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REMARKS ON CONFORMAL TRANSFORMATIONS

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1. Introduction

Let (M_1, g_1) and (M_2, g_2) be two connected riemannian manifolds. A diffeomorphism from M_1 to M_2 will be said to be conformal if the pull back of the metric g_2 is proportional to g_1. For the two-dimensional case, these diffeomorphisms are just holomorphic transformations between the underlying complex structures of M_1 and M_2. For dimension greater than two, under the condition that $(M_1, g_1) = (M_2, g_2)$ various authors have been trying to find conditions for a one-parameter group of conformal transformations to be actually a one-parameter group of isometries. It seems that the basic philosophy for such a phenomena is similar to that of Schur's theorem, which states that if the sectional curvature of a connected riemannian manifold of dimension greater than two is constant at every point, then the manifold has constant curvature; throughout this paper by curvature alone we always mean sectional curvature. To verify this principle, we modify the Schur's type argument to generalize and simplify some known theorems. Some new phenomena are also obtained.

In § 1, using a result of H. Omori we prove that if M is complete with the sectional curvature bounded from below and the scalar curvature bounded above by a negative constant, then every conformal transformation on M preserving the scalar curvature is an isometry. This result was obtained by M. Obata [12] in case M is compact.

In § 2, we prove that if M is einsteinian and dim $M \geq 3$, then either M has constant curvature or every conformal transformation is a homothety. This is true even for pseudoriemannian manifolds. Kulkarni [8] proved this fact under some additional assumption, namely, at a generic point the curvature function (of the grassmannian of two planes) has only nondegenerate critical points. For the four-dimensional case, he assumed the manifold to be nowhere constantly curved. If M is complete, the general result was obtained by Yano and Nagano [16], and Nagano [11]. A special case of [11] was reproved in [8].

In § 3, we study the totally geodesic submanifolds of a conformally flat manifold. Using these results we are able to prove that a nontrivial riemannian product cannot be conformally flat unless both factors have constant curvature.

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We believe that this is a new phenomena. Using this and the results of Cheeger and Gromoll [2], we are able to classify complete conformally flat manifolds with nonnegative curvature and compact conformally flat manifolds with nonnegative ricci curvature. We remark that it has only been proved before by Bochner that the latter manifolds are homology spheres. At last, we classify all complete locally homogeneous manifolds each of which admits a nonhomothety conformal transformation. This result was obtained by Goldberg and Kobayashi [5] and Barbance [1] if M is compact and has dimension ≥ 4.

Finally, we would like to thank W. C. Hsiang and Kulkarni for discussions.

2. Local formulas

Let (M_1, g_1), (M_2, g_2) be two connected riemannian manifolds with metrics g_1 and g_2 respectively, and F be a mapping of M_1 into M_2. We say F is conformal if there is a function ρ on M_1 such that $F^*g_2 = e^{2\rho}g_1$. Except in Theorem 1 below, we shall assume $\rho > -\infty$.

Let $\omega_1, \cdots, \omega_n$ be a local coframe field on M_1. Then the structure equations are

$$(2.1) \qquad d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0 ,$$

$$(2.2) \qquad \begin{aligned} d\omega_{ij} &= -\sum_k \omega_{ik} \wedge \omega_{kj} + \Omega_{ij} , \\ \Omega_{ij} &= \frac{1}{2} \sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l , \end{aligned}$$

where $1 \leq i, j \leq n$.

Let $\omega_i^* = e^\rho \omega_i$, $1 \leq i \leq n$. From now on, if Φ is an object on (M_1, g_1), then Φ^* will be the corresponding object on M_1 with the metric $e^{2\rho}g_1$. Thus

$$d\omega_i^* = -e^\rho \sum_j \omega_{ij} \wedge \omega_j + e^\rho \sum_i \rho_j\omega_j \wedge \omega_i = -\sum_j \omega_{ij}^* \wedge \omega_j^* ,$$

where

$$(2.3) \qquad \omega_{ij}^* = \omega_{ij} + \rho_j\omega_i - \rho_i\omega_j .$$

Furthermore, by using (2.1), (2.2), (2.3) and directly computing $d\omega_{ij}^* + \sum_k \omega_{ik}^* \wedge \omega_{kj}^*$ we can easily obtain

