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NORMAL FILTRATION OF $H^1(M, \theta)$

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Dedicated to Professor V. Ganapathy Iyer on his 65th birthday

Introduction

The Kodaira-Spencer-Kuranishi theory [3], [6], [7], [10], [11], [12], [13], [14] of deformations concerns itself mainly with the variation of the complex analytic structure on a compact complex analytic manifold M. Denoting the sheaf of germs of holomorphic vector fields by θ, the group of infinitesimal deformations is shown to be $H^1(M, \theta)$. It is therefore quite natural to expcet that $H^1(M, \theta)$ would control the local deformations of M. In fact Frölicher-Nijenhuis [2] discovered in 1957 that if $H^1(M, \theta) = 0$, any family of deformations is locally trivial at the reference point. There is a natural quadratic map $H^1(M, \theta) \to H^2(M, \theta)$ which assigns to every infinitesimal deformation, the obstruction to prolonging it one step. If such a prolongation is possible, then one meets another obstruction which also lies in $H^2(M, \theta)$. Subsequent obstructions also lie in $H^2(M, \theta)$. If one can pass all these obstructions, one can construct formally a family of deformations of M. But then, one meets the significant analytic question of convergence; the formal deformations constructed apriori need not converge and therefore need not define a genuine deformation of M. In the special case when $H^2(M, \theta) = 0$, there is no serious difficulty in constructing formally a family of deformations. Kodaira-Nirenberg-Spencer [5] proved that in this case, the formally constructed family actually converges to a genuine family of deformations of M. Moreover, Kodaira-Spencer [8] proved that this family is universal and effective.

In [9] Kuranishi introduced the notion of a normal family of deformations of a compact complex manifold and proved the existence of a holomorphic normal family of deformations for any given compact complex manifold. This family, constructed by Kuranishi, is more general than the one constructed by Kodaira-Nirenberg-Spencer and reduces to their family when $H^2(M, \theta) = 0$.

In [10], [12] Kuranishi proved the fundamental existence theorem of deformation theory, namely, the existence of a universal and effective family of deformations for any compact complex manifold.

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In order to define and prove the existence of a normal family of deformations, Kuranishi defined a decreasing filtration of the first cohomology group $H^1(A)$ of the complex $A = \sum_{p\geq 0} A^p$, with respect to the exterior differential operator $\bar\partial$. Here A^p denotes the space of differential forms on M of type $(0, p)$ with values in the complex vector bundle of tangent vectors of type $(1, 0)$. We call this filtration the normal filtration of $H^1(A)$.

In order to construct a universal family, Kuranishi constructed an analytic injective map $\Phi\colon W \to A^1$, W being an open neighborhood of the origin in $H^1(M, \theta)$. This mapping plays the crucial role in the construction of universal family. We call this mapping Φ the canonical universal map.

The main theorem of this paper gives a characterization of the normal filtration of $H^1(M, \theta)$ in terms of the canonical universal map Φ.

This paper consists of two parts. In Part I, there are three sections; the first section contains known facts on complex manifolds, which are needed for our purpose; the second section gives the definition of the normal filtration and the third section describes how and in what context the mapping Φ was constructed by Kuranishi. The second part consists of four sections. The first section gives the statement of our main theorem; the second gives the statements of a lemma and a proposition on which the proof of the main theorem mainly rests, the third section gives the proof of the main theorem and the final section gives the proofs of the lemma and the proposition of § 2.

PART I. NORMAL FILTRATION OF $H^1(M, \theta)$

1. Some facts on compact complex manifolds

In this section we briefly mention some well known facts which we need, on compact complex manifolds. Let M be a compact complex manifold, and let $\boldsymbol{M}$ denote the underlying C^∞ differential manifold. Let $T'M$ and $T''M$ be the complex vector bundles of type $(1, 0)$ and $(0, 1)$, respectively, of M. Then $T\boldsymbol{M} \otimes_{\boldsymbol{R}} \boldsymbol{C} = T'M \oplus T''M$ where $T\boldsymbol{M} \otimes_{\boldsymbol{R}} \boldsymbol{C}$ is the complex tangent bundle of $\boldsymbol{M}$. $T'M$ is the complex conjugate of $T''M$. Let A^p denote the space of C^∞ differential forms of M of type $(0, p)$ with values in $T'M$. If $\theta \in A^p$, in terms of complex analytic local coordinates $Z = (Z^1, \cdots, Z^n)$ of M, we have

$$\theta = \sum_{\alpha_1, \cdots, \alpha_p} \theta_{\alpha_1, \cdots, \alpha_p} d\bar{Z}^{\alpha_1} \wedge \cdots \wedge d\bar{Z}^{\alpha_p} ,$$

where $\theta_{\alpha_1, \cdots, \alpha_p}$ are vector fields of type $(1, 0)$ and are skewsymmetric in $\alpha_1, \cdots, \alpha_p$. We have the exterior derivative $\bar\partial\colon A^p \to A^{p+1}$ defined by

$$\bar\partial\theta = \sum \frac{\partial\theta_{\alpha_1, \cdots, \alpha_p}}{\partial\bar{Z}^\beta} d\bar{Z}^\beta \wedge d\bar{Z}^{\alpha_1} \wedge \cdots \wedge d\bar{Z}^{\alpha_p} .$$

Also we have the bracket operator $[\ \ ]: A^p \times A^q \to A^{p+q}$. If $\theta \in A^p$, $\phi = \sum_{\beta_1,\cdots,\beta_q} \phi_{\beta_1,\cdots,\beta_q} d\bar{Z}^{\beta_1} \wedge \cdots \wedge d\bar{Z}^{\beta_q} \in A^q$, then locally

$$[\theta,\phi] = \sum_{\substack{\alpha_1,\cdots,\alpha_p \\ \beta_1,\cdots,\beta_q}} [\theta_{\alpha_1,\cdots,\alpha_p}, \phi_{\beta_1,\cdots,\beta_q}] d\bar{Z}^{\alpha_1} \wedge \cdots \wedge d\bar{Z}^{\alpha_p} \wedge d\bar{Z}^{\beta_1} \wedge \cdots \wedge d\bar{Z}^{\beta_q} .$$

Since the change of charts are complex analytic, it can be checked easily that this definition of $[\ \ ]$ is independent of the chart Z. Then $A = \sum_{p\geq 0} A^p$ is a graded Lie algebra complex with respect to $\bar\partial$ and $[\ \ ]$. This means that $\bar\partial$ is linear, $\bar\partial\cdot\bar\partial = 0$, $[\ \ ]$ is bilinear, and the following three formulas hold for $\theta \in A^p$, $\phi \in A^q$, $\psi \in A^r$:

$$(1.1) \qquad [\theta,\phi] = (-1)^{pq+1}[\phi,\theta] ,$$

$$(1.2) \qquad \bar\partial[\theta,\phi] = [\bar\partial\theta,\phi] + (-1)^p[\theta,\bar\partial\phi] ,$$

$$(1.3) \qquad (-1)^{pr}[\theta,[\phi,\psi]] + (-1)^{qp}[\phi,[\psi,\theta]] + (-1)^{rq}[\psi,[\theta,\phi]] = 0 .$$

Let $H^{0,p}_{\bar\partial}(A)$ be the p-dimensional cohomology group of the complex A with respect to the differential operator $\bar\partial$, and $H^p(M,\theta)$ be the p-dimensional cohomology group of M with coefficients in the sheaf θ of germs of holomorphic vector fields of M. Then we have the Dolbeault isomorphism

$$(1.4) \qquad H^{0,p}_{\bar\partial}(A) \simeq H^p(M,\theta) .$$

We are mainly interested in A^1 as it plays a very important role in deformation theory of compact complex manifolds. This is due to the following Proposition 1.1. An almost complex structure on M is, by definition, a C^∞ vector subbundle over C, say T'', of $TM \otimes_R C$ such that we have a direct sum decomposition

$$(1.5) \qquad TM \otimes_R C = T'' \oplus \bar{T}'' ,$$

where $\bar{T}''$ denotes the conjugate of T''. Every complex structure M on M induces on M an almost complex structure in a canonical way since $TM \otimes_R C = T''M \oplus T'M$ and $T'M$ is the conjugate of $T''M$. Let $\rho'(M)$ (resp. $\rho''(M)$) denote the projection of $TM \otimes_R C$ onto $T'M$ (resp. $T''M$) with kernel $T'M$ (resp. $T'M$).

Definition (1.1). An almost complex structure T'' on M is said to be of finite distance from the complex structure M on M if and only if $\rho''(M)$ induces an isomorphism of T'' onto $T''M$.

Then we have the important proposition.

Proposition (1.1) [4], [13]. *There is a bijective correspondence between almost complex structures on M having finite distance to the complex structure*

M on M and the elements of $A^1(M)$ sufficiently closed to zero in the C' topology.

We give now a brief explanation of the bijective correspondence. Let T'' be an almost complex structure on M having finite distance to M, and β be the inverse of the isomorphism $\rho''(M)\colon T'' \to T''M$.

Let $\omega = -\rho'(M) \circ \beta$. Then $\omega\colon T''M \to T'M$ is a C^∞ homomorphism of vector bundles and thus can be considered as an element of A^1. Hence $T'' = \{L - \omega(L)\,; L \in T''M\}$. Conversely, let ω be a C^∞ homomorphism of vector bundles $T''M \to T'M$. Then $\{L - \omega(L)\,; L \in T''M\}$ defines a C^∞ vector subbundle T'' of $TM \otimes_R C$ which is an almost complex structure having finite distance to M.

Another important fact which should be mentioned in this connection is the integrability condition. An almost complex structure on M is said to be integrable if it is induced by a complex structure on M. Let $\omega \in A^1(M)$ give rise to an almost complex structure M_ω on M. If M_ω is integrable, then it is easily seen that we must have

$$\bar\partial\omega - \tfrac{1}{2}[\omega, \omega] = 0\ . \tag{1.6}$$

Newlander and Nirenberg [15] proved that the converse is also true. It should be remarked that the proof of the converse is difficult. Kuranishi has given in [13] a proof of the converse for the real analytic case under a more general formulation.

