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ON THE RADIUS OF THE SMALLEST BALL CONTAINING A COMPACT MANIFOLD OF POSITIVE CURVATURE

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Theorem. *Let $M \subset E^n$ be a compact Riemannian manifold of dimension $n - 1 \geq 2$ with sectional curvatures $K(\pi) \geq 1/c^2$ for all tangent plane sections π. Then M is contained in a ball of radius $R < \frac{1}{2}\pi c$ and this bound is best possible.*

According to a well-known result of Bonnet, any two points on M can be joined by a minimizing geodesic of length less than or equal to πc. Hence the crude bound $R < \pi c$ follows. The proof we give of the theorem was inspired by the interesting note of Nitsche [1].

Proof. Let B be the closed ball of smallest radius containing M. We choose coordinates so that $B = \{x = (x_1, \cdots, x_n) : |x| \leq R\}$. The set of points of $M \cap \partial B$ must "support" B, that is, each closed half space $x \cdot v \geq 0$ (v a constant vector) must contain at least one such point. Let $C = M \cap \partial B$.

If two points of C are antipodal on ∂B, their distance apart, $2R$, must be less than the length of a minimizing geodesic joining these points. Hence by the theorem of Bonnet mentioned above $R < \frac{1}{2}\pi c$.

Now suppose C contains no pair of antipodal points of ∂B. Then C contains at least three points. If P_1 is one such point, a second point P_2 must lie in the half-space $x \cdot P_1 \leq 0$. We choose a coordinate system in which these points are

$$P_1 = (0, \cdots, 0, (R^2 - \alpha^2)^{1/2}, \alpha), \qquad P_2 = (0, \cdots, 0, -(R^2 - \alpha^2)^{1/2}, \alpha)$$

where $0 \leq \alpha < R/\sqrt{2}$. Still another point $P_3 = (x_1, \cdots, x_n)$ must lie in the half-space $x_n \leq 0$. Let L be the perimeter of the triangle determined by P_1, P_2, P_3. Then

$$L = 2(R^2 - \alpha^2)^{1/2} + [2R^2 - 2(R^2 - \alpha^2)^{1/2}x_{n-1} - 2\alpha x_n]^{1/2} + [2R^2 + 2(R^2 - \alpha^2)^{1/2}x_{n-1} - 2\alpha x_n]^{1/2} .$$

For fixed α and $x_n \leq 0$, the right hand side is minimized for $x_{n-1} = \pm(R^2 - x_n^2)^{1/2}$. Hence

$$L \geq 2(R^2 - \alpha^2)^{1/2} + [2R^2 - 2(R^2 - \alpha^2)^{1/2}(R^2 - x_n^2)^{1/2} - 2\alpha x_n]^{1/2} + [2R^2 + 2(R^2 - \alpha^2)^{1/2}(R^2 - x_n^2)^{1/2} - 2\alpha x_n]^{1/2} .$$

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Keeping α fixed the right hand side is minimized when $x_n = 0$. Hence

$$L \geq 2(R^2 - \alpha^2)^{1/2} + [2R^2 - 2R(R^2 - \alpha^2)^{1/2}]^{1/2} + [2R^2 + 2R(R^2 - \alpha^2)^{1/2}]^{1/2} .$$

Finally, the right hand side is minimized when $\alpha = 0$. This gives $L \geq 4R$. Now consider a (minimizing) geodesic triangle on M determined by the points P_1, P_2, P_3. According to a theorem of Toponogov [2], the perimeter P of any such triangle satisfies $P \leq 2\pi c$. Since $P > L$, we have $R < \frac{1}{2}\pi c$. This is the required estimate. The following example shows that this estimate is best possible.

Example. Suppose first $n = 3$. Consider the surfaces of revolution of constant Gauss curvature $1/c^2$. The generating curves of these surfaces form a one paramete family of curves starting from a semi-circle of diameter $2c$ and eventually stretching out to a "needle" of diameter $2\pi c$. If we "round off the corners" and revolve the modified generating curve, we obtain a sequence of compact surfaces satisfying $K \geq 1/c^2$ with Euclidean diameter tending to $2\pi c$.

To obtain an example for $n > 3$, let $x_2 = f(x_1)$ describe the modified generating curve described above. Then the hypersurface determined by the relation $x_n^2 + \cdots + x_2^2 = f(x_1)^2$ gives the repuired example.

Remark. If $n = 2$ and M is a *simple* plane curve with curvature $\geq 1/c > 0$, then it is easy to show that M is contained in a circle of radius c. If $n = 3$ and M is an ellipsoid satisfying $K \geq 1/c^2$, then it is easy to check that M is contained in a ball of radius c.

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References

[1] J. C. C. Nitsche, *The smallest sphere containing a rectifiable curve,* Amer. Math. Monthly **27** (1971) 881–882.

[2] V. A. Toponogov, *Riemannian spaces having their curvature bounded below by a positive number,* Amer. Math. Soc. Transl. **37** (1964) 291–336.

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