$$(2.4) \qquad \begin{aligned} \Omega_{ij}^* = \Omega_{ij} &- \sum_k (\rho_{jk} - \rho_k\rho_j)\omega_i \wedge \omega_k \\ &- \sum_k (\rho_{ik} - \rho_i\rho_k)\omega_k \wedge \omega_j - \sum_k \rho_k^2\omega_i \wedge \omega_j , \end{aligned}$$

where (ρ_{jk}) is the hessian of ρ and is defined by

$$(2.5) \qquad \sum_k \rho_{jk}\omega_k = d\rho_j - \sum_k \rho_k\omega_{kj} .$$

On the other hand, by letting $u = e^{-\rho}$, we find

$$u_i = -e^{-\rho}\rho_i \,, \tag{2.6}$$

$$u_{ij} = -e^{-\rho}(\rho_{ij} - \rho_i\rho_j) \,. \tag{2.7}$$

We shall call u the *associated function* of the conformal transformation F. From (2.4), (2.6) and (2.7) we have

$$\Omega^*_{ij} = \Omega_{ij} + e^{\rho} \sum_k u_{jk}\omega_i \wedge \omega_k + e^{\rho} \sum_k u_{ik}\omega_k \wedge \omega_j - e^{2\rho} \sum_k u_k^2\omega_i \wedge \omega_j \,. \tag{2.8}$$

Thus (2.4) and (2.8) give

$$e^{2\rho}R^*_{ijij} = R_{ijij} + \rho_j^2 - \rho_{jj} + \rho_i^2 - \rho_{ii} - \sum_k \rho_k^2 \,, \tag{2.9}$$

$$R^*_{ijij} = u^2R_{ijij} + uu_{ii} + uu_{jj} - \sum_k u_k^2 \,, \tag{2.10}$$

whenever $i \neq j$. Also, when $\{i, j, k\}$ are distinct, we have

$$e^{2\rho}R^*_{ijik} = R_{ijik} - \rho_{jk} + \rho_k\rho_j \,, \tag{2.11}$$

$$R^*_{ijik} = u^2R_{ijik} + uu_{jk} \,. \tag{2.12}$$

Let the ricci curvature in direction j be denoted by $\mathrm{Ric}\,(j)$, and the scalar curvature by R. Then

$$\mathrm{Ric}\,(j) = \sum_{i \neq j} R_{ijij}, \qquad R = \sum_j \mathrm{Ric}\,(j) \,,$$

and we can easily obtain

$$e^{2\rho}\,\mathrm{Ric}^*\,(j) = \mathrm{Ric}\,(j) - \Delta\rho - (n-2)\sum_k \rho_k^2 - (n-2)(\rho_{jj} - \rho_j^2) \,, \tag{2.13}$$

$$\mathrm{Ric}^*\,(j) = u^2\,\mathrm{Ric}\,(j) + u\Delta u + (n-2)uu_{jj} - (n-1)\sum_k u_k^2 \,, \tag{2.14}$$

$$e^{2\rho}R^* = R - 2(n-1)\Delta\rho - n(n-2)\sum_k \rho_k^2 \,, \tag{2.15}$$

$$R^* = u^2R + 2(n-1)u\Delta u - n(n-1)\sum_k u_k^2 \,, \tag{2.16}$$

where $\Delta u = \sum_i u_{ii}$ and $\Delta\rho = \sum_i \rho_{ii}$ are the laplacians of the functions u and ρ respectively. When $n \geq 3$ and $j \neq k$, we have

$$\begin{aligned} e^{2\rho}\,\mathrm{Ric}^*\,(j,k) &= e^{2\rho} \sum_{i \neq j,\, k} R^*_{ijik} \\ &= \mathrm{Ric}\,(j,k) - (n-2)\rho_{jk} + (n-2)\rho_j\rho_k \,, \end{aligned} \tag{2.17}$$

$$\mathrm{Ric}^*\,(j,k) = u^2\,\mathrm{Ric}\,(j,k) + (n-2)uu_{jk} \,. \tag{2.18}$$

Recall that the Weyl conformal tensor C_{ijkl} is defined for $\dim M_1 \geq 4$ as follows:

$$C_{ijkl} = R_{ijkl} - \frac{1}{n-2}\{\mathrm{Ric}\,(i,k)\delta_{jl} - \mathrm{Ric}\,(j,k)\delta_{il} + \delta_{ik}\,\mathrm{Ric}\,(j,l)$$
$$- \delta_{jk}\,\mathrm{Ric}\,(i,l)\} + \frac{R}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}) \,.$$

It is well known that if the conformal tensor vanishes, then the manifold is locally conformally euclidean or, in other words, conformally flat.