2. Normal filtration of $H^1(A)$

Let $u_1, \cdots, u_l$ be an ordered sequence of indeterminates; we keep these fixed in our discussion here. Then $B(A^p, u_1, \cdots, u_l)$ denotes the space of polynomials ϕ of the form

$$\phi = \sum_{q=1}^{l} \sum_{1\le i_1<,\cdots,<i_q\le l} \phi_{i_1,\cdots,i_q} u_{i_1}\cdots u_{i_q}\ , \qquad \phi_{i_1,\cdots,i_q} \in A^p\ . \tag{2.2}$$

If $\phi \in B(A^p, u_1, \cdots, u_l)$, then $\bar\partial\colon A^p \to A^{p+1}$ can be extended to

$$B(A^p, u_1, \cdots, u_l) \to B(A^{p+1}, u_1, \cdots, u_l)$$

by defining

$$\bar\partial\phi = \sum_{p=1}^{l} \sum_{1\le i_1<,\cdots,<i_q\le l} (\bar\partial\phi_{i_1,\cdots,i_q}) u_{i_1}\cdots u_{i_q}\ . \tag{2·2}$$

The bracket operator [ ] can also be extended as follows. Let $\phi \in B(A^p\,; u_i, \cdots, u_l)$, $\psi \in B(A^q\,; u_i, \cdots, u_l)$. Then $[\phi, \psi]$ is defined to be the element θ in $B(A^{p+q}\,; u_1, \cdots, u_l)$ such that

$$\begin{aligned}(2.3)\qquad &\theta_i = 0\ ,\\ &\theta_{i_1,\cdots,i_q} = \sum_{1\le a(1)<\cdots<a(s)\le q} \tfrac{1}{2}[\phi_{i_{a(1)},\cdots,i_{a(s)}}, \psi_{i_1\cdots\hat{i}_a\cdots i_q}]\ ,\end{aligned}$$

for $q \geq 2$ and $1 \leq i_1 < \cdots < i_q \leq l$, where $\hat{i}_a$ means omission of $i_{a(1)}, \cdots, i_{a(s)}$.

It should be remarked, however, that this definition of extension of the bracket [ ], is dependent of the order of the indeterminates $u_1, \cdots, u_l$. We are mainly concerned with the space $B(A^1; u_1, \cdots, u_l)$.

Definition (2.1). An element $\phi \in B(A^1; u_1, \cdots, u_l)$ is said to be distinguished if and only if $\bar{\partial}\phi - [\phi, \phi] = 0$.

Definition (2.2). Take a subsequence $1 \leq i_1 \leq \cdots \leq i_q \leq l$. We then have the projection opepator $\rho[u_{i_1}, \cdots, u_{i_q}]$ of $B(A^p; u_1, \cdots, u_l)$ onto $B(A^p; u_{i_1}, \cdots, u_{i_q})$ defined by

$$\rho[u_{i_1}, \cdots, u_{i_q}](\phi) = \sum_{1\le k_1,\cdots,k_s\le q} \phi_{a(k_1)\cdots a(k_s)} u_{a(k_1)} \cdots u_{a(k_s)}$$

where $a(k) = i_k$. It is easy to check that if ϕ is distinguished, so is $\rho[u_{i_1}, \cdots, u_{i_q}](\phi)$.

In [9] Kuranishi defined a subset $G^{(h,l)}(u_1, \cdots, u_{h-1}; u_h, \cdots, u_l)$ of $B(A^1; u_h, \cdots, u_l)$ for $1 \leq h \leq l$. From the definition this subset is independent of $u_1, \cdots, u_{h-1}$ preceeding $u_h, \cdots, u_l$. Hence we can write $G^{(h,l)}(u_h, \cdots, u_l)$ instead of $G^{(h,l)}(u_1, \cdots, u_{h-1}; u_h, \cdots, u_l)$. When there is no possibility of confusion, we denote this subset by $G^{(h,l)}$ also. The definition is by double induction of (h, l).

$G^{(1,1)}(u_1)$ is defined to be the subspace of distinguished elements in $B(A^1; u_1)$. Assume $G^{(h',l')}$ is defined for $1 \leq h' \leq h \leq l' \leq l$. Then $G^{(h,l+1)}$ is defined by induction of h:

(1) $G^{(1,l+1)}$ is the space of distinguished elements in $B(A^1; u_1, \cdots, u_{l+1})$.

(2) Assuming that $G^{(h',l+1)}$ is defined for $1 \leq h' \leq h$, $G^{(h+1,l+1)}$ is the set of all elements ϕ in $B(A^1; u_{h+1}, \cdots, u_{l+1})$ satisfying the following four conditions:

(i) For any $h + 1 \leq j_1 < \cdots \leq j_s \leq l + 1$ with $h + s \leq l$,

$$\rho[u_{j_1}, \cdots, u_{j_s}](\phi) \in G^{(h+1,h+s)}(u_{j_1}, \cdots, u_{j_s})\ .$$

(ii) Identify $B(A^1; u_{h+1}, \cdots, u_{l+1})$ canonically with a subset of $B(A^1; u_h, \cdots, u_{l+1})$. Then

$$\phi \in G^{(h,l+1)}(u_h, \cdots, u_{l+1})\ .$$

(iii) Consider $u_1, \cdots, u_{h-1}, u_{h+1}, \cdots, u_{l+1}$ instead of $u_1, \cdots, u_{l+1}$. Then

$$\phi \in G^{(h,l)}(u_{h+1}, \cdots, u_{l+1})\ .$$

(iv) For any i, $h+1 \leq i \leq l+1$, and any θ in $G^{(h,l)}(u_h, \cdots, \hat{u}_i, \cdots, u_{l+1})$ such that

$$\rho[u_{h+1}, \cdots, \hat{u}_i, \cdots, u_{l+1}](\phi) = \rho[u_{h+1}, \cdots, \hat{u}_i, \cdots, u_{l+1}](\phi) ,$$

there exists $\phi^* \in G^{(h,l+1)}(u_h, \cdots, u_{l+1})$ such that

$$\rho[u_{h+1}, \cdots, u_{l+1}](\phi^*) = \phi , \qquad \rho[u_h, \cdots, \hat{u}_i, \cdots, u_{l+1}](\phi^*) = \theta .$$

Definition (2.3). $Z^{(l)} = \{\alpha \in A^1 \mid \alpha u_l \in G^{(l,l)}(u_l)\}$.

The following proposition follows from Kuranishi [9].

Proposition (2.1). (i) *$Z^{(l)}$ is a vector subspace of $Z^{(l-1)}$.*

(ii) *$\bar{\partial} A^0 \subseteq Z^{(l)}$ for all l.*

In view of this proposition, we can define

$$H^{(l)}(A) = Z^{(l)}/\bar{\partial} A^0 .$$

Thus we get a filtration of $H^1(A)$:

$$H^1(A) = H^{(1)} \supseteq H^{(2)} \supseteq \cdots , \tag{2.4}$$

which we call the normal filtration of $H^1(A)$. It should be remarked that the proof of the above proposition is quite complicated.

Let $L(H^{(1)}, \cdots, H^{(q)}; A^1)$ denote the space of multilinear mappings of $H^{(1)} \times \cdots \times H^{(q)}$ into A^1. A mapping $f \in L(H^{(1)}, \cdots, H^{(q)}; A^1)$ is said to be symmetric if f can be extended to a symmetric q-ple multilinear mapping of $H^{(1)} \times \cdots \times H^{(1)}$ into A^1. The space of symmetric multilinear mappings of $H^{(1)} \times \cdots \times H^{(q)}$ into A^1 is denoted by $SL(H^{(1)}, \cdots, H^{(q)}; A^1)$. Let $R_1 \colon H^{(1)} \to A^1$ be such that $R_1(x) = x$ for all $x \in H^{(1)}$, and let $R_q \in L(H^{(1)}, \cdots, H^{(q)}; A^1)$, $2 \leq q \leq n$. We then denote the sequence $\{R_1, \cdots, R_n\}$ by R.

Definition (2.4). $R = \{R_1, \cdots, R_n\}$ is called an *ND*-sequence of length n over $H^{(1)}$ if and only if, for every $(t_1, \cdots, t_n) \in H^{(1)} \times \cdots \times H^{(n)}$,

$$T_R(t_1, \cdots, t_n) = \sum_{q=1}^{n} \sum_{1 \leq i_1 < \cdots < i_q \leq n} R_q(t_{i_1}, \cdots, t_{i_q}) u_{i_1} \cdots u_{i_q}$$

is a distinguished element.

In [9, p. 287] Kuranishi proved that there exists an *ND*-sequence of any given length over $H^{(1)}$. Moreover, these R_q can be taken to be symmetric in view of Theorem 3 [9, p. 291].

For $(t_1, \cdots, t_n, t_{n+1}) \in H^{(1)} \times \cdots \times H^{(n)} \times H^{(n)}$ and $(\zeta_2, \cdots, \zeta_n) \in H^{(1)} \times \cdots \times H^{(n-1)}$, set

$$\begin{aligned}
&K_R(t_1, \cdots, t_n, t_{n+1}; \zeta_2, \cdots, \zeta_n) \\
&= \sum_{q=0}^{n-1} \sum_{1\le i_1<\cdots<i_q\le n} [R_{q+1}(t_{i1}, \cdots, t_{i_q}, t_{n+1}), \\
&\qquad R_{n-q}(t_1, \cdots, \hat{t}_{i_1}, \cdots, \hat{t}_{i_q}, \cdots, t_n)] \\
&+ \sum_{k=1}^{n=1} \sum_{q=0}^{k-1} \sum_{1\le i_1<\cdots<i_q\le k} [R_{q+1}(t_{i_1}, \cdots, t_{i_q}, \cdots, \zeta_{k+1}), \\
&\qquad R_{k-q}(t_i, \cdots, \hat{t}_{i_1}, \cdots, \hat{t}_{i_q}, \cdots, t_k] .
\end{aligned} \tag{2.5}$$

Also set $K_R(t_1, t_2) = [t_1, t_2]$ for any $(t_1, t_2) \in H^{(1)} \times H^{(2)}$. It is to be noted that $K_R(t_1, t_2)$ does not depend on R. The following theorem of Kuranishi [9, p. 287] gives a characterization of the filtration $H^{(1)} \supseteq H^{(2)} \supseteq \cdots$.

Theorem (2.1). *Let $R = \{R_1, \cdots, R_n\}$ be a symmetric ND-sequence of length n over $H^{(1)}$. Then*

$$\begin{aligned}
H^{(n+1)} = \{x \in H^{(n)} \mid &S_k^x \text{ in } L(H^{(k)}, \cdots, H^{(n)}; H^{(k-1)}), \qquad k = 2, \cdots, n, \\
&\text{and } S^x \text{ in } L(H^{(1)}, \cdots, H^{(n)}; A^1) \text{ such that} \\
&S^x(t_1, \cdots, t_n) + K(t_1, \cdots, t_n, x; S_2^x(t_2, \cdots, t_n), \cdots, S_n^x(t_n)) = 0 \\
&\text{for all } (t_1, \cdots, t_n) \in H^{(1)} \times \cdots \times H^{(n)}\} .
\end{aligned}$$

3. Kuranishi space

Here, after stating some definitions concerning deformations of compact complex manifolds, we state the fundamental existence theorem of Kuranishi.

Definition (3.1). By a family of compact complex analytic manifolds we mean a triple (X, π, V) of reduced analytic spaces X, V and a proper surjective holomorphic map $\pi: X \to V$ satisfying the following property (*): For each point $p \in X$ we can find an open analytic subspace Y of X containing p, an open analytic subspace V' of V containing $\pi(p)$, a domain U' in a complex Euclidian space C^n, and a complex analytic isomorphism $h: Y \to U' \times V'$ such that $\pi(q) = \rho(h(q))$ for all q in Y, where ρ is the projection to V' of $U' \times V'$.

When π satisfies the above property (*), we say it is simple. Thus a family of compact complex manifolds means a triple (X, π, V) of reduced analytic spaces X, V and a proper, surjective, simple holomorphic map $\pi: X \to V$. It can be seen that $\pi^{-1}(t)$ for each $t \in V$ is a compact complex manifold.

Definition (3.2). A family of deformations of a fixed compact complex manifold M is a family (X, π, V) with a distinguished point $0 \in V$, together with an isomorphism $i: M \to \pi^{-1}(0)$.

Definition (3.3). A family of deformations (X, π, V) is said to be effectively parametrized at $0 \in V$ if the Kodaira-Spencer map $\rho_0: T_0V \to H^1(\pi^{-1}(0), \theta_\pi - 1_{(0)})$ is injective, where T_0V is [1] the Zariski tangent space of V at 0, and $\theta_\pi - 1_{(0)}$ is the sheaf of germs of holomorphic vector fields on $\pi^{-1}(0)$.

Definition (3.4). A family (X, π, V) of deformations of M is said to be a complex analytic universal family at the reference point 0 if for any family (X', π', V') of deformations of M with reference point $0'$, there are holomorphic maps $f\colon U \to V$, $\tilde{f}\colon \pi'^{-1}(U) \to X$ such that $\tilde{f}$ maps $\pi'^{-1}(t)$ holomorphically isomorphic to $\pi^{-1}(f(t))$ for each $t \in U$, where U is an open neighborhood of $0'$ in V' such that $f(0') = 0$.

The existence of a holomorphic family of deformations of a compact complex manifold M for which $H^2(M, \theta) = 0$ was proved by Kodaira-Nirenberg-Spencer [5]:

Theorem (3.1) *Kodaira-Nirenberg-Spencer*). *Let M be a compact complex manifold. If $H^2(M, \theta) = 0$, then there exists a complex analytic family $(X, \pi, V, 0)$ of deformations of M such that ρ_0 maps the tangent space T_0V isomorphically onto $H^1(M, \theta)$, where X and V are complex manifolds.*

Kodaira-Spencer [8] proved that the above family is universal at 0. Kuranishi obtained a theorem in [9] more general than Theorem (3.1) of Kodaira-Nirenberg and Spencer. This theorem is concerned with normal families of deformations, a concept which we define below.

Definition (3.5). Let (X, π, V) be a family of compact complex manifolds where X and V are complex manifolds. Let $M_t = \pi^{-1}(t)$ for $t \in V$, and assume for each t a subset $K(t)$ of $H^p(A(M_t))$ is given. Then the family $\{K(t)\}$ is said to be coherent at t_0 if and only if the following condition is satisfied: For each cocycle β in $A^p(M_{t_0})$ representing an element in $K(t_0)$, there is a family β_t depending differentiably no t and defined for t sufficiently near t_0 such that $\beta_{t_0} = \beta$ and that β_t is a cocycle representing an element in $K(t)$.

Definition (3.6). Let (X, π, V) be a family of compact complex manifolds where X and V are complex manifolds. Let $M_t = \pi^{-1}(t)$, $t \in V$. Let $H^1(A(M_t)) \supseteq H^2(A(M_t)) \supseteq \cdots$ be the normal filtration of $H^1(A(M_t))$. Then the family (X, π, V) is said to be normal at $t_0 \in V$ if and only if the family $\{H^{(r)}(A(M_t))\}$ is coherent at t_0 for each $r = 1, 2, \cdots$.

Kuranishi [9] obtained the following very interesting properties of normal families of deformations:

Theorem (3.2). *Let (X, π, V) be a family of compact complex manifolds which is normal at $t_0 \in V$. Then the image of the infinitesimal deformation of the family at t_0 is in $H^*(M_{t_0}) = \bigcap_{r=1}^{\infty} H^{(r)}(A(M_{t_0})$.*

Theorem (3.3). *Let M be a compact complex manifold. If $H^2(M, \theta) = 0$, then any family $(X, \pi, V, 0)$ of deformations of M is normal at $0 \in V$.*

The following theorem gives a generalization of Theorem (3.1):

Theorem (3.4). *For any compact complex manifold M, there exists a holomorphic family $(X, \pi, V, 0)$ of deformations of M, which is normal at 0, such that the infinitesimal deformation at 0 is a bijective mapping to $H^*(M) = \bigcap_{r=1}^{\infty} H^{(r)}(A(M))$.*

The most important existence theorem in deformation theory is the theorem on the existence of a universal and effective family of deformations for a compact complex manifold. Kuranishi proved this theorem in [10], and gave a different proof in [12]. One may find the proof of this theorem with complete details in [13].

Theorem (3.5) (*Kuranishi*), (*Fundamental existence theorem of deformation theory*). *For any compact complex manifold there exists a universal and effective family of deformations.*

It becomes necessary for us to state the main ideas which Kuranishi used to prove this theorem, as our main theorem is based on these.

Let us fix a hermitian metric on M, and let (θ, ψ) be the L_2 inner product of θ, $\psi \in A^p$. With respect to the fixed hermitian metric, we have the formal adjoint operator δ of $\bar{\partial}$, which is characterized by $(\bar{\partial}\theta, \psi) = (\theta, \delta\psi)$. We then define the complex Laplace-Beltrami operator $\square = \bar{\partial}\delta + \delta\bar{\partial}$, which plays an important role. The fact that the metric is hermitian implies that $\square\theta$ is of the same type as θ. A form θ is said to be harmonic if $\square\theta = 0$; or equivalently $\bar{\partial}\theta = 0 = \delta\theta$. The fact that the operator $\square$ is a strongly elliptic second order operator implies that the space H^p of harmonic forms in A^p is finite dimensional. Also we can establish the existence of the harmonic projection operator P and the corresponding Green's operator G, yielding the Hodge decomposition:

$$\theta = P\theta + \bar{\partial}Q\theta + Q\bar{\partial}\theta \ , \qquad \text{where } Q = \delta G,\ \theta \in A^p \ . \tag{3.1}$$

Also with respect to the Sobolov's k-norm, we have for any $\theta \in A^p$

$$|P\theta|_k \leq c\,|\theta|_k \ , \qquad |Q\theta|_{k+1} \leq c\,|\theta|_k \ . \tag{3.2}$$

where c is a constant.

Using these tools and the implicit mapping theorem in Banach spaces, Kuranishi proved that there exist an open neighborhood W of the origin in H^1 and a complex analytic injective mapping $\Phi\colon W \to A^1$ such that $\{\Phi(t), t \in T\}$, where T is the analytic set in W defined by

$$T = \{s \in W \,|\, P[\Phi(s), \Phi(s)]\} = 0 \ , \tag{3.3}$$

represents a complex analytic universal and effective family of deformations of M. T is called the Kuranishi space.

The mapping Φ is of vital importance to us, and is called the canonical universal map. Our main theorem gives a characterization of the normal filtration of $H^1(M, \theta)$ in terms of Φ. We give below some important properties of this map Φ.

By construction, Φ satisfies the following identity:

$$\Phi(t) - \tfrac{1}{2}Q[\Phi(t), \Phi(t)] \equiv t \qquad \text{for all } t \text{ in } W \ . \tag{3.4}$$

Let U be an open subset of M with complex analytic chart $Z = (Z^1, \cdots, Z^n)$. Then locally

$$(3.5) \qquad \Phi(t) = \sum_{\alpha,\beta} \Phi_\alpha^\beta(Z,t) d\bar{Z}^\alpha \cdot \partial/\partial Z^\beta ,$$

where $\Phi_\alpha^\beta(Z,t)$ is C^∞ in Z and t. Also the local expression for $[\Phi(t),\Phi(t)]$ is given by

$$(3.6) \qquad [\Phi(t),\Phi(t)] = \sum_{\alpha_1,\alpha_2,\beta} \Phi_{\alpha_1,\alpha_2}^\beta(Z,t) d\bar{Z}^{\alpha_1} \wedge d\bar{Z}^{\alpha_2} \cdot \partial/\partial Z^\beta ,$$

where $\Phi_{\alpha_1,\alpha_2}^\beta(Z,t)$ is C^∞ in Z and t, and is skewsymmetric in α_1 and α_2.

Lemma (3.1). *For any integer $l \geq 1$, we have*

$$\bar{\partial}\Phi(t) - \tfrac{1}{2}[\Phi(t),\Phi(t)] \equiv \tilde{Q}_{\Phi(t)}^l \bar{\partial}\Phi(t) - \tfrac{1}{2}\sum_{k=0}^{l-1} Q_{\Phi(t)}^k P[\Phi(t),\Phi(t)] ,$$

where the operator $\tilde{Q}_\xi$ is defined by $\tilde{Q}_\xi \eta = -Q[\eta,\xi]$ and $\eta \in A^p$, and Q_ξ^k is the k-fold composition of $\tilde{Q}_\xi$.

Proof. By (3.4) we have for all t in W

$$\bar{\partial}\Phi(t) - \tfrac{1}{2}\bar{\partial}Q[\Phi(t),\Phi(t)] \equiv 0 .$$

Use of the Hodge decomposition (3.1) gives

$$\begin{aligned}\bar{\partial}\Phi(t) - \tfrac{1}{2}[\Phi(t),\Phi(t)] &\equiv -\tfrac{1}{2}Q\bar{\partial}[\Phi(t),\Phi(t)] - \tfrac{1}{2}P[\Phi(t),\Phi(t)] \\ &\equiv -Q[\bar{\partial}\Phi(t),\Phi(t)] - \tfrac{1}{2}P[\Phi(t),\Phi(t)] \\ &\qquad\qquad \text{(by (1.1) and (1.2))} \\ &\equiv \tilde{Q}_{\Phi(t)}^1 \bar{\partial}\Phi(t) - \tfrac{1}{2}P[\Phi(t),\Phi(t)] .\end{aligned}$$

Now writing

$$\bar{\partial}\Phi(t) \equiv \tfrac{1}{2}[\Phi(t),\Phi(t)] + \tilde{Q}_{\Phi(t)}^1 \bar{\partial}\Phi(t) - \tfrac{1}{2}P[\Phi(t),\Phi(t)]$$

and observing that $[[\Phi(t),\Phi(t)],\Phi(t)] = 0$, we have

$$\bar{\partial}\Phi(t) - \tfrac{1}{2}[\Phi(t),\Phi(t)] \equiv \tilde{Q}_{\Phi(t)}^2 \bar{\partial}\Phi(t) - \tfrac{1}{2}\sum_{k=0}^{1} \tilde{Q}_{\Phi(t)}^k P[\Phi(t),\Phi(t)] .$$

On iteration we get the required result.

PART II. CHARACTERIZATION OF THE NORMAL FILTRATION OF $H^1(M,\theta)$

1. Statement of the main theorem

Throughout this part M will denote a compact complex manifold, and θ the sheaf of germs of holomorphic vector fields. In § 2 of Part I we have defined the normal filtration of $H^1(M,\theta)$:

$$H^1(M,\theta) = H^{(1)} \supseteq H^{(2)} \supseteq \cdots .$$

In § 3 of Part I we have defined the canonical universal map $\Phi: W \to A^1$, where W is an open neighborhood of the origin in $H^1(M,\theta)$. Our main theorem gives a characterization of the normal filtration of $H^1(M,\theta)$ in terms of Φ.

We know that $H^1(M,\theta)$ is a finite dimensional vector space. Let m be the dimension of $H^1(M,\theta)$, and $(S^1, \cdots, S^m)$ a linear chart of $H^1(M,\theta)$. For $t = (t^1, \cdots, t^m) \in H^1(M,\theta)$ we define a differential operator $D_t = t^1\partial/\partial S^1 + \cdots + t^m\partial/\partial S^m$. It is well known that D_t is defined independently of the choice of the chart $(S^1, \cdots, S^m)$.

Define a new filtration of $H^1(M,\theta)$:

$$H^1(M,\theta) = H^{[1]} \supseteq H^{[2]} \supseteq \cdots$$

and a sequence of maps $\Phi^{(n-2)}: H^{[1]} \to A^1$, $n = 1, 2, \cdots$ by induction of n as follows:

Define $H^{[1]} = H^1(M,\theta)$ and $\Phi^{(-1)} = \Phi$, and assume that $H^{[1]}, \cdots, H^{[n]}$; $\Phi^{(-1)}, \Phi^{(0)}, \cdots, \Phi^{(n-2)}$ are defined. We are going to construct $H^{[n+1]}$ and $\Phi^{(n-1)}$.

Definition (1.1). A linear subspace L of $H^{[n]}$ is called an allowable subspace when there is a polynomial map

$$\mu: H^{[1]} \to H^{[1]} \tag{1.1}$$

of the form

$$\mu(t) = t + \mu_2^{(n-1)}(t) + \cdots + \mu_n^{(1)}(t) , \tag{1.2}$$

where $\mu_r^{(s)}(t)$ is a homogeneous map of degree r with values in $H^{[s]}$ satisfying the following condition:

$$PD_{t_1,\cdots,t_{n+1}}[\Phi^{(n-2)} \circ \mu, \Phi^{(n-2)} \circ \mu] = 0 \tag{1.3}$$

for all $(t_1, \cdots, t_{n+1}) \in H^{[1]} \times \cdots \times H^{[n]} \times L$.

Then we prove

Propositin (1.1). *If L_1 and L_2 are allowable subspaces of $H^{[n]}$, then $L_1 + L_2$ is also an allowable subspace of $H^{[n]}$.*

By the Proposition there is therefore the unique maximal allowable subspace of $H^{[n]}$ which we define to be $H^{[n+1]}$. Let μ^{n-1} be a map as in (1.1) corresponding to $H^{[n+1]}$, and set $\Phi^{(n-1)} = \Phi^{(n-2)} \circ \mu^{n-1}$. Then our main theorem is the following:

Main Theorem. *The two filtrations*

$$H^1(M,\theta) = H^{(1)} \supseteq H^{(2)} \supseteq \cdots , \qquad H^1(M,\theta) = H^{[1]} \supseteq H^{[2]} \supseteq \cdots$$

coincide.

2. Statements of a lemma and a proposition

To prove our main theorem we need a lemma and a proposition, which are stated in this section and are proved in § 4.

Lemma (2.1). *Let $H^{[s]}$, $\Phi^{(s-2)}$ be defined for $s \leq n$ as in* § 1. *Assume that $H^{[s]} = H^{(s)}$ for all $s \leq n$. Let*

$$\theta^{r-2}\colon H^{(1)} \to H^{(1)} \qquad (r = 3, \cdots, n)$$

be a polynomial map of the form

$$\theta^{r-2}(t) = t + \theta_2^{(r-2)}(t) + \cdots + \theta_{r-1}^{(1)}(t) , \tag{2.1}$$

where $\theta_l^{(s)}(t)$ is a homogeneous map of degree l with values in $H^{(s)}$. Assume that

$$PD_{t_1,\cdots,t_r}[\Psi^{(r-2)}, \Psi^{(r-2)}] = 0 \tag{2.2}$$

for all $(t_1, \cdots, t_r) \in H^{(1)} \times \cdots \times H^{(r)}$, $(r = 1, \cdots, n)$, where

$$\Psi^{(r-2)} = \Phi \circ \theta^1 \circ \cdots \circ \theta^{r-2} , \qquad \Psi^{(0)} = \Psi^{(-1)} = \Phi . \tag{2.3}$$

Then the sequence

$$D_{t_1}\Psi^{(n-2)}, \cdots, D_{t_1,\cdots,t_n}\Psi^{(n-2)} \tag{2.4}$$

is an ND-sequence of length n over $H^{(1)}$.

Proposition (2.1). *Let $H^{[s]}$, $\Phi^{(s-2)}$ be defined for $s \leq n$ as in* § 1. *Assume that $H^{[s]} = H^{(s)}$ for all $s \leq n$. Let $\theta^{r-2}\colon H^1 \to W$ $(r = 3, \cdots, n)$ be a map satisfying* (2.1) *and* (2.2). *Assume that the sequence* (2.4) *is an ND-sequence of length r. Then for $r = 1, \cdots, n$ and $(t_1, \cdots, t_{r+1}) \in H^{(1)} \times \cdots \times H^{(r)} \times H^{(r)}$, we have*

$$\begin{aligned} D_{t_1,\cdots,t_{r+1}}[\Psi^{(r-1)}, \Psi^{(r-1)}] \simeq 2K_R(t_1, \cdots, t_{r+1}; \theta_r^{(1)}(t_2, \cdots, t_{r+1}) , \\ \cdots, \theta_2^{(r-1)}(t_r, t_{r+1}) \qquad (\mathrm{mod}\, B^r). \end{aligned} \tag{2.3}$$

Here B^r is the subspace of A^2 generated by all elements of the form $D_{s_1,\cdots,s_q}[\Psi^{(q-2)}, \Psi^{(q-2)}]$ where $(s_1, \cdots, s_q) \in H^{(1)} \times \cdots \times H^{(q)}$, $q \leq r$.

3. Construction of the filtration $H^1(M,\theta) = H^{[1]} \supseteq H^{[2]} \supseteq \cdots$ and proof of the main theorem

By definition $H^{[1]} = H^1(M,\theta)$ and $\Phi^{(-1)} = \Phi$.

Construction of $H^{[2]}$ and proof of $H^{[2]} = H^{(2)}$. By our construction, $H^{[2]}$ is the maximal subspace of $H^{[1]}$ such that for all $(t_1, t_2) \in H^{[1]} \times H^{[2]}$, $PD_{t_1,t_2}[\Phi, \Phi] = 0$ and $\Phi^{(0)} = \Phi$. According to Kuranishi's construction,

$$(3.1) \qquad H^{(2)} = \{t_2 \in H^{(1)} \mid P[t_1, t_2] = 0\} \qquad \text{for all } t_1 \in H^{(1)} .$$

Since $D_{t_1,t_2}[\Phi, \Phi] = 2[t_1, t_2]$, it is clear that $PD_{t_1,t_2}[\Phi, \Phi] = 0$ if and only if $P[t_1, t_2] = 0$. Hence we have proved that $H^{[2]} = H^{(2)}$. In other words, we have

Proposition (3.1). *$t \in H^{[2]} = H^{(2)}$ if and only if $P[t, u] = 0$ for every $u \in H^{(1)}$.*

Construction of $H^{[3]}$. We have $H^{[1]} = H^{(1)}$, $H^{[2]} = H^{(2)}$ and $\Phi^{(-1)} = \Phi^{(0)} = \Phi$. We are going to construct $H^{[3]}$ and $\Phi^{(1)}$. Let $V \subseteq H^{[2]}$ be an allowable subspace. This means, according to Definition (1.1), that there exists a map:

$$(3.2) \qquad \mu\colon H^{(1)} \to H^{(1)}$$

of the form

$$(3.3) \qquad \mu(t) = t + \mu_2^{(1)}(t) .$$

where $\mu_2^{(1)}(t)$ is a homogeneous map of degree 2 with values in H^1, such that

$$(3.4) \qquad PD_{t_1,t_2,t_3}[\Phi \circ \mu, \Phi \circ \mu] = 0 .$$

Now by Lemma (2.1), $\{D_{t_1}\Phi, D_{t_1,t_2}\Phi\}$ is an ND-sequence $R = \{R_1, R_2\}$ of length 2 over H^1. Thus according to Proposition (2.1), for all $(t_1, t_2, t_3) \in H^{(1)} \times H^{(2)} \times V$. we have

$$(3.5) \qquad D_{t_1,t_2,t_3}[\Phi \circ \mu, \Phi \circ \mu] \simeq 2K_R(t_1, t_2, t_3\,;\, \mu_2^{(1)}(t_2, t_3)) \qquad (\text{mod } B^2) .$$

By observing the definition of B^2 and applying the previous case (namely, the case $n = 2$), we find that $P(B^2) = 0$. Hence

$$(3.6) \qquad PD_{t_1,t_2,t_3}[\Phi \circ \mu, \Phi \circ \mu] = 0 \qquad \rightleftarrows PK_R(t_1, t_2, t_3\,;\, \mu_2^{(1)}(t_2, t_3)) = 0 .$$

Expanding $K_R(t_1, t_2, t_3\,;\, \mu_2^{(1)}(t_2, t_3))$ by means of (2.5) of Part I, we see that V is an allowable subspace of $H^{(2)}$ if and only if there exists a symmetric bilinear map $\mu_2^{(1)}\colon H^{(2)} \times V \to H^{(1)}$ such that

$$(3.7) \qquad -P \sum [R_1(t_1), R_2(t_2, t_3)] = P[t_1, \mu_2^{(1)}(t_2, t_3)] \quad \text{for all } (t_2, t_3) \in H^{(2)} \times V .$$

By denoting the left hand side of (3.7) by $\Delta(t_1, t_2, t_3)$, we have a symmetric trilinear map

$$\Delta\colon H^{(1)} \times H^{(2)} \times V \to H^2(M,\theta) .$$

Thus we have proved that V is an allowable subspace of $H^{(2)}$ if and only if there exists a symmetric bilinear map $\mu_2^{(1)}\colon H^{(2)} \times V \to H^{(1)}$ such that

$$\Delta(t_1, t_2, t_3) = P[t_1, \mu_2^{(1)}(t_2, t_3)] \qquad \text{for all } (t_2, t_3) \in H^{(2)} \times V . \tag{3.8}$$

We want to prove now the following proposition.

Proposition (3.2). *If V_1, V_2 are allowable subspaces of $H^{(2)}$, then $V_1 + V_2$ is also an allowable subspace of $H^{(2)}$.*

This proposition follows immediately from the following Lemma.

Lemma (3.1). *V is an allowable subspace of $H^{(2)}$ if and only if $\nabla(H^{(2)} \times V) \subseteq \Omega$, where*

$$\nabla\colon H^{(2)} \times H^{(2)} \to L(H^{(1)}, H^2(M,\theta)) \tag{3.9}$$

is a symmetric bilinear map defined by $\nabla(t_2, t_3)(t_1) = \Delta(t_1, t_2, t_3)$ for all $(t_2, t_3) \in H^{(2)} \times H^{(2)}$ and all $t_1 \in H^{(1)}$, and Ω is a linear subspace of $L(H^{(1)}, H^2(M,\theta))$ defined by

$$\Omega = \{\lambda \in L(H^{(1)}, H^2(M,\theta)) \mid \text{there is } h_1 \text{ in } H^{(1)} \text{ such that } \lambda(t) = P[t, h_1]\} . \tag{3.10}$$

Proof of Lemma (3.1). Let V be an allowable subspace of $H^{(2)}$. Then by the definitions of ∇ and Ω, we get

$$\nabla(H^{(2)} \times V) \subseteq \Omega . \tag{3.11}$$

Conversely, let $V \subseteq H^{(2)}$ satify the above condition. We have to prove that there exists $\mu_2^{(1)}\colon H^{(2)} \times V \to H^{(1)}$ such that (3.8) is satisfied. We now define a map $\sigma\colon H^{(1)} \to \Omega$ by

$$\sigma(h_1)(h) = P[h, h_1] \qquad \text{for all } h,\ h_1 \in H^{(1)} . \tag{3.12}$$

Since this is a surjective linear mapping, there exists a linear mapping $\tilde{\mu}\colon \Omega \to H^{(1)}$ such that $\sigma \circ \tilde{\mu} = $ identity. Extend $\tilde{\mu}$ to $L(H^{(1)}; H^2(M,\theta))$, and denote the extension by μ. By assumption, for all $(t_2, t_3) \in H^{(2)} \times V$ we have $\nabla(t_2, t_3) \in \Omega$. Hence we can define a map $\mu_2^{(1)}\colon H^{(2)} \times V \to H^{(1)}$ by

$$\mu_2^{(1)}(t_2, t_3) = \mu \circ \nabla(t_2, t_3) . \tag{3.13}$$

As ∇ is symmetric, $\mu_2^{(1)}$ is also symmetric, and by its construction we easily see that it satisfies (3.8). This completes the roof of Lemma (3.1).

Proposition (3.2) is an immediate consequence of Lemma (3.1). From Proposition (3.2) it follows that there is the unique maximal allowable subspace of $H^{(2)}$, which is defined to be $H^{[3]}$. Let μ^1 be a map as in (3.2) corresponding to $H^{[3]}$. Then define $\Phi^{(1)} = \Phi^{(0)} \circ \mu^1$.

Proof of $H^{[3]} = H^{(3)}$. Let $t_3 \in H^{[3]}$. Then there exists a map $\mu_2^{(1)}: H^{(2)} \times H^{[3]} \to H^{(1)}$ such that

$$PK_R(t_1, t_2, t_3; \mu_2^{(1)}(t_2, t_3)) = 0 , \qquad (t_1, t_2) \in H^{(1)} \times H^{(2)} .$$

Define $S_2^{t_3}: H^{(2)} \to H^{(1)}$ by $S_2^{t_3}(t_2) = \mu_2^{(1)}(t_2, t_3)$. Then Theorem (2.1) of Part I is satisfied, and hence $t_3 \in H^{(3)}$.

By retracing the steps the converse is easily seen. Thus $H^{[3]} = H^{(3)}$, and hence we have proved

Proposition (3.3). *$t \in H^{[3]} = H^{(3)}$ if and only if there exists a map $\mu^1: H^{(1)} \to H^{(1)}$ of the form $\mu^1(t) = t + \mu_2^{(1)}(t)$ such that $PD_{t_1,t_2,t_3}[\Phi \circ \mu^1, \Phi \circ \mu^1] = 0$, for all $(t_1, t_2) \in H^{(1)} \times H^{(2)}$.*

General case. We assume that $H^{[1]}, \cdots, H^{[n]}; \Phi^{(-1)}, \cdots, \Phi^{(n-2)}$ are constructed, and also that $H^{[r]} = H^{(r)}$ for $1 \leq r \leq n$. We are going to construct $H^{[n+1]}$ and $\Phi^{(n-1)}$ and prove $H^{[n+1]} = H^{(n+1)}$.

Remark. The construction of $H^{[3]}$ does not clearly indicate how the construction should go in the general case. The construction becomes quite complicated even in the case of $H^{[4]}$. Once we construct $H^{[4]}$, we see the general pattern of construction for $H^{[n+1]}$. Hence it should be remarked that we get the motivation for the various steps of construction of $H^{[n+1]}$ from the corresponding steps for the construction of $H^{[4]}$.

Let $V \subseteq H^{(n)}$ be an allowable subspace of $H^{(n)}$. This means that there exists a map

$$\mu: H^{(1)} \to H^{(1)} \tag{3.14}$$

of the form

$$\mu(t) = t + \mu_2^{(n-1)}(t) + \cdots + \mu_n^{(1)}(t) \tag{3.15}$$

such that for $(t_1, \cdots, t_n, t_{n+1}) \in H^{(1)} \times \cdots \times H^{(n)} \times V$

$$PD_{t_1,\cdots,t_{n+1}}[\Phi^{(n-2)} \circ \mu, \Phi^{(n-2)} \circ \mu] = 0 . \tag{3.16}$$

Since all the assumptions of Lemma (2.1) are satisfied, $D_{t_1}\Phi^{(n-2)}$ and $D_{t_1,\cdots,t_n}\Phi^{(n-2)}$ form a symmetric ND-sequence $R = \{R_1, \cdots, R_n\}$ of length n over $H^{(1)}$. Thus by Proposition (2.1) for all $(t_1, \cdots, t_{n+1}) \in H^{(1)} \times H^{(2)} \times \cdots \times H^{(n)} \times V$,

$$\begin{aligned} D_{t_1,\cdots,t_{n+1}}[\Phi^{(n-2)} \circ \mu, \Phi^{(n-2)} \circ \mu] \simeq 2K_R(t_1, \cdots, t_{n+1}; \mu_n^{(1)}(t_2, \cdots, t_{n+1}), \\ \cdots, \mu_2^{(n-1)}(t_n, t_{n+1})) , \qquad \bmod (B^2, \cdots, B^n) . \end{aligned} \tag{3.17}$$

By induction assumptions we have $P(B^r) = 0$ for all r, $2 \leq r \leq n$, so that

$$\begin{aligned} PD_{t_1,\cdots,t_{n+1}}[\Phi^{(n-2)} \circ \mu, \Phi^{(n-2)} \circ \mu] = 0 \\ \rightleftarrows PK_R(t_1, \cdots, t_{n+1}; \mu_n^{(1)}(t_2, \cdots, t_{n+1}), \cdots, \mu_2^{(n-1)}(t_n, t_{n+1})) = 0 . \end{aligned} \tag{3.18}$$

By expanding K_R and using (2.5) of Part I, we see that V is an allowable subspace of $H^{(n)}$ if and only if there exist symmetric multilinear maps

$$(3.19)\quad \mu_{n-k+2}^{(k-1)}: H^{(k)} \times \cdots \times H^{(n)} \times V \to H^{(k-1)}\,, \qquad k = 2, \cdots, n$$

such that

$$(3.20)\quad \begin{aligned} -P\sum_{p=0}^{n-1} \sum_{1\le i_1<\cdots<i_q\le n} [R_{q+1}(t_{i_1}, \cdots, t_{i_q}, t_{n+1}), R_{n-q}(t_i, \cdots, t_{i_q}, \cdots, t_n)] \\ = P\sum_{k=1}^{n-1}\sum_{q=0}^{k-1} \sum_{1\le i_1<\cdots<i_q\le k} [R_{q+1}(t_{i_1}, \cdots, t_{i_q}, \mu_{n-k+1}^{(k)}(t_{k+1}, \cdots, t_{n+1}))\,, \\ R_{k-q}(t_1, \cdots, \hat{t}_{I_q}, \cdots, t_k)]\,, \end{aligned}$$

where $\hat{t}_{I_q}$ means the omission of $t_{i_1}, \cdots, t_{i_q}$.

Denoting the expression on the left hand side of (3.20) by $\Delta(t_1, \cdots, t_{n+1})$, we see that

$$(3.21)\quad \Delta: H^{(1)} \times \cdots \times H^{(n)} \times V \to H^2(M, \theta)$$

is a symmetric $(n+1)$-linear map. Thus $V \subseteq H^{(n)}$ is an allowable subspace if and only if there exist symmetric multilinear maps

$$(3.22)\quad \mu_{n-k+2}^{(k-1)}: H^{(k)} \times \cdots \times H^{(n)} \times V \to H^{(k-1)}\,, \qquad k = 2, \cdots, n\,,$$

such that

$$(3.23)\quad \begin{aligned} \Delta(t_1, \cdots, t_{n+1}) = P\sum_{k=1}^{n-1}\sum_{q=0}^{k-1} \sum_{1\le i_1<\cdots<i_q\le k} [R_{q+1}(t_{i_1}, \cdots, t_{i_q}\,, \\ \mu_{n-k+1}^{(k)}(t_{k+1}, \cdots, t_{n+1})), R_{k-q}(t_1, \cdots, \hat{t}_{I_q}, \cdots, t_k)]\,, \end{aligned}$$

where $\hat{t}_{I_q}$ means the omission of $t_{i_1}, \cdots, t_{i_q}$.

Proposition (3.3). *If V_1, V_2 are two allowable subspaces of $H^{(n)}$, then $V_1 + V_2$ is also an allowable subspace of $H^{(n)}$.*

This proposition follows from the following Lemma 3.3.

Lemma (3.3). *V is an allowable subspace of $H^{(n)}$ if and only if $\nabla(H^{(n)} \times V) \subseteq \Omega$ where*

$$\nabla: H^{(n)} \times H^{(n)} \to L(H^{(1)} \times \cdots \times H^{(n-1)}, H^2(M, \theta))$$

is defined by

$$(3.24)\quad \nabla(t_n, t_{n+1})(t_1, \cdots, t_{n-1}) = \Delta(t_1, \cdots, t_{n+1})$$

for all $(t_n, t_{n+1}) \in H^{(n)} \times H^{(n)}$ and $(t_1, \cdots, t_{n-1}) \in H^{(1)} \times \cdots \times H^{(n-1)}$, and $\Omega \subseteq L(H^{(1)} \times \cdots \times H^{(n-1)}, H^2(M, \theta))$ is defined by

$$\Omega = \Big\{\lambda \,|\, h_k \in SL(H^{(k+1)} \times \cdots \times H^{(n-1)}, H^{(k)}),\ k = 1, \cdots, n-2$$

$$\text{and } h_{n-1} \in H^{(n-1)} \text{ such that } \lambda(t_1, \cdots, t_{n+1})$$

$$(3.25) \qquad = P \sum_{k=1}^{n-2} \sum_{q=0}^{k-1} \sum_{1 \le i_1 < \cdots < i_q \le k} [R(t_{i_1}, \cdots, t_{i_q}, h_k(t_{k+1}, \cdots, t_{n-1})) ,$$

$$R(t_1, \cdots, \hat{t}_{I_q}, \cdots, t_k)] + P \sum_{q=0}^{n-2} \sum_{1 \le i_1 < \cdots < i_q \le n-1} [R(t_{i_1}, \cdots, t_{i_q}, h_{n-1}) ,$$

$$R(t_1, \cdots, \hat{t}_{I_q}, \cdots, t_{n-1})]\Big\} .$$

Proof of Lemma (3.3). Let V be an allowable subspace of $H^{(n)}$. Then from the definitions of ∇, Ω and (3.23) it follows that $\nabla(H^n \times V) \subseteq \Omega$. Conversely, assume that V satisfies this condition. Then we prove that V is allowable by proving that there exist symmetric multilinear maps $\mu^{(k-1)}_{n-k+2}$ of the form (3.22), which satisfy (3.23).

By the definition of Ω we obtain a surjective linear map

$$(3.26) \qquad \begin{aligned} \sigma\colon H^{(n-1)} \oplus SL(H^{(n-1)}, H^{(n-2)}) \oplus SL(H^{(n-1)} \times H^{(n-2)}, H^{(n-3)}) \\ \oplus \cdots \oplus SL(H^{(n-1)} \times \cdots \times H^{(2)}, H^{(1)}) \to \Omega . \end{aligned}$$

Thus there exists a map

$$(3.27) \qquad \begin{aligned} \tilde{\mu}\colon \Omega \to H^{(n-1)} \oplus SL(H^{(n-1)}, H^{(n-2)}) \\ \oplus \cdots \oplus SL(H^{(n-1)} \times \cdots \times H^{(2)}, H^{(1)}) \end{aligned}$$

such that $\sigma \circ \tilde{\mu} =$ identity. Extend $\tilde{\mu}$ to $(H^{(1)} \times \cdots \times H^{(n-1)}, H^2(M, \theta))$ and denote the extension by μ. By assumption, $\nabla(t_n, t_{n+1}) \in \Omega$ for all $(t_n, t_{n+1}) \in H^{(n)} \times V$. Hence $\mu \circ \nabla(t_n, t_{n+1})$ is well defined. Let

$$(3.28) \qquad \mu \circ \nabla(t_n, t_{n+1}) = ((\mu \circ \nabla(t_n, t_{n+1}))_1, \cdots, (\mu \circ \nabla(t_n, t_{n+1}))_{n-1}) ,$$

where $(\mu \circ \nabla(t_n, t_{n+1}))_1 \in H^{(n-1)}$, and

$$(\mu \circ \nabla(t_n, t_{n+1}))_r \in SL(H^{(n-1)} \times \cdots \times H^{(n-r+1)}, H^{(n-r)}) \quad \text{for } r = 2, \cdots, n-1 .$$

Then we define

$$(3.29) \qquad \begin{aligned} \mu_2^{(n-1)}(t_n, t_{n+1}) &= (\mu \circ \nabla(t_n, t_{n+1}))_1 , \\ \mu_r^{(n-r+1)}(t_{n-r+2}, \cdots, t_{n+1}) &= (\mu \circ \nabla(t_n, t_{n+1}))_{(r-1)}(t_{n-r+2}, \cdots, t_{n-1}) \\ &\qquad \text{for } r = 3, \cdots, n . \end{aligned}$$

From the construction of these maps $\mu^{(k)}_{n-k+1}$ it follows that they satisfy (3.23) but may not be symmetric. We should note that $\mu^{(k)}_{n-k+1}(t_{k+1}, \cdots, t_{n+1})$ is symmetric in the last two variables t_n, t_{n+1} since ∇ is so, and it is also symmetric in the first $n-k-1$ variables since $(\mu \circ \nabla(t_n, t_{n+1}))_k$ is so. Hence, if only we can interchange t_s and t_r where $k+1 \leq s \leq n-1$ and $n \leq r \leq n+1$, then $\mu^{(k)}_{n-k+1}$ will be symmetric as seen from the following lemma.

Lemma (3.4). *Let $E^1, E^2, \cdots, E^n, F$ be vector spaces, and $f \in L(E^1, \cdots, E^n; F)$ be such that*

$$f(x_1, \cdots, x_r, \cdots, x_s, \cdots, x_n) = f(x_1, \cdots, x_s, \cdots, x_r, \cdots, x_n)$$

for $1 \leq r \leq s \leq n$ and $x_r, x_s \in E^{(s)}$. Then there exists a symmetric multilinear map

$$\bar{f}: E^1 \times \cdots \times E^1 \ (n \text{ factors}) \to F$$

such that $\bar{f}(t_1, \cdots, t_n) = f(t_1, \cdots, {}_n)$ for $(t_1, \cdots, t_n) \in E^1 \times \cdots \times E^n$.

We give a proof of this lemma in § 4. Hence to complete the proof of the lemma we have to prove the following proposition.

Proposition (3.4). *There exists a solution $\{\mu^{(k)}_{n-k+1}, k = 1, \cdots, n-1\}$ of* (3.23) *such that $\mu^{(k)}_{n-k+1}$ are all symmetric.*

Proof of Proposition (3.4). (3.23) can be written in the following form:

$$(3.30) \qquad \Delta(t_1, \cdots, t_{n+1}) = \sum_{k=1}^{n-1} M_k(t_1, \cdots, t_k, \mu^{(k)}_{n-k+1}(t_{k+1}, \cdots, t_{n+1})) ,$$

where

$$M_k: H^{(1)} \times \cdots \times H^{(k)} \times H^{(k)} \to H^2(M, \theta)$$

is defined by

$$M_k(t_1, \cdots, t_k, t'_k) = P \sum_{q=0}^{k-1} \sum_{1 \leq i_1 < \cdots < i_q \leq k} [R_{q+1}(t_1, \cdots, t_{i_q}, t'_k) , \\ R_{k-q}(t_1, \cdots, \hat{t}_{i_1}, \cdots, \hat{t}_{i_q}, \cdots, t_k)] .$$

For a fixed $\alpha \in H^{(k)}$, we define

$$M^\alpha_k: H^{(1)} \times \cdots \times H^{(k)} \to H^2(M, \theta)$$

by

$$M^\alpha_k(t_1, \cdots, t_k) = M_k(t_1, \cdots, t_k, \alpha) .$$

Then it is clear that $M^\alpha_k \in SL(H^{(1)}, \cdots, H^k; H^2(M, \theta))$. We now claim that we can find ${}^*\mu^{(k)}_{n-k+1} \in SL(H^{(k+1)}, \cdots, H^{(n)}, V; H^{(k)})$ such that they satisfy (3.30).

Take a basis $e_1, \cdots, e_m$ of $H^1(M, \theta)$ such that $e_{a(j)+1}, \cdots, e_m$ is a basis of $H^{(j)}$ and $e_{a(n+1)+1}, \cdots, e_m$ is a basis of V for a sequence $a(1) \leq \cdots \leq a(n+1)$.

Extend $\mu^{(k)}_{n-k+1}$ to multilinear mappings of $H^1 \times \cdots \times H^1$ ($n-k+1$ factors) into $H^{(k)}$. Then there exists $^*\mu^{(k)}_{n-k+1} \in SL(H^{(1)} \times \cdots \times H^{(1)}; H^{(k)})$, ($n-k+1$ factors of $H^{(1)}$), such that

$$(3.31) \qquad {}^*\mu^{(k)}_{n-k+1}(e_{\lambda_1}, \cdots, e_{\lambda_{n-k+1}}) = \mu^{(k)}_{n-k+1}(e_{\lambda_1}, \cdots, e_{\lambda_{n-k+1}})$$

for any $1 \leq e_{\lambda_1} < \cdots < e_{\lambda_{n-k+1}} \leq m$. We prove that the restriction of $^*\mu^{(k)}_{n-k+1}$ to $H^{(k+1)} \times \cdots \times H^{(n-1)}$ satisfy (3.30). Let $t_i \in H^{(i)}$, $i = 1, \cdots, n$, and $t_{n+1} \in V$. Let $t_i = \sum_{\lambda_i} c_i^{\lambda_i} e_{\lambda_i}$. Then

$$(3.32) \qquad \begin{aligned} \varDelta(t_1, \cdots, t_{n+1}) &= \sum_{\lambda_1, \cdots, \lambda_{n+1}} c_1^{\lambda_1} \cdots c_{n+1}^{\lambda_{n+1}} \varDelta(e_{\lambda_1}, \cdots, e_{\lambda_{n+1}}) \\ &= \sum_{1 \leq \mu_1 < \cdots < \mu_{n+1} \leq m} \left(\sum_{\lambda_1, \cdots, \lambda_{n+1}} c_1^{\lambda_1} \cdots c_{n+1}^{\lambda_{n+1}} \right) \varDelta(e_{\mu_1}, \cdots, e_{\mu_{n+1}}) , \end{aligned}$$

where $(\lambda_1, \cdots, \lambda_{n+1})$ is a permutation of $(\mu_1, \cdots, \mu_{n+1})$.

There exists a permutation π of $(\mu_1, \cdots, \mu_{n+1})$ such that $\lambda_i = \pi(\mu_i)$ and $\lambda_i > a(i)$. This implies $\mu_i > a(i)$ for all i. Thus from (3.30) we have

$$(3.33) \qquad \begin{aligned} \varDelta(e_{\mu_1}, \cdots, e_{\mu_{n+1}}) &= \sum_{k=1}^{n-1} M_k(e_{\mu_1}, \cdots, e_{\mu_k}, \mu^{(k)}_{n-k+1}(e_{\mu_{k+1}}, \cdots, e_{\mu_{n+1}})) \\ &= \sum_{k=1}^{n-1} M_k(e_{\mu_1}, \cdots, e_{\mu_k}, {}^*\mu^{(k)}_{n-k+1}(e_{\mu_{k+1}}, \cdots, e_{\mu_{n+1}})) . \end{aligned}$$

Now if $(\lambda_1, \cdots, \lambda_k)$ is a permutation of $(\mu_1, \cdots, \mu_k)$, then $(\lambda_{k+1}, \cdots, \lambda_{n+1})$ must also be a permutation of $(\mu_{k+1}, \cdots, \mu_{n+1})$.

Since $^*\mu^{(k)}_{n-k+1}$ and $M_k{}^*\mu^{(k)}_{n-k+1}$ are symmetric, we have

$$(3.34) \qquad \varDelta(e_{\mu_1}, \cdots, e_{\mu(n+1)}) = \sum_{k=1}^{n-1} M_k(e_{\lambda_1}, \cdots, e_{\lambda_k}, {}^*\mu^{(k)}_{n-k+1}(e_{\lambda_{k+1}}, e_{\lambda_{n+1}})) .$$

If $(\lambda_1, \cdots, \lambda_k)$ is not a permutation of $(\mu_1, \cdots, \mu_k)$, then by induction assumption both $M_k(e_{\lambda_1}, \cdots, e_{\lambda_k}, {}^*\mu^{(k)}_{n-k+1}(e_{\lambda_{k+1}}, e_{\lambda_{n+1}}))$ and $M_k(e_{\mu_1}, \cdots, e_{\mu_k}, {}^*\mu^{(k)}_{n-k+1}(e_{\mu_{k+1}}, \cdots, e_{\mu_{n+1}}))$ vanish. Thus (3.34) holds in all cases. This shows that $^*\mu^{(k)}_{n-k+1}$ satisfy (3.30), and therefore Proposition (3.4) is proved.

By Proposition (3.3), there exists a unique maximal allowable subspace of $H^{(n)}$, which is defined to be $H^{[n+1]}$. Let μ^{n-1} be a map as in (3.14) corresponding to $H^{[n+1]}$, and define $\Phi^{(n-1)} = \Phi^{(n-2)} \circ \mu^{n-1}$. This completes the construction of $H^{[n+1]}$ and $\Phi^{(n-1)}$.

Proof of $H^{[n+1]} = H^{(n+1)}$. Let $t_{n+1} \in H^{[n+1]}$. Then for every $(t_1, \cdots, t_n) \in H^{(1)} \times \cdots \times H^{(n)}$ we have

$$PK_R(t_1, \cdots, t_{n+1}; \mu^{(1)}_n(t_2, \cdots, t_{n+1}), \cdots, \mu^{(n-1)}_2(t_n, t_{n+1})) = 0 .$$

Define

$$S_k^{t_{n+1}}: H^{(k)} \times \cdots \times H^{(n)} \to H^{(k-1)}, \qquad k = 2, \cdots, n,$$

by

$$S_k^{t_{n+1}}(t_k, \cdots, t_n) = \mu_{n-k+2}^{(k-1)}(t_k, \cdots, t_{n+1}), \qquad k = 2, \cdots, n.$$

Then by Theorem 2.1 of Part I we see that $t_{n+1} \in H^{(n+1)}$. Conversely, if $t_{n+1} \in H^{(n+1)}$, using Theorem 2.1 of Part I we can construct symmetric maps $\mu_{n-k+2}^{(k-1)}$ satisfying (3.23) proving that $t_{n+1} \in H^{[n+1]}$. Hence we have proved the following proposition.

Proposition (3.5). *$t \in H^{[n+1]} = H^{(n+1)}$ if and only if there exists a map $\mu: H^{(1)} \to H^{(1)}$ of the form* (3.15) *such that* (3.16) *holds for all*

$$(t_i, \cdots, t_n) \in H^{(1)} \times \cdots \times H^{(n)}.$$

This completes the proof of our main theorem.

Remark. (1) Since $H^1(M, \theta)$ is of finite dimension, the normal filtration of $H^1(M, \theta)$ must terminate. Hence there exists a positive integer l_0 such that

$$H^{(1)} \supseteq H^{(2)} \supseteq \cdots \supseteq H^{(l_0)} = H^{(l_0+1)} = \cdots.$$

(2) The maps $\mu^{n-1}(t) = t + \mu_2^{(n-1)}(t) + \cdots + \mu_{n-s+1}^{(s)}(t) + \cdots + \mu_n^{(1)}(t)$ can be so chosen that $\mu_{n-s+1}^{(s)}(t)$ is in the orthogonal complement of $H^{(s+1)}$ in $H^{(s)}$. Hence for any $l \geq l_0$, μ^{l-2} can be so chosen that $\mu_{l-l_0-r}^{(l_0+r)}(t) = 0$ for all $r = 0, 1, \cdots, l - l_0 - 2$.

4. Proofs of Lemma (2.1) and Proposition (2.1)

The proof of our main theorem depends very much on Lemma (2.1) and Proposition (2.1). We prove these as well as Lemma (3.4) in this section. The proof of Lemma (3.4) is quite elementary; however for the sake of completeness we give it here

Proof of Lemma (3.4). Let $\{e_1, \cdots, e_{r_1}\}$ be a basis of E^1 such that $\{e_1, \cdots, e_{r_k}\}$ is a basis of E^k. Define $\bar{f}$ such that $\bar{f}(e_{\nu_1}, \cdots, e_{\nu_n}) = f(e_{\nu_{\pi(1)}}, \cdots, e_{\nu_{\pi(n)}})$ if there exists a permutation π of $(1, \cdots, n)$ such that $(e_{\nu_{\pi(1)}}, \cdots, e_{\nu_{\pi(n)}}) \in E^1 \times \cdots \times E^n$. $\bar{f}$ is defined to be zero otherwise. The proof is complete, once we show that $\bar{f}$ is well defined. Let π, π' be two permutations of $\{1, \cdots, n\}$ such that both $(e_{\nu_{\pi(1)}}, \cdots, e_{\nu_{\pi(k)}})$ and $(e_{\nu_{\pi'(1)}}, \cdots, e_{\nu_{\pi'(n)}}) \in E^1 \times \cdots \times E^n$. Consider $f(e_{\nu_{\pi'(1)}}, \cdots, e_{\nu_{\pi'(n)}})$. We can assume that $\pi'(n) = \pi(n)$ by hypothesis. Let $\pi'(s) = \pi(s)$ for $s = r, \cdots, n$. Then using the hypothesis it is easily seen that $\pi'(r-1) = \pi(r-1)$. Thus

$$f(e_{\nu_{\pi(1)}}, \cdots, e_{\nu_{\pi(n)}}) = f(e_{\nu_{\pi'(1)}}, \cdots, e_{\nu_{\pi'(n)}}),$$

and $\bar{f}$ is well defined.

We now introduce some notations. Let $\Phi\colon H^{(1)} \to W$ be the canonical universal map. Then Φ can be written as

$$\Phi(t) = t + \Phi_2(t) + \Phi_3(t) + \cdots .$$

where $\Phi_r(t)$ is homogeneous of degree r. Let $\mu\colon H^{(1)} \to H^{(1)}$ be any polynomial map. Then we can write

$$\mu(t) = t + \mu_2(t) + \cdots + \mu_l(t) ,$$

where $\mu_r(t)$ is homogeneous of degree r. Let $(S^1, \cdots, S^m)$ be a chart of $H^{(1)}$, and let $t = (t_1^1, \cdots, t_1^m) \in H^{(1)}$. Consider the differential operator $D_{t_1} = t_1^1\partial/\partial S^1 + \cdots + t_1^m\partial/\partial S^m$. We denote $(D_{t_1}\mu)(0)$ by $\mu(t_1)$, and similarly for $t_1, \cdots, t_n$ in $H^{(1)}$, $\mu(t_1, \cdots, t_n)$ denotes $(D_{t_1,\cdots,t_n}\mu)(0)$. Also we recall that if f is any homogeneous map of degree l of $H^{(1)}$, then there exists a unique symmetric l-linear map F such that $F(t, \cdots, t) = l!f(t)$. Also it is easy to check that $F(t_1, \cdots, t_l) = (D_{t_1,\cdots,t_l}f)(0)$. By a decomposition $\underline{J}$ of $(t_1, \cdots, t_l)$ into k subsets we mean a collection $(J_1, \cdots, J_k)$ of k subsets of $(t_1, \cdots, t_l)$ such that they are disjoint and their union is $(t_1, \cdots, t_l)$. We consider two such decompositions $\underline{J}$, $\underline{J}'$ to be the same if and only if, as sets, $J_r = J'_r$ for $r = 1, \cdots, k$, where $\underline{J}' = (J'_1, \cdots, J'_k)$. Define an equivalence relation in the set of all decompositions of $(t_1, \cdots, t_l)$ into k subsets as follows. We say $\underline{J}$, $\underline{J}'$ are equivalent if and only if there exists a permutation π of $(1, \cdots, k)$ such that, as sets, $J_r = J'_{\pi(r)}$ for $r = 1, \cdots, k$. Denote by $\underline{I}$ an equivalence class of this type. $|J_r|$ denotes the number j_r of elements in J_r, and $J!$ denotes $j_1!\cdots j_k!$. Then we have the following result.

Lemma (4.1).

$$D_{t_1,\cdots,t_l}(\Phi\circ\mu) = \sum_{J} (1/J!)D_{\mu(\underline{J})}\Phi = \sum_{I} D_{\mu(I)}\Phi . \tag{4.1}$$

Proof. We can assume that Φ is homogeneous of degree h. Then there exists a unique symmetric h-linear map $\tilde{\Phi}$ such that $\Phi(t) = (1/h!)\tilde{\Phi}(t, \cdots, t)$. Thus $\Phi\circ\mu(t) = (1/h!)\tilde{\Phi}(\mu(t), \cdots, \mu(t))$. Similarly let $\tilde{\mu}_r$ denote the symmetric r-linear map corresponding to μ_r. Let $(\Phi\circ\mu(t))_l$ denote the homogeneous part of degree l of $\Phi\circ\mu(t)$. Then

$$(\Phi\circ\mu(t))_l = \sum_{j_1+\cdots+j_h=l} (1/h!)\tilde{\Phi}(\mu_{j_1}(t), \cdots, \mu_{j_h}(t)) . \tag{4.2}$$

Consider

$$N(t_1, \cdots, t_l) = \sum_{\underline{J}} (1/h!)\tilde{\Phi}(\tilde{\mu}(J_1), \cdots, \tilde{\mu}(J_h)) . \tag{4.3}$$

Since this is a symmetric l-linear map, there exists a unique symmetric map of degree l such that $N(t, \cdots, t) = l!f(t)$. We claim that $f(t) = (\Phi\circ\mu(t))_l$. For, we have

$$
\begin{aligned}
N(t, \cdots, t) &= \sum_{J} (J!/h!)\tilde{\Phi}(\mu_{j_1}(t), \cdots, \mu_{j_h}(t)) \\
&= \sum_{j_1+\cdots+j_h=l} \frac{1}{h!}\Big(\sum_{J \,|\, |J_r|=j_r} J!\Big)\tilde{\Phi}(\mu_{j_1}(t), \cdots, \mu_{j_h}(t)) \qquad (4.4) \\
&= l!(\Phi \circ \mu(t))_l \ ,
\end{aligned}
$$

and therefore

$$D_{t_1,\cdots,t_l}(\Phi \circ \mu) = N(t_1, \cdots, t_l) = \sum_{J} (1/J!)D_{\mu(J)}\Phi \ .$$

Now we are in a position to prove Lemma (2.1) and Proposition (2.1).

Proof of Lemma (2.1). The proof is by induction on the length of the ND-sequence. Using the fact that $P[t_1, t_2] = 0$ for all $(t_1, t_2) \in H^{(1)} \times H^{(2)}$, it is quite easy to check that $D_{t_1}\Phi$, $D_{t_1,t_2}\Phi$ form an ND-sequence of length 2 over $H^{(1)}$. Now assume that $H^{[s]}$, $\Phi^{(s-2)}$ are defined for $s \leq r$ and that $H^s = H^{(s)}$ for $s \leq r$. We further assume that there exist polynomial maps $\theta^1, \cdots, \theta^{r-2}$ of the form (2.1) such that

$$PD_{t_1,\cdots,t_s}[\Psi^{(s-2)}, \Psi^{(s-2)}] = 0 \ , \qquad (4.5)$$

for all $(t_1, \cdots, t_s) \in H^{(1)} \times \cdots \times H^{(s)}$, $s = 2, \cdots, r$, and that $D_{t_1}\Psi^{(s-2)}, \cdots, D_{t_1,\cdots,t_s}\Psi^{(s-2)}$ form an ND-sequence of length s for $s = 2, \cdots, r-1$.

We prove now that $D_{t_1}\Psi^{(r-2)}, \cdots, D_{t_1,\cdots,t_r}\Psi^{(r-2)}$ form an ND-sequence of length r. Observing the definitions (2.1) and (2.4) of Part I, we note that it is sufficient to prove that

$$\bar{\partial} D_{t_{i_1},\cdots,t_{i_q}}\Psi^{(r-2)} = \tfrac{1}{2}D_{t_{i_1},\cdots,t_{i_q}}[\Psi^{(r-2)}, \Psi^{(r-2)}] \qquad (4.6)$$

for all $1 \leq i_1 < \cdots < i_q \leq r$. By Lemma (3.1) of Part I, we have

$$
\begin{aligned}
&\bar{\partial} D_{t_{i_1},\cdots,t_{i_q}}\Psi^{(r-2)} - \tfrac{1}{2}D_{t_{i_1},\cdots,t_{i_q}}[\Psi^{(r-2)}, \Psi^{(r-2)}] \qquad (4.7) \\
&= D_{t_{i_1},\cdots,t_{i_q}}\tilde{Q}^{(r-1)}_{\Psi^{(r-2)}}(\bar{\partial}\Psi^{(r-2)}) + D_{t_{i_1},\cdots,t_{i_q}}\sum_{k=0}^{r-2}\tilde{Q}^{k}_{\Psi^{(r-2)}}\tfrac{1}{2}P[\Psi^{(r-2)}, \Psi^{(r-2)}] \ .
\end{aligned}
$$

Noting that $\tilde{Q}^{(r-1)}_{\Psi^{(r-2)}(t)}(\bar{\partial}\Psi^{(r-2)}(t)) = 0(t^{r+1})$, we find that the first term on the right hand side of (4.7) vanishes. Also the induction hypothesis implies that the second term vanishes. Hence we have the required result.

Remark. Lemma (3.4) implies that the ND-sequence $D_{t_1}\Psi^{(r-2)}, \cdots, D_{t_1,\cdots,t_r}\Psi^{(r-2)}$ is symmetric.

Now we prove Proposition (2.1).

Proof of Proposition (2.1). Once again the proof is by induