i-1}, U_p)$. From (5.2) we get

$$D_{i-1}\bar{K}_i + \bar{K}_{i+1}D_i = 1 - S_i ,$$

where $S_i = D_{i-1}\bar{S}_i + \bar{S}_{i+1}D_i$ is smoothing and commutes with D. Moreover, since $\bar{K}_i$ maps compactly supported sections into compactly supported sections and the D_i do not increase supports, it follows that

$$S_i \colon \Gamma_c(E_i, U_p) \longrightarrow \Gamma_c(E_i, U_p) . \tag{5.5}$$

Let $b_i(x, y) \in \Gamma(E_{i-2} \boxtimes E_i', U_p \times U_p - \Delta)$ be the kernel of $A_{i-1}\bar{K}_i$. Then on $\Gamma_c(E_i, U_p)$ we have

$$A_i\bar{K}_{i+1}D_i = A_i - A_iD_{i-1}\bar{K}_i - A_iS_i = A_i + D_{i-2}A_{i-1}\bar{K}_i - A_iS_i .$$

In the last equality we used the fact that $\bar{K}_i u$ has compact support if u does. If $c_i(x, y)$ denotes the kernel of A_iS_i, which is smooth since A_iS_i has order $-\infty$, (ii) of Lemma 7 gives

$$D_{i-2}(x)b_i(x, y) - D_i(y)'b_{i+1}(x, y) = a_i(x, y) - c_i(x, y) .$$

Taking alternating sums we get

$$\hat{D}_{-2}b(x, y) = a(x, y) - c(x, y) .$$

Finally, $D_{i-1}A_iS_i = -A_{i+1}D_iS_i = -A_{i+1}S_{i+1}D_i$, where in the second equality we used (5.5). Therefore

$$D_{i-1}(x)c_i(x, y) + D_i'(y)c_{i+1}(x, y) = 0 ,$$

and taking alternating sums we get $\hat{D}c(x, y) = 0$.

Theorem 3. *Suppose E is sheaf exact. Then for U_p sufficiently small there exist pseudo-differential operators $K_i \colon \Gamma_c(E_i, U_p) \to \Gamma(E_{i-1}, U_p)$ such that*

(i) *$D_{i-1}K_i + K_{i+1}D_i = 1$ on $\Gamma_c(E_i, U_p)$,*

(ii) *if k_i is the kernel of K_i, then $\mu(k)$ is closed and*

$$\int_{S_\varepsilon} \mu(k) = \nu(p) .$$

Moreover, if $\tilde{K}_i$ are any pseudo-differential operators satisfying (i) *with kernel $\tilde{k}_i$, then $\mu(\tilde{k})$ is cohomologous to $\mu(k)$.*

Proof. Only the uniqueness part remains to be proved. We apply Proposition 3 to $A = K - \tilde{K}$. $\mu(c)$ is a *closed* form on all of V_p by (5.1), hence $\int_{S_\varepsilon} \mu(c) = 0$. Now fix S_ε in $V_p - p$ and take $\bar{b}(x, y) \in \Gamma((E \boxtimes E')_{-2}, U_p \times U_p)$ which agrees with $b(x, y)$ outside a sufficiently small neighborhood of Δ so that $\mu(\hat{D}\bar{b}) = \mu(\hat{D}b)$ on S_ε. $\mu(\hat{D}\bar{b})$ is again a closed form, smooth on all of V_p. Since

$$\int_{S_\varepsilon} \mu(\hat{D}\bar{b}) = \int_{S_\varepsilon} \mu(\hat{D}b) ,$$

this shows that the right hand side vanishes. Therefore $\int_{S_\varepsilon} \mu(a) = 0$, and the proof is complete.

Remarks. (1) Observe that the same argument shows that the cohomology class of $\mu(k)$ depends only on the cohomology class of k on $U_p \times U_p - \Delta$. Thus even though the expression for $\nu(p)$ was derived from the fact that k is the kernel of an operator, k can be altered in its cohomology class to a section which is not necessarily the kernel of an operator, and we still get the local index. This is another degree of flexibility gained from "working away from the singularity".