Now formula (2.8) shows

$$e^{2\rho}C^*_{ijkl} = C_{ijkl} \,, \tag{2.19}$$

$$C^*_{ijkl} = u^2 C_{ijkl} \tag{2.20}$$

for all $\{i,j,k,l\}$. Note that in (2.19) and (2.20) the left hand side is evaluated at $F(x)$ whereas the right hand side is evaluated at x.

In the 3-dimensional case, the Weyl conformal tensor should be replaced by

$$C_{ijk} = \frac{1}{n-2}(R_{ijk} - R_{ikj}) - \frac{1}{2(n-1)(n-2)}(\delta_{ij}R_k - \delta_{ik}R_j) \,,$$

where

$$\sum_k R_{ijk}\omega_k = dR_{ij} - R_{kj}\omega_{ki} - R_{ik}\omega_{kj}$$

is the covariant derivative of the Ricci tensor. In this case the vanishing of such a conformal tensor is a necessary and sufficient condition for the manifold to be conformally flat. Moreover, we can prove

$$C^*_{ijk} = u^3 C_{ijk} \,. \tag{2.21}$$

3. Manifolds with nonpositive scalar curvature

In this section, we shall assume that the manifold M_1 is complete and has curvature bounded from below (not on M_2). The last assumption is made for the purpose of using a theorem of H. Omori. It seems reasonable that the theorems still hold without this assumption.

First of all, let us introduce the terminology of the following two functions which the author knows from Kulkarni [8]:

(a) K is a function defined on M_1 which associates to every point the infimum of all the sectional curvatures at that point.

(b) Ric is a function defined on M_1 which associates to every point the infimum of the ricci curvature at that point.

Now we can have our theorem:

Theorem 1. *Let $F\colon M_1 \to M_2$ be a conformal mapping such that $F^*g_2 = e^{2\rho}g_1$, and suppose ρ is not identically $-\infty$.*

(i) *Then* $\inf R \geq 0$ *implies* $\sup R^* \geq 0$.

(ii) *If* $\inf R < 0$, *and* $\sup R^* < 0$, *then*

$$F^*g_2 \leq \frac{\inf R}{\sup R^*} g_1 \,. \tag{3.1}$$

(iii) *If $R = R^*$, and R is bounded from above by a negative constant, then*

$$F^*g_2 \leq g_1 \,. \tag{3.2}$$

Furthermore, the above assertions remain true if we replace R by K or Ric.

Proof. Let $u = e^{-\rho}$. Then u is a positive function satisfying (2.16). Let ε be an arbitrary positive number. Then a slight modification of Theorem A' of [14] gives a point q in M_1 such that

$$\sum_i u_i^2 < \varepsilon \,, \quad \Delta u > -\varepsilon \,; \quad u^2(q) < \inf u^2 + \varepsilon \,, \quad u(q) < \inf u + \varepsilon \,.$$

Formula (2.16) then implies

$$\begin{aligned} R^*(q) &\geq u^2(q)R(q) - 2(n-1)u(q)\varepsilon - n(n-1)\varepsilon \\ &\geq u^2(q) \inf R - 2(n-1)(\inf u)\varepsilon - (2+n)(n-1)\varepsilon \\ &\geq (\inf u^2) \inf R - \varepsilon \inf R - 2(n-1)(\inf u)\varepsilon - n(n-1)\varepsilon \,. \end{aligned} \tag{3.3}$$

Assertion (i) then follows by letting $\varepsilon \to 0$ in (3.3). Assertion (ii) can be proved similarly.

Assertion (iii) is equivalent to $\inf u^2 \geq 1$. If this is not true, then $\inf u^2 < 1$, and therefore we can find points $\{q_i\}$ such that

$$R(q_i)(1 - u^2(q_i)) \geq -(n-1)(2u(q_i)\varepsilon + n)\varepsilon, \qquad \inf u(q_i) < 1 \,.$$

This is impossible if R is bounded above by a negative constant. The final remarks are proved by using (2.10) and (2.14) instead of (2.16).

Corollary 1.1. *Suppose $R > 0$. Then there exists no nondegenerate conformal mapping of M_1 into a manifold with the scalar curvature bounded from above by a negative constant.*

Note that Corollary 1.1 implies that there is no nondegenerate conformal mapping of euclidean space or euclidean space minus a point into the ball because the first two spaces admit conformal metric which has nonnegative curvature. For the two-dimensional case, the last assertion is the Liouville theorem.

Corollary 1.2. *Let M_1, M_2 be two complete riemannian manifolds with the* *sectional curvature bounded from below and the scalar curvature bounded from above by a negative constant. Then any conformal diffeomorphism between the two manifolds, which preserves the scalar curvature, must be an isometry. The same conclusion holds if we replace the scalar curvature by K or* Ric.

In case $M_1 = M_2 =$ a compact manifold and the scalar curvature is a negative constant, Corollary 1.2 was obtained by Lichnerowicz [10] for an infinitesimal conformal transformation. Later Obata [12] generalized it to a conformal diffeomorphism. Kulkarni [8] also obtained the above corollary by assuming the manifold to be compact and those objects to be constant. All of these ideas are similar.

If the manifold M_1 is compact, we can improve (i) as follows:

Theorem 2. *Let M_1 be a compact manifold of* dimension ≥ 3 *and with nonpositive total scalar curvature, and M_2 be another manifold with nonnegative scalar curvature. If F is a conformal diffeomorphism between M_1 and M_2, then F must be a homothety. Furthermore, both M_1 and M_2 should have zero scalar curvature.*

Proof. From formula (2.15), we have

$$\begin{aligned} R &= 2(n-1)\Delta\rho + n(n-2)\sum_k \rho_k^2 + e^{2\rho}R^* \\ &\geq 2(n-1)\Delta\rho + n(n-2)\sum_k \rho_k^2 \end{aligned} \tag{3.4}$$

since M_1 is compact. Integrating (3.4) over M_1 gives

$$0 \geq \int_{M_1} R \geq n(n-2)\int_{M_1} \sum_k \rho_k^2 \,. \tag{3.5}$$

Since $n \geq 3$, from (3.5) it is seen that ρ is constant and F is a homothety. The last assertion follows from the fact that now (3.4) and (3.5) become equalities. q.e.d.

In view of Theorems 1 and 2, we have

Corollary 2.1. *Let g_1 and g_2 be two conformally equivalent metrics on a compact manifold. If both metrics have constant scalar curvatures, then the curvatures have the same sign.*

Theorem 2 was obtained by Obata [12] under the assumption that M_1 has nonpositive scalar curvature, and by Lichnerowicz [10] and Yano [15] under the assumption that the scalar curvatures of both M_1 and M_2 are constant.

4. Einstein manifolds

In this section, we shall prove a local theorem, i.e., no completeness or compactness will be assumed in case M_1 and M_2 are einsteinian. Precisely, we have

Theorem 3. *Let M_1, M_2 be two einstein manifolds of* dimension ≥ 3, *and* $F: M_1 \to M_2$ *be a conformal diffeomorphism. Then either F is a homothety or both M_1 and M_2 have constant curvature.*

Proof. From (2.14) we have

$$u_{jj} = \mathrm{Ric}^*(j) - u^2 \mathrm{Ric}(j) - u\Delta u + (n-1) \sum_k u_k^2 .$$

As both M_1 and M_2 are einsteinian, $\mathrm{Ric}^*(j)$ and $\mathrm{Ric}(j)$ are constants. Thus u_{jj} is independent of j, so that formula (2.10) gives

$$R^*_{ijij} - u^2 R_{ijij} = 2u\Delta u/n - \sum_k u_k^2 . \tag{4.1}$$

On the other hand, formula (2.18) implies that $u_{jk} = 0$ whenever $j \neq k$ since both R^*_{jk} and R_{jk} are zero whenever $j \neq k$. Thus from (2.12) and (2.8) we have $R^*_{ijij} = u^2 R_{ijik}$ and

$$\Omega^*_{ij} - \Omega_{ij} = h\omega_i \wedge \omega_j \tag{4.2}$$

for some function h.

We are going to exteriorly differentiate (4.2). First of all, exterior differentiation of (2.2) gives

$$d\Omega_{ij} = \Omega_{ik} \wedge \omega_{kj} - \omega_{ik} \wedge \Omega_{kj} , \tag{4.3}$$

Similarly,

$$d\Omega^*_{ij} = \Omega^*_{ik} \wedge \omega^*_{kj} - \omega^*_{ik} \wedge \Omega^*_{kj} . \tag{4.4}$$

Thus from (4.2) we have

$$\begin{aligned} &-\Omega_{ik} \wedge \omega_{kj} + \omega_{ik} \wedge \Omega_{kj} + \Omega^*_{ik} \wedge \omega^*_{kj} - \omega^*_{ik} \wedge \Omega^*_{kj} \\ &\quad = dh \wedge \omega_i \wedge \omega_j - h\omega_{ik} \wedge \omega_k \wedge \omega_j + h\omega_i + \omega_{jk} \wedge \omega_k , \end{aligned}$$

Using (2.3), (4.2) and simplifying, we obtain

$$\begin{aligned} &\sum_k \rho_j \Omega^*_{ik} \wedge \omega_k - \sum_k \rho_k \Omega^*_{ik} \wedge \omega_j - \sum_k \rho_k \omega_i \wedge \Omega^*_{kj} \\ &\quad + \sum_k \rho_i \omega_k \wedge \Omega^*_{jk} = \sum_k h_k \omega_k \wedge \omega_i \wedge \omega_j . \end{aligned} \tag{4.5}$$

Equating the coefficients of $\omega_k \wedge \omega_i \wedge \omega_j$ on both sides of (4.5) thus gives

$$\rho_k R^*_{ikik} + \rho_k R^*_{kjkj} = h_k \tag{4.6}$$

for distinct $\{i, j, k\}$. Note that by taking the exterior derivative for the formula corresponding to (2.1) we have the Bianchi identity $\sum \Omega^*_{ik} \wedge \omega^*_k = 0$.

Now in (4.6), summing on i first and then on j we obtain

$$(4.7)\quad (n-1)\rho_k \operatorname{Ric}^*(k) + (n-3)\rho_k \operatorname{Ric}^*(k) = (n-1)(n-2)h_k\ ,$$

which, together with (4.7), implies

$$(4.8)\qquad \rho_k(R^*_{ikik} + R^*_{jkjk} - 2\operatorname{Ric}^*(k)/(n-1)) = 0\ ,$$

whenever $\{i, j, k\}$ are distinct.

Now write $M_1 = M_1' \cup M_1''$ such that $\sum_k \rho_k^2 = 0$ on M_1' and $\sum_k \rho_k^2 \neq 0$ on M_1''. Let us restrict (4.8) to the interior of M_1'', and fix a point p in M_1'', so that the vector grad ρ is not zero at this point. Choose a frame $e_1, \cdots, e_n$ such that at p,

$$(4.9)\qquad \rho_i = \langle \operatorname{grad} \rho, e_i \rangle \neq 0.$$

Since M_2 is einsteinian, $\operatorname{Ric}^*(i)$ is independent of i. Thus (4.8), (4.9) imply that $R^*_{ikik} + R^*_{jkjk}$ is independent of $\{i, j, k\}$ whenever they are distinct. From this, one easily finds that R^*_{ijij} is independent of i, j whenever $i \neq j$. Hence from (4.9) again, $R^*_{ijij} = \operatorname{Ric}^*(i)/(n-1) = \text{constant} = c^*$ (say). We have therefore proved that the sectional curvature associated with the plane spanned by e_i, e_j is a constant c^* whenever e_i, e_j can be completed into a frame satisfying (4.9). Hence by continuity, the sectional curvature at p is actually a constant equal to c^*.

Now (4.2) gives

$$(4.10)\qquad \Omega_{ij} = (c^* e^{2\rho} - h)\omega_i \wedge \omega_j\ .$$

Schur's theorem or an argument similar to the above then shows that $c^* e^{2\rho} - h = c$ is a constant. This constant is the sectional curvature of M_1'' which in turn equals $\operatorname{Ric}(i)/(n-1)$.

Finally, we claim that M_1' is equal to M_1 or M_1' consists of isolated points of M_1. This fact will, of course, conclude the proof of the theorem.

We recall that we have

$$(4.11)\qquad u_{ij} = 0 \qquad \text{if } i \neq j\ ,$$

$$(4.12)\qquad 2uu_{ii} = \sum_k u_k^2 + c^* - u^2 c\ .$$

Let p be a boundary point of M_1' if it is not empty, and g be a geodesic through p parametrized by the arc length s. Choose a frame $\{e_1, \cdots, e_n\}$ such that e_1 is tangent to g and the frame is parallel along g. (4.11) and (4.12) then give

$$(4.13)\qquad 2uu_{11} = u_1^2 + c^* - u^2 c$$

along g with $u_1 = 0$ at p. Since $u > 0$ along g, (4.13) is a regular ordinary differential equation with analytic coefficients. Hence the unique solution of

(4.13) is analytic on g. In particular, u_1 is analytic on g. This implies that on g, u_1 either is identially zero or has only isolated zeroes. Let us write (4.13) in the following form:

$$(4.14) \qquad (2\sqrt{u})_{11} = \tfrac{1}{2}(c^* - u^2c)u^{-3/2}.$$

If $c^* - u^2c \neq 0$ at p, then $(\sqrt{u})_1$ and hence u_1 cannot be zero in a small geodesic ball around the point p. The radius of such a ball can be estimated from the value $c^* - u^2(p)c$. (Note that in a general riemannian manifold, one can always find a geodesically convex neighborhood around a point.) Hence p is isolated.

Now suppose $c^* - u^2(p)c = 0$. Let q be any point on g such that $u_1(q) = 0$. Since $u_1(p) = 0$, one can find a point q_1 such that $p < q_1 < q$ and $(\sqrt{u})_{11}(q_1) = 0$. (4.14) then implies that $c^* - u^2(q_1)c = 0$. If $c = 0$, then $c^* = 0$, and (4.14) implies $u_1 \equiv 0$ on g. So assume $c \neq 0$. From this and $c^* - u^2(p) = 0$, one sees $u(p) = u(q)$. This again implies that there is a point q_2 such that $p < q_2 < q_1$ and $u_1(q_2) = 0$. Continuing this process, we find a sequence of distinct points $\{q_i\}$ converging to p and such that $u_1(q_i) = 0$. By the real analyticity of $u_1, u_1 \equiv 0$ on g. Let N be a geodesically convex neighborhood around p. We have therefore proved $M_1' \cap N$ is geodesically convex. By a theorem of Cheeger and Gromoll, we know that M_1' is actually a convex manifold with boundary. By taking a geodesic transversal to the boundary and applying the above argument, one sees that M_1' must be a point. This finishes the proof of Theorem 3.

When $M_1 = M_2$ and dimension $M_1 = 4$, R. Kulkarni [8] proved the following statement: If $M_1 = M_2$ is einsteinian and nowhere constantly curved, then every conformal diffeomorphism of M_1 is a homothety. This statement is, of course, a special case of Theorem 3. We note that the proof of Theorem 3 works for pseudoriemannian manifolds.

Corollary 4.1. *Let M_1, M_2 be einstein manifolds of* dimension ≥ 3. *If M_1 is complete, then either every conformal diffeomorphism of M_1 onto M_2 is a homothety or M_1 and M_2 are isometric to spheres.*

Proof. If the conformal diffeomorphism is not a homothety, Theorem 3 implies that both M_1 and M_2 have constant curvature. Theorem 1 and the discussions below will show that both M_1 and M_2 are spheres.

If $M_1 = M_2$, this corollary was first obtained by Nagano and Yano [16] for a one-parameter group of conformal diffeomorphisms. Nagano [11] then generalized it to the case where $M_1 = M_2 =$ a complete manifold with parallel ricci tensor and the one-parameter group replaced by a conformal diffeomorphism. The special case where M_1 and M_2 are coincident complete einstein manifolds was reproved by R. S. Kulkarni [8].

From the proof of Theorem 3, we have the following

Proposition 1. *Let M_1, M_2 be two constantly curved manifolds, and F be a* *conformal diffeomorphism of M_1 onto M_2. Then the gradient of the associated function u of F can have only isolated zeroes in M_1.*

Let us now examine more closely the case where M_1 and M_2 are simply connected open domains in euclidean space. In this case, we fix a global coordinate system $\{x^1, \cdots, x^n\}$ in the euclidean space, and write $\omega^i = dx^i$.

(4.11) and (4.12) imply that u_i is independent of x^j when $j \neq i$ and that $u_{ii} = u_{jj}$ for all i, j. Hence actually $u_{ii} = u_{jj} = a$, where a is a constant. Solving these equations, we know

$$u = a \sum_i x^{i2} + \sum_i b_i x^i + d \tag{4.15}$$

for some constants $\{b_i\}$ and d. (4.12) then gives

$$4ua = \sum_i (2ax^i + b_i)^2 ,$$

which is reduced to, in consequence of (4.15),

$$4ad = \sum_i b_i^2 . \tag{4.16}$$

Now there are two cases. If $a = 0$, (4.16) implies $b_i = 0$ for all i and hence $u = d =$ constant. In this case, the transformation is obviously a composite of a euclidean motion and a standard homothety with constant d. The second case is $a \neq 0$. Then (4.15) and (4.16) together imply

$$u = a \sum_i (x^i - \tfrac{1}{2}b_i/a)^2 ,$$

so that by choosing coordinate suitably we may assume

$$u = \mathrm{a} \sum_i x^{i2} .$$

In this case, it is easy to see that the transformation F is given by a euclidean motion, a homothety and an inversion. One may prove this fact in the following way: composing F with a homothety and an inversion, one can get an isometry which must be the euclidean motion. In this second case, F cannot be defined at some point. Hence the only conformal transformation which is globally defined in the euclidean space is a homothety. The other cases of Corollary 3.1 follow either from Theorem 1 or the proof of Theorem 3.

5. Locally product manifolds, nonnegative curved manifolds and locally homogeneous manifolds

In this section, we shall see that if a conformally flat manifold is a product, then actually it is a product of two manifolds with constant curvature. We first prove the following

Proposition 2. *Let M be a conformally flat manifold, and N a connected* *totally geodesic submanifold of M. Suppose at each point of N we can find a normal e_n such that the sectional curvature associated with the plane spanned by e_n and any vector in the tangent plane of N is zero. Then N has constant curvature.*

Proof. First of all, suppose $\dim N \geq 3$. Then from (2.10), we have

$$R_{jkjk} - R_{ijij} + R_{ikik} = 2uu_{jj} - \sum_k u_k^2 , \tag{5.1}$$

whenever $\{i, j, k\}$ are distinct. Taking j to be the normal direction e_n mentioned in the proposition, one sees from (5.1) that the sectional curvature of N is constant at every point. Schur's theorem then implies that N has constant curvature.

For the two-dimensional case, we note that $ds^2 = \omega_1^2 + \omega_2^2$ is a flat metric and that $u^{-2}ds^2 = (u^{-1}\omega_1)^2 + (u^{-1}\omega_2)^2$ is the metric of the manifold. Formula (2.14) then shows $u_{11} = u_{22}$ and

$$u(u_{11} + u_{22}) - u_1^2 - u_2^2 - u_3^2 = u(u_{11} + u_{22}) - u_1^2 - u_2^2 \tag{5.2}$$

is the Gauss curvature of N. Hence $u_3 = 0$. On the other hand (2.18) gives $u_{12} = u_{21} = 0$. The equation $u_{11} = u_{22}$ then implies $u_{11} = u_{22} =$ constant on N and a straightforward computation from (5.2) thus shows the Gauss curvature of N to be constant.

Proposition 3. *Suppose, in Proposition* 2, *M is of* dimension ≥ 4. *Then the sectional curvature of M associated with the plane spanned by any two vectors normal to N is a constant c, and $c = -c^*$ where c^* is the constant sectional curvature of N.*

Proof. Let e_i and e_k run over the vectors tangent to N. Summing on i and k in (5.1), one finds $2m(2uu_{jj} - \sum_k u_k^2)$ is the scalar curvature of N, where m is the dimension of N. Thus u_{jj} is constant at a point as far as e_j is normal to N. Hence at every point the sectional curvature of M associated with the plane spanned by any two vectors normal to N is a constant, and this constant is equal to $-c^*$ due to the vanishing of the Weyl conformal tensor.

Theorem 4. *Let M be a conformally flat manifold of* dimension ≥ 3. *Suppose M is a nontrivial product $N_1 \times N_2$. Then both N_1 and N_2 have constant curvature, and if both N_1 and N_2 are of* dimension ≥ 2, *then the curvature of N_1 and N_2 just differ by a sign.*

The case $\dim M = 3$ and the last fact of Theorem 4 were pointed out to the author by Kulkarni. Now it is possible to use a strong result of Cheeger and Gromoll [2] and Kuiper [7] to prove the following

Theorem 5. *Let M be a complete conformally flat manifold with nonnegative curvature. Then either one of the following holds:*

(i) *M is conformally equivalent to an elliptic space form,*

(ii) *M is covered by the isometric product of a line and a manifold of constant curvature, and the covering transformation is a local isometry,*

(iii) *M is contractible.*

Proof. If M is compact and has finite fundamental group, then (i) is a theorem of Kuiper [7]. So suppose the universal cover $\tilde{M}$ of M is noncompact. By Theorems 3.1 and 9.1 of [2], there is a compact toally convex manifold S in $\tilde{M}$ such that all sectional curvatures vanish for planes spanned by a tangent vector and a normal vector of S. Furthermore, if S is empty, then either $\tilde{M}$ is isometric to euclidean space or $M = \tilde{M}$ is contractible. If S is nonempty, Proposition 1 above shows that S has constant curvature. Since S is compact and simply connected, it must be the standard sphere. Thus Proposition 3 shows codim $S = 1$, and Theorem 4.2 of [2] completes the proof of the theorem.

Remark. Examples show that (iii) is best possible, i.e., M need not be isometric to euclidean space. In fact, M may be the product of a complete positively curved open surface and a Euclidean space. Actually we can construct such a manifold with positive curvature.

Another strong result of Cheeger and Gromoll enables us to prove the following

Theorem 6. *Let M be a compact conformally flat manifold with nonnegative ricci curvature. Then M is either conformally equivalent to an elliptic space or isometrically covered by the direct product of a straight line and a manifold with constant curvature.*

Proof. The proof is similar to that of Theorem 5 except that here we use the result of [3].

Remark. Bochner (cf. [4]) proved before that a compact conformally flat manifold with positive ricci curvature is a homology sphere.

Finally, let us consider locally homogeneous manifolds. First of all, we define such a manifold. A manifold M is said to be locally homogeneous if for any two points p, q in M, there exists an isometry of a neighborhood of p onto a neighborhood of q carrying p into q.

Proposition 4. *Let M be a locally homogeneous manifold of* dimension ≥ 3. *If M admits a nonisometric conformal transformation, then M is conformally flat. If M is further complete, then M has nonnegative curvature.*

Proof. Since M is locally homogeneous, the conformal tensors C_{ijkl}, C_{ijk} have constant length. (2.20) and (2.21) then show that either $u \equiv 1$ or the conformal tensor vanishes. If M is complete, Theorem 1 is applicable since the function K is a constant. Hence the manifold has nonnegative curvature.

Remark. In case dim $M \geq 4$, the proposition was obtained by Yano [15] for a one-parameter group of conformal transformations and by Kulkarni [8] for a nowhere conformally flat M. The idea of using the local homogeneity to get the constancy of the length of the Weyl conformal tensor has already been used by Barbance, Hsiung, Lichnerowicz and others.

Finally, by using Theorem 5 we have

Theorem 7. *Let M be a complete locally homogeneous manifold of dimen-*

sion ≥ 3. If M admits a nonhomothetic conformal transformation, then M is isometrically covered by

(i) *euclidean space,*

(ii) *the direct product of a line and the sphere,*

(iii) *sphere.*

Proof. Since every complete locally homogeneous manifold is covered by a homogeneous one, we may assume M to be homogeneous. By Theorem 5, we have only to prove that if M is contractible, then M is isometric to euclidean space. This is a special case of a theorem of Cheeger and Gromoll [3].

Remark. For a compact M of dim ≥ 4, Theorem 7 was obtained by Goldberg-Kobayashi [5] for the continuous case, and by C. Barbance [1] for the discrete case. Barbance's result was reproved by Kulkarni [8].

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