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SPECIAL CONNECTIONS AND ALMOST FOLIATED METRICS

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On manifolds with a complex almost-product structure, we study some special connections related to the parallelism and integrability of the distributions and to a complex symmetric bilinear form (pseudo-metric) compatible with the structure, and establish the notion of almost-foliated metric which includes as a particular case the metric of a foliated type on a foliated manifold. (For Reinhart spaces see [6].)

1. Adapted connections

Let V be a differentiable manifold of class C^∞ and dimension n, and let $T^C(V) = T(V) \otimes_R C$ denote the complexified space of the tangent space $T(V)$ of the manifold. A complex almost-product structure defined on V gives two C^∞-fields T^1 and T^2 of supplementary subspaces, with respect to the Whitney sum, of $T^C(V)$ ($\dim T^1 = n_1$, $\dim T^2 = n_2, n_1 + n_2 = n$). If $x \in V$, then every vector $X \in T^C_x$ is the sum of two vectors $PX \in T^1_x$ and $QX \in T^2_x$, so that $T^1_x + T^2_x = T^C_x$, $P + Q = I$ (identity), P, Q being the projection tensors associated with T^1 and T^2.

The complex almost-product structure is determined by a vectorial form H such that $H^2 = I$ gives $H = P - Q$ in T^C. It is equivalent to the reduction of the structural group $GL(n, C)$ of the fibration $T^C(V)$. The principal fibration associated with $T^C(V)$ has, as a structural group, the subgroup of the complex linear group $GL(nC)$ of the form

$$(1) \qquad \begin{pmatrix} GL(n_1, C) & 0 \\ 0 & GL(n - n_1, C) \end{pmatrix},$$

The structure determined by the operator $H = P - Q$, such that $H^2 = I$, comprises as particular cases: the almost-complex structure when n is even and $J = iP - iQ$, $\bar{P} = iP$, $\bar{Q} = iQ$ are conjugate operators; and the real almost-product structure when P, Q are real.

We represent by $A(V)$ the fibration of the complex references of T^C with $GL(n, C)$ as the structural group, and by $A'(V)$ the subfibration of the linear references adapted to the complex almost-product structure with (1) as the structural group.

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Definition 1. A connection is said to be adapted if it preserves the complex almost-product structure.

We can easily see that these adapted connections make H parallel; that is, $\nabla H = 0$ for an adapted connection, and deduce that the adapted connections are the infinitesimal connections on $A'(V)$. These connections generalize the almost-complex connections of A. Lichnerowicz [4] and the connections of Schouten [7], which are the connections established by I. Cattaneo-Gasparini [1] and by Legrand [3]. For arbitary vector fields X, Y in T^C, in the same way as for the real case we define a torsion tensor N for the complex almost-product structure by

$$(2) \qquad N(X, Y) = \tfrac{1}{4}([HX, HY] + [X, Y] - H[HX, Y] - H[X, HY]) ,$$

where we write, for a tensor β of type (1, 2),

$$\beta(HX, Y) = \beta H(X, Y) , \qquad \beta(X, HY) = \beta \cdot H(X, Y) .$$

Proposition 1. *If α is a tensor of type $(1,2)$, β a tensor of type $(1,1)$ and ∇ a symmetric connection, then $\nabla' = \nabla + \alpha$ is a connection such that when applied to β we have $\nabla'\beta = \nabla\beta + \alpha*\beta = \nabla\beta + \alpha\cdot\beta - \beta\alpha$.*

Proposition 2. *For a symmetric connection ∇ in T^C, all the connections adapted to the complex almost-product structure defined by the tensor H are given by*

$$(3) \qquad \nabla' = \nabla - \tfrac{1}{2}\nabla H \cdot H + \beta$$

with the condition $\beta \cdot H - H\beta = 0$.

Proof. Since $\nabla(HH) = \nabla H \cdot H + H\nabla H = 0$, and $H\nabla H \cdot H = -\nabla H$, we obtain

$$\nabla' H = \nabla H - \tfrac{1}{2}(\nabla H \cdot H)*H + \beta*H = \nabla H - \tfrac{1}{2}\nabla H + \tfrac{1}{2}H\nabla H \cdot H = 0 .$$

Definition 2. For the adapted connections ∇' and the torsion tensor N of the structure, we define the connections

$$(4) \qquad E = \nabla' - \tfrac{1}{2}N = \nabla - \tfrac{1}{2}\nabla H \cdot H + \beta - \tfrac{1}{2}N .$$

Proposition 3. *$N = HEH$.*

Proof. Since

$$EH = \nabla' H - \tfrac{1}{2}N*H = \tfrac{1}{2}(-N \cdot H + HN) , \quad HEH = \tfrac{1}{2}(-HN \cdot H + N) ,$$
$$N(X, Y) = \tfrac{1}{2}[(\nabla_{HX}H)Y - (\nabla_{HY}H)X - H(\nabla_X H)Y + H(\nabla_Y H)X] ,$$

we have $-HN \cdot H(X, Y) = N(X, Y)$, and hence the proposition.

It is well known that if the complex almost-product structure is integrable, then there exists a symmetric connection which makes it parallel. However, the following immediate proposition, the E connections represent all the connections such that if H is parallel with respect to them then it is integrable, and conversely.

Proposition 4. *A necessary and sufficient condition for the complex almost-product structure determined by H to be integrable is that H be parallel with respect to an E connection.*

In the case of a real almost-product structure, the connections L of Walker [10] are defined in the form $L = D + N$ such that they make H parallel, D being a symmetric connection. Then $L \subset \nabla', D \subset E$.

2. Connections in relation with a pseudo-metric adapted to the complex almost-product structure

Given the complex almost-product manifold V, whose characteristic tensor is H, let g be a C-bilinear symmetric form of a complex pseudo-metric C^∞ defined on V. We say that g is adapted to the complex almost-product structure if

$$g(HX, HY)_p = g(X, Y)_p \ , \quad \forall p \in V \ , \quad \forall X, Y \in T^C \ .$$

For the two subspaces T^1 and T^2 of T^C determined by H, the condition for the pseudo-metric to be adapted to this decomposition is that T^1 and T^2 be orthogonal with respect to g at every point p.

In accordance with Proposition 2, by taking different expressions for β we can determine the adapted connections with certain special properties as in the following proposition.

Proposition 5. *There exists a unique connection on $T^C(V)$ with the following conditions*:

(a) *It is adapted to the structure H.*

(b) *The connection induced in T^1 (or T^2) is compatible with g.*

(c) *The first n_1 components of the torsion are of type $(0, 2)$, and the last $n - n_1$ are of type $(2, 0)$.*

This connection (called the second connection) is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \tfrac{1}{4}[(\nabla_{HY} H)X + H((\nabla_Y H)X) + 2H((\nabla_X H)Y)] \ . \tag{5}$$

Lemma 1. *Suppose $\nabla' = \nabla + \alpha$, where α is a tensor of type $(1, 2)$, and let g be a tensor of type $(0, 2)$. Then*

$$\begin{aligned} (\nabla' g)(X, Y, Z) &= (\nabla g)(X, Y, Z) + (\alpha * g)(X, Y, Z) \ , \\ (\alpha * g)(X, Y, Z) &= -g(\alpha(X, Y), Z) - g(Y, \alpha(X, Z)) \ . \end{aligned} \tag{6}$$

Proof. Since

$$\nabla'_X(g(Y, Z)) = Xg(Y, Z) = (\nabla'_X g)(Y, Z) + g(\nabla'_X Y, Z) + g(Y, \nabla'_X Z) \ ,$$
$$\nabla_X(g(Y, Z) = Xg(Y, Z) = (\nabla_X g)(Y, Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \ ,$$

substration of these two equations gives the second equation of (6) immediately.

Proof of Proposition 5. a) Since

$$(\nabla I)Y = (\nabla(HH))Y = (\nabla H)HY + H(\nabla H)Y = 0 \ ,$$
$$H(\nabla H)HY = -(\nabla H)Y \ ,$$

in accordance with Proposition 1 we obtain

$$(\tilde{\nabla}_X H)Y = (\nabla_X H)Y + \tfrac{1}{4}((\nabla_Y H)X + H(\nabla_{HY}H)X + 2H(\nabla_X H)HY$$
$$- H(\nabla_{HY}H)X - (\nabla_Y H)X - 2((\nabla_X H)Y) = 0 \ .$$

b) Since ∇ and g are compatible with the complex almost-product structures,

$$4(\tilde{\nabla}_{PX}g)(PY, PZ) = 4(\nabla_{PX}g)(PY, PZ) + [(\nabla_{HY}H)X + H((\nabla_Y H)X)$$
$$+ 2H(\nabla_X H)Y]*g(PX, PY, PZ) \ .$$

Since $\nabla g = 0$, $H(\nabla H)PX = -(\nabla H)PX$ and $(\nabla H)P = 2Q\nabla P$, by Lemma 1 we obtain

$$4(\tilde{\nabla}_{PX}g)(PY, PZ) = -g((\nabla_{PY}H)PX + H((\nabla_{PY}H)PX + 2H(\nabla_{PX}H)PY, PZ)$$
$$- g(PY, (\nabla_{PZ}H)PX + H(\nabla_{PZ}H)PX + 2H(\nabla_{PX}H)PZ)$$
$$= -g(2H(\nabla_{PX}H)PY, PZ) - g(PY, 2H(\nabla_{PX}H)PZ) \ .$$

On the other hand, from $\nabla(HP) = (VH)P + H\nabla P = \nabla P$ it follows $P(\nabla H)P = 0$ and therefore

$$H(\nabla_{PX}H)PY = P(\nabla_{PX}H)PY - Q(\nabla_{PX}H)PY = -Q(\nabla_{PX}H)PY \ .$$

Thus

$$g(2H(\nabla_{PX}H)PY, PZ) = -2g(Q(\nabla_{PX}H)PY, PZ) = 0 \ .$$

On account of the orthogonality of T^1 and T^2, we hence have $(\tilde{\nabla}_{PX}g)(PY, PZ) = 0$, which is similarly true with P replaced by Q.

c) We must show that the first components of the torsion of $\tilde{\nabla}$ are of type $(0, 2)$ and the second ones are of type $(2, 0)$, that is,

$$P \operatorname{Tor}_{\tilde{\nabla}}(PY, PZ) = 0 \ , \quad P \operatorname{Tor}_{\tilde{\nabla}}(PY, QZ) = 0 \ , \quad Q \operatorname{Tor}_{\tilde{\nabla}}(QY, QZ) = 0 \ .$$

For this purpose, it sufficies to observe that the torsion of $\tilde{\nabla}$ is the Nijenhuis tensor except for a sign so that

$$PN(PY, PZ) = PQN(Y, Z) = 0 \ , \qquad N(PY, QZ) = 0 \ .$$

Similarly, $QN(QY, QZ) = 0$.

To prove that $\tilde{\nabla}$ is the only connection satisfying a), b) and c), we shall prove that if a connection $\nabla = \tilde{\nabla} + \beta$, β being a tensor of type $(1, 2)$ satisfies a), b) and c), then $\beta(Y, Z) = 0$, where Y, Z are arbitrary.

From a) we have $\beta * H = 0$, that is, $\beta(Y, HZ) - H\beta(Y, Z) = 0$, from which follow

$$P\beta(Y, HZ) - P\beta(Y, Z) = 0 , \qquad Q\beta(Y, HZ) + Q\beta(Y, Z) = 0 .$$

Moreover,

$$P\beta(Y, QZ) = 0 , \qquad Q\beta(Y, PZ) = 0 . \tag{7}$$

By b) we obtain $\beta * g(PY, PX, PZ) = 0$, $\beta * g(QY, QX, QZ) = 0$, from the first of which it follows

$$-g(\beta(PY, PX), PZ) - g(PX, \beta(PY, PZ)) = 0 .$$

Putting $X = Z$ for arbitrary Z in the above equation yields

$$g(\beta(PY, PZ), PZ) = 0 ,$$

which implies

$$P\beta(PY, PZ) = 0 . \tag{8}$$

In a similar way, we obtain

$$Q\beta(QY, QZ) = 0 . \tag{9}$$

From c) follow

$$P\beta(PY, QZ) - P\beta(QZ, PY) = 0 , \qquad Q\beta(QY, PZ) - Q\beta(PZ, QY) = 0 , \tag{10}$$

which, together with (7), (8), (9), hence give $\beta(Y, Z) = 0$.

The coefficient of this connection was obtained by Vaismann [8] for real almost-product Riemannian manifolds, and in the case of almost-complex manifolds this connexion coincides with that introduced in [2, p. 143].

Proposition 6. *There exists a connection ∇' on a complex almost-product manifold adapted to the structure such that its torsion is*

$$\mathrm{Tor}_{\nabla'}(X, Y) = \tfrac{1}{2}[(\nabla_Y H)HX - (\nabla_X H)HY] . \tag{11}$$

This connection has also the property that the connections induced in T^1 and T^2 are compatible with the metric induced in T^1 and T^2.

For the connection ∇ corresponding to a g pseudo-metric adapted to the complex almost-product structure, we have

Proposition 7. *If the connection ∇ makes T^1 parallel, it also makes T^2*

parallel, and consequently both T^1 *and* T^2 *are integrable.*

Proof. Since g is adapted to the structures, ∇ is the metric connection and ∇ makes T^1 parallel, we have, respectively, $g(PY, QZ) = 0$, $\nabla g = 0$ and $Q\nabla P = 0$, the last of which implies $\nabla P = P\nabla P$. Thus

$$\begin{aligned}\nabla(g(PY, QZ)) &= (\nabla g)(PY, QZ) + g(\nabla PY, QZ) + g(PY, \nabla QZ)\\ &= g(P(\nabla P)Y, QZ) + g(PY, (\nabla Q)Z)\\ &= g(PY, (\nabla Q)Z)) = 0\ .\end{aligned}$$

Hence $(\nabla Q)Z \in T^2$ implies $P(\nabla Q)Z = 0$, which is the condition for ∇ to make T^2 parallel.

The integrability is a consequence of the parallelism with respect to a symmetric connection.

Definition 2. Let ∇ be a symmetric connection. Then a connection is a C-connection if it is of the form

$$C = \nabla - Q\nabla P + QN + \gamma\ , \qquad Q\gamma \cdot P = 0\ . \tag{12}$$

Proposition 8. *A necessary and sufficient condition for* T^1 *to be integrable is that it be parallel with respect to a C-connection.*

Proof. If T^1 is integrable, then $QN = 0$, and the expression of C is reduced to the expression of the connection which makes T^1 parallel. Conversely, $QCP = Q\nabla P - Q\nabla P + QN \cdot P + Q\gamma \cdot P = 0$ implies that $QN \cdot P = 0$ and therefore that $Q[P, P] \cdot P = Q[P, P] = 0$.

Corollary.

$$Q\ \mathrm{Tor}_C\,(PX, PY) = 0\ . \tag{13}$$

3. Almost-foliated pseudo-metrics

Definition 3. Let V be a C^∞ manifold with a complex almost-product structure, g a complex pseudo-metric, and $\tilde{\nabla}$ the second connection given by $\tilde{\nabla} = \nabla + \alpha/4$, where ∇ is the metric connection. Then g is said to be almost-foliated if

$$(\tilde{\nabla}_{PX} g)(QY, QZ) = 0\ , \qquad \forall X, Y, Z \in T^C(V)\ . \tag{14}$$

Proposition 9. *A necessary and sufficient condition for the form* g *to be almost-foliated is that*

$$(\alpha * g)(PX, QY, QZ) = 0\ .$$

Proposition 10. *If the form* g *is almost-foliated, then the fields of* T^2 *parallel with respect to the connection* $\tilde{\nabla}$ *along any curve preserve their length.*

Proof. From Proposition 5 and (14) we obtain $(\tilde{\nabla}_X g)(QY, QZ) = 0$.

4. Real foliated manifolds

If we consider a real foliated manifold, then the almost-foliated metric contains the fibre-like metric (Reinhart spaces [6]) as a special case in accordance with the following proposition.

Proposition 11. *Given a real foliated Riemannian manifold* (V, T^1, T^2), T^1 *being integrable, a necessary and sufficient condition for the metric to be fibre-like is that it be almost-foliated.*

Proof. Suppose on the manifold there exists a fibre-like metric, ∇ is the metric connection, and taking references adapted to the foliation $(\partial x^a, Y_u)$, (θ^a, dy^u), $(a, b = 1, \cdots, n_1;\ u, v = n_1 + 1, \cdots, n)$, we have [5]

$$ds^2 = g_{ab}(x, y)\theta^a\theta^b + G_{uv}(y)dy^u dy^v \ . \tag{15}$$

Then the condition of fibre-like metric is expressed as

$$\nabla_{\partial_a}(g(Y_u, Y_v)) = \partial_a G_{uv} = 0 \ , \tag{16}$$

that is,

$$g(\nabla_{\partial_a}Y_u, Y_v) + g(Y_u, \nabla_{\partial_a}Y_v) = 0 \ . \tag{17}$$

We must prove that in this case $(\tilde{\nabla}_{PX}g)(QY, QZ) = 0$. For this purpose we shall first demonstrate

$$(\tilde{\nabla}_{\partial_a}g)(Y_u, Y_v) = (\nabla_{\partial_a}g)(Y_u, Y_v) + \tfrac{1}{4}(\alpha * g)(\partial_a, Y_u, Y_v) = 0 \ .$$

$(\nabla g) = 0$, since ∇ is the metric connection and

$$\begin{aligned} -(\alpha * g)(\partial_a, Y_u, Y_v) &= g(\alpha(\partial_a, Y_u), Y_v) + g(Y_u, \alpha(\partial_a, Y_v)) \\ &= g((\nabla_{-Y_u}H)\partial_a + H(\nabla_{Y_u}H)\partial_a + 2H(\nabla_{\partial_a}H)Y_u, Y_v) \\ &\quad + g(Y_v, (\nabla_{-Y_v}H)\partial_a + H(\nabla_{Y_v}H)\partial_a + 2H(\nabla_{\partial_a}H)Y_v) \ . \end{aligned}$$

On the other hand,

$$(\nabla H)P = 2Q\nabla P \ , \qquad (\nabla H)Q = -2P\nabla Q \ .$$

Since $g(PY, QZ) = 0$,

$$-(\alpha * g)(\partial_a, Y_u, Y_v) = -4(g(Q\nabla_{Y_u}\partial_a, Y_v) + g(Y_v, Q\nabla_{Y_v}\partial_a)) \ ,$$

or

$$(\alpha * g)(\partial_a, Y_u, Y_v) = 4(g(\nabla_{Y_u}\partial_a, Y_v) + g(Y_v, \nabla_{Y_v}\partial_a)) \ . \tag{18}$$

Since ∇ is symmetric and $[\partial_a, Y_u] \in T^1$, by (17) we finally obtain

$$(\alpha * g)(\partial_a, Y_u, Y_v) = 4(g(\nabla_{\partial_a}Y_u, Y_v) + g(Y_u, \nabla_{\partial_a}Y_v)) = 0 \ . \tag{19}$$

To prove that

$$(\tilde{\nabla}_{\partial_a} g)(Y_u, Y_v) = 0 \text{ implies } (\tilde{\nabla}_{PX} g)(QY, QZ) = 0 ,$$

it suffices to consider

$$\begin{aligned}(\tilde{\nabla}_{PX} g)(QY, QZ) &= \tilde{\nabla}_{PX}(g(QY, QZ)) - g(\tilde{\nabla}_{PX} PY, QZ) - g(QY, \tilde{\nabla}_{PX} QZ) \\ &= \tilde{\nabla}_{C^a \partial_a}(g(\Gamma^u Y_u, \Gamma^v Y_v)) - g(\tilde{\nabla}_{C^a \partial_a} \Gamma^u Y_u, \Gamma^v Y_v) \\ &\quad - g(\Gamma^u Y_u, \tilde{\nabla}_{C^a \partial_a} \Gamma^v Y_v) .\end{aligned}$$

Conversely, if the metric is almost-foliated and T^1 is integrable, then the metric is fibre-like. In fact, since the metric is almost-foliated we have $(\alpha * g)(PX, QY, QZ) = 0$. For the foliated manifold V, by taking adapted references we thus obtain (19), which is equivalent to $\partial_a G_{uv} = o$.

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