(2) Of course it is natural to ask whether the $\hat{D}$ cohomology class of k itself is determined by (i), even though for the local index only the cohomology class "modulo smooth sections on $U_p \times U_p$" is relevant. From Proposition 3 it is immediate that the class of k is unique if $E \boxtimes E'$ is sheaf exact in dimension -1. This is of course the case both for the de Rham complex and the Dolbeault complex. In the first case $E \boxtimes E'$ is the de Rham complex of $M \times M$ and hence locally exact. In the second case, $E \boxtimes E'$ is a subcomplex of the Dolbeault complex of $M \times M$ given by a certain condition on the holomorphic coordinates (see § 2). One just has to check that the usual maps constructed to show exactness [9], [12] leave this subcomplex invariant. But this is immediate because these maps do not affect the holomorphic coordinates since they involve only the anti-holomorphic part.

6. The classical complexes

6.1. The de Rham complex. Let $k_q(x, y)$ be the double form on $\boldsymbol{R}^n \times \boldsymbol{R}^n - \Delta$, of bidegree $(q - 1, n - q)$, defined by

$$k_q(x, y) = \gamma_n |x - y|^{-n} \sum_I \sum_k (-1)^{k-1}(x_{i_k} - y_{i_k}) dx^{i_1} \cdots \widehat{dx^{i_k}} \cdots dx^{i_q} * dy^I ,$$

where γ_n is the reciprocal of the volume of the unit $(n-1)$-sphere. We use the usual multi-index notation: $I = (i_1, \cdots, i_q)$, $i_1 < \cdots < i_q$. If α is a q-form with compact support in $\boldsymbol{R}^n$, let

$$K_q\alpha(x) = (-1)^{q(n-q)} \int_y k_q(x, y)\alpha(y) \ .$$

The reason for the sign $(-1)^{q(n-q)}$ is that to apply the pairing (2.1) in the second variable α should come before the dy-part of k_q. Then $K_q\colon \Gamma_c(\Lambda^q\boldsymbol{R}^n) \to \Gamma(\Lambda^{q-1}\boldsymbol{R}^n)$ and satisfies

$$dK_q + K_{q+1}d = 1 \qquad \text{on } \Gamma_c(\Lambda^q\boldsymbol{R}^n) \ .$$

It is enough to check this identity for $\alpha = \varphi dx^I$. Thus

$$K_q\alpha(x) = \gamma_n \sum_k (-1)^{k-1} \int \frac{(x_{i_k} - y_{i_k})\varphi(y)dy}{|x-y|^n} dx^{i_1} \cdots \widehat{dx^{i_k}} \cdots dx^{i_q} \ ,$$

$$\begin{aligned}
\gamma_n^{-1} dK_q\alpha &= \sum_{k=1}^q \frac{\partial}{\partial x^{i_k}} \int \frac{(x_{i_k} - y_{i_k})\varphi(y)dy}{|x-y|^n} dx^I \\
&\quad + \sum_{k=1}^q \sum_{j \neq i_k} (-1)^{k-1} \frac{\partial}{\partial x^j} \int \frac{(x_{i_k} - y_{i_k})\varphi(y)dy}{|x-y|^n} dx^j dx^{i_1} \widehat{dx^{i_k}} dx^{i_q} \ , \\
\gamma_n^{-1} K_{q+1} d\alpha &= \sum_{j \notin I} \int \frac{(x_j - y_j)(\partial\varphi/\partial y^j)dy}{|x-y|^n} dx^I \\
&\quad + \sum_{k=1}^q \sum_{j \neq i_k} (-1)^k \int \frac{(x_{i_k} - y_{i_k})(\partial\varphi/\partial y^{i_k})dy}{|x-y|^n} dx^j dx^{i_1} \widehat{dx^{i_k}} dx^{i_q} \ .
\end{aligned}$$

Differentiating the first expression under the integral sign by making the substitution $y = x + z$ as in Lemma 5, we see that the second terms are cancelled and

$$\begin{aligned}
dK_q\alpha + K_{q+1}d\alpha &= \gamma_n \int \frac{\sum_{j=1}^n (x_j - y_j)(\partial\varphi/\partial y^i)dy}{|x-y|^n} dx^I \\
&= \gamma_n \lim_{|y-x| \geq \epsilon} \int d_y\{\varphi(y) \sum (-1)^{k-1}(x_j - y_j) \\
&\qquad \cdot dy^1 \cdots \widehat{dy^j} \cdots dy^n\} dx^I \\
&= \varphi(x) dx^I \ .
\end{aligned}$$

Therefore $k = \sum (-1)^q k_q$ is a local Thom class for the de Rham complex in the sense that it satisfies (i) of Theorem 3. From our general theory we know that $k(x, y)$ is *closed* on $\boldsymbol{R}^n \times \boldsymbol{R}^n - \Delta$. To find its cohomology class, let $i_x\colon \boldsymbol{R}^n - 0 \to \boldsymbol{R}^n \times \boldsymbol{R}^n - \Delta$ be the inclusion of the fibre at x, i.e., $i_x(y) = (x, y)$. Then

$$i_x * k(x, y) = -i_x * k_1(x, y) = W_x ,$$

where W_x is the volume form of the sphere centered at x:

$$W_x = \gamma_n |x - y|^{-n} \sum (-1)^{k-1}(y_k - x_k) dy^1 \cdots \widehat{dy^k} \cdots dy^n .$$

In other words, $k(x, y)$ represents the usual topological Thom class. But so does $\delta * W_0$, where $\delta(x, y) = y - x$ is the difference map. By the uniqueness of the topological Thom clase, $k(x, y)$ must be cohomologous to $\delta * W_0$ (it is in fact *equal*, but we do not need this). From this we get the usual local index $\nu(0)$ for the de Rham complex:

$$\nu(0) = \int_{S^{n-1}} \Delta^*(f \times 1) * k = \int_{S^{n-1}} (1 - f) * W_0 = \deg_0 (1 - f) .$$

6.2. The Dolbeault complex. The formula for the local index for the de Rham complex as a local degree is very special. It is a consequence of the fact that the local cohomology of the complex is one-dimensional. For the Dolbeault complex the situation is more interesting because the local cohomology is *infinite dimensional* (the space of local holomorphic forms).

We work first with the complex $\Lambda^{0,*}$ of forms of type $(0, q)$ on C^n. The local kernel can be guessed easily from our experience with the de Rham complex:

$$k_q(z, \zeta) = C_n |z - \zeta|^{-2n} \sum_I \sum_k (-1)^{k-1} \overline{(z_{i_k} - \zeta_{i_k})} d\bar{z}^{-i_1} \cdots \widehat{d\bar{z}^{-i_k}} \cdots d\bar{z}^{-i_q} * d\bar{\zeta}^I ,$$

where $C_n = (-1)^{\frac{1}{2}n(n-1)}(n - 1)! \, (2\pi i)^{-n}$, and $*$ denotes the duality operator defined by

$$d\bar{\zeta}^I(*d\bar{\zeta}^I) = d\zeta^1 \cdots d\zeta^n d\bar{\zeta}^1 \cdots d\bar{\zeta}^n .$$

k_q is a double form on $C^n \times C^n - \Delta$ of type $(0, q - 1)$ in z and $(n, n - q)$ in ζ. The associated integral operator

$$K_q : \Gamma_c(\Lambda^{0,q} C^n) \longrightarrow \Gamma(\Lambda^{0,q-1} C^n)$$

satisfies

$$\bar{\partial} K_q + K_{q+1} \bar{\partial} = 1 \qquad \text{on } \Gamma_c(\Lambda^{0,q} C^n) .$$

This can be checked in the same way as we did for the de Rham complex.

It follows that $k = \sum (-1)^q k_q$ is $\bar{\partial}$-closed in $C^n \times C^n - \Delta$. Let $i_z : C^n - 0 \to C^n \times C^n - \Delta$ be the inclusion of the fibre over z: $i_z(\zeta) = (z, \zeta)$. Then

$$i_z^* k(z, \zeta) = -i_z^* k_1(z, \zeta) = W_z(\zeta) ,$$

where $W_z(\zeta)$ is the Bochner-Martinelli kernel

$$W_z(\zeta) = C_n|z-\zeta|^{-2n} \sum (-1)^{k-1}(\overline{\zeta_k - z_k}) d\zeta d\bar{\zeta}^1 \cdots \widehat{d\bar{\zeta}^k} \cdots d\bar{\zeta}^n .$$

The basic property of this kernel [5] is that if f is holomorphic on the unit ball in C^n, then

$$f(z) = \int_{S^{2n-1}} W_z(\zeta) f(\zeta) .$$

Note the analogy with the real case. In both cases the restriction of k to a fibre gives the evaluation functional on the local cohomology of the fibre by integration over the boundary.

From the general theory we get the formula for the local index

$$\nu(0) = \int_{S_\varepsilon^{2n-1}} \Delta^*(f \times 1)^* k(z, \zeta) . \tag{6.1}$$

If $n = 1$ the Dolbeault complex consists of just one operator $\bar{\partial}\colon \Gamma(\Lambda^{0,0}) \to \Gamma(\Lambda^{0,1})$, and the local kernel is just

$$k_1(z, \zeta) = \frac{1}{2\pi i} \frac{d\zeta}{z - \zeta} .$$

Therefore

$$\nu(0) = \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{dz}{z - fz} = \operatorname{Res}_0 \left\{\frac{dz}{z - fz}\right\} .$$

In higher dimensions there is an analogous formula in terms of the Cauchy kernel and residues, rather than (6.1) which involves the Bochner-Martinelli kernel and integration over spheres. We discuss briefly this alternative expression.

Recall that $\Lambda^{0,*} \boxtimes (\Lambda^{0,*})'$ is the subcomplex of $\Lambda^{n,*}(C^n \times C^n)$ consisting of forms of degree zero in dz and n in $d\zeta$. We denote this subcomplex by $\Lambda^{0;*}_{0;n}$ and its homology by $H^{0;*}_{0;n}$. $\Lambda^{0;*}_{0;n}$ then gives a resolution of the sheaf $\Omega^{0,n}$ of germs of holomorphic forms on $C^n \times C^n$ of degree 0 in z and n in ζ. We can interpret the Cauchy kernel

$$\left\{\left(\frac{1}{2\pi i}\right)^n \frac{d\zeta^1 \cdots d\zeta^n}{(\zeta^1 - z^1) \cdots (\zeta^n - z^n)}\right\} \tag{6.2}$$

as an element of the cohomology group $H^{n-1}(C^n \times C^n - \Delta, \Omega^{0,n})$. More precisely, if $U \subset C^n$ is a ball about 0 (any Stein neighborhood of 0 would do), and $V \subset U$ is another such neighborhood of 0 such that $fV \subset U$, then we interpret the Cauchy kernel as an element of $H^{n-1}(V \times U - \Delta, \Omega^{0,n})$ as follows: Let $W = \{W_j\}$ be the open cover of $V \times U - \Delta$ given by $W_j = \{z_j - \zeta_j \neq 0\}$.

Then each W_j is a domain of holomorphy (cf. Theorem 2.5.14 of [9]), hence $\bar{\partial}$-acyclic. From standard sheaf theory it follows that the Čech cohomology of this cover is the same as the cohomology of the space $V \times U - \Delta$. Now (6.2) defines an $(n-1)$-cochain of this cover, trivially a cocycle since there are no n-cochains, and hence a cohomology class $c \in H^{n-1}(V \times U - \Delta, \Omega^{0,n})$.

The basic fact that we need is the following proposition. It can be proved by the same argument as Theorem 2.2 of [7].

Proposition. *Let* $D\colon H^{n-1}(V \times U - \Delta, \Omega^{0,n}) \xrightarrow{\approx} H^{n,n-1}_{0,n}(V \times U - \Delta)$ *be the Dolbeault isomorphism. Then* $Dc = \{k(z, \zeta)\}$.

Granting this, the naturality of the Dolbeault isomorphism gives a commutative diagram

$$\begin{array}{ccc} H^{n-1}(V - 0, \Omega^n) & \xleftarrow{\Delta^*(f \times 1)^*} & H^{n-1}(V \times U - \Delta, \Omega^{0,n}) \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle D} \\ H^{n,n-1}(V - 0) & \xleftarrow{\Delta^*(f \times 1)^*} & H^{n,n-1}_{0,n}(V \times U - \Delta) . \end{array}$$

Therefore

$$\nu(0) = \int_{|z|=\varepsilon} D\left\{\frac{1}{(2\pi i)^n} \frac{dz^1 \cdots dz^n}{(z^1 - f^1) \cdots (z^n - f^n)}\right\} .$$

The expression in brackets is interpreted as a cochain of the cover $\Delta^{-1}(f \times 1)^{-1}W$ of $V - 0$, hence a cohomology class since the sets $\Delta^{-1}(f \times 1)^{-1}W_j$ are again domains of holomorphy.

But the composition

$$H^{n-1}(V - 0, \Omega^n) \xrightarrow{D} H^{n,n-1}(V - 0) \xrightarrow{\int} \mathbf{C}$$

is the *Grothendieck residue* (cf. [4]). It has a computational algorithm similar to that of the residue calculus in one variable. Thus we can state the final form of the residue formula for the local index as

$$\text{(6.3)} \qquad \nu(0) = \operatorname{Res}_0 \left\{\frac{dz^1 \cdots dz^n}{(z^1 - f^1) \cdots (z^n - f^n)}\right\} .$$

In particular, the properties of Res_0 imply that $\nu(0)$ depends only on a finite (but large!) number of derivatives of f at 0.

More generally we could consider a holomorphic vector bundle E over the compact complex analytic manifold M, and a holomorphic bundle map $\varphi\colon f^*E \to E$. There $\Lambda^{0,q}(df) \otimes \varphi$ gives a lift of f to the complex $\Lambda^{0,*}(E)$ of forms of type $(0, q)$ with values in E. Taking tensor products we get the analogous formula for the local index in this situation:

$$\nu(0) = \operatorname{Res}_0 \left\{ \frac{\operatorname{tr}_C \varphi(z) dz^1 \cdots dz^n}{(z^1 - f^1) \cdots (z^n - f^n)} \right\} . \tag{6.4}$$

Here we take the trace of the matrix of $\varphi(z)$ even though strictly speaking φ is not an endomorphism for $z \neq 0$. Note that if 0 is a simple fixed point, from the properties of Res_0, (6.4) reduces to

$$\frac{\operatorname{tr}_C \varphi(0)}{\det_C (1 - df_0)} ,$$

which is the formula of Atiyah-Bott [3].

If $E = \Lambda^{p,0}(M)$, the bundle of holomorphic p-forms, then $\Lambda^{0,*}(E)$ is just the Dolbeault complex $\Lambda^{p,*}(M)$ of forms of type $(p, *)$. All holomorphic maps lift naturally to this complex, and if we denote the local index in this case by $\nu_p(0)$, then

$$\nu_p(0) = \operatorname{Res}_0 \left\{ \frac{\operatorname{tr}_C \Lambda^{p,0}(df) dz^1 \cdots dz^n}{(z^1 - f^1) \cdots (z^n - f^n)} \right\} .$$

This can be written in the following alternative form, where $I = (i_1, \cdots, i_p)$, $J = (j_1, \cdots, j_{n-p})$ is the multi-index complementary to I and $\varepsilon_I = \delta^{1 \cdots n}_{i_1 \cdots i_p j_1 \cdots j_{n-p}}$:

$$\nu_p(0) = \operatorname{Res}_0 \left\{ \frac{\sum \varepsilon_I df^I dz^J}{(z^1 - f^1) \cdots (z^n - f^n)} \right\} . \tag{6.5}$$

Observe that

$$\sum (-1)^p \nu_p(0) = \operatorname{Res}_0 \left\{ \frac{d(z^1 - f^1) \cdots d(z^n - f^n)}{(z^1 - f^1) \cdots (z^n - f^n)} \right\} = \deg_0 (1 - f) , \tag{6.6}$$

where the last equality follows from the properties of Res_0. Thus even though each $\nu_p(0)$ is not expressible as a local degree, their alternating sum is. Therefore $\sum (-1)^p \nu_p(0)$ is a topological, rather than holomorphic, invariant of f at 0. This is not surprising because for the Lefschetz numbers it is easy to see that

$$L(f, \text{de Rham}) = \sum (-1)^p L(f, \Lambda^{p,*}) .$$

(6.6) just says that this relation also holds for the local indices.

There is a formula for the local coincidence index $\nu_{f,g}(0)$ (cf. § 3) with respect for $\Lambda^{p,*}$ similar to (6.5):

$$\nu_{f,g}(0) = \operatorname{Res}_0 \left\{ \frac{\sum \varepsilon_I df^I dg^J}{(g^1 - f^1) \cdots (g^n - f^n)} \right\} .$$

In this context it is also natural to consider *meromorphic maps* (or correspondences). By this we mean an irreducible n-dimensional analytic subvariety $G \subset M \times M$ such that there exists a subvariety $V \subset M$ and a holomorphic map $f: M - V \to M$ with

$$G \cap M \times (M - V) = \{(fx, x) : x \in M - V\} .$$

Then we can define the Lefschetz number of G to be

$$L(G, \Lambda^{p,*}) = \int_G h(z, \zeta) ,$$

where $h(z, \zeta) = \sum (-1)^q h_q(z, \zeta)$ is the harmonic kernel of $\Lambda^{p,*}$ considered simply as an (n, n)-form on $M \times M$. Note that if G is a manifold, then its Lefschetz number is just the coincidence number of the projections on each factor:

$$L(G, \Lambda^{p,*}) = L(\pi_1, \pi_2 ; \Lambda^{p,*}) .$$

We can still talk about fixed points of G, the intersections of G with Δ. If G is a map near the diagonal, and its fixed points are isolated, then $L(G, \Lambda^{p,*})$ is a sum of local indices given by (6.5).

We conclude with the following elementary corollary of these observations, which does not require to know the nature of the fixed-point coincidence formula. Suppose $H^{0,q}(M) = 0$ for $q > 0$.

(a) If $\dim \overline{M} = \dim M$, $g: \overline{M} \to M$ is holomorphic, and $\deg g \neq 0$, then any holomorphic map $f: \overline{M} \to M$ has a coincidence with g.

(b) Any meromorphic map G of M has a fixed point.

For the proof, observe that the harmonic kernel $h(z, \zeta)$ of M is just $\pi_2^* \omega$ where $\omega \in \Gamma(\Lambda^{n,n} M)$ is a volume element. It follows easily that $L(f, g ; \Lambda^{0,*}) = \deg g$ and $L(G ; \Lambda^{0,*}) = 1$.

A Kähler manifold M of positive sectional curvature satisfies $H^{0,q}(M) = 0$ for $q > 0$. In this connection see Theorem 4 of [6].

Added in proof. Since this paper was written, another proof of the holomorphic formula and a detailed account of the Grothendieck residue have been given respectively in: L. Sibner & R. Sibner, *A note on the Atiyah-Bott fixed point formula*, to appear in Pacific J. Math.; Y. L. L. Tong, *Integral representation formulae and Grothendieck residue symbol*, to appear in Amer. J. Math.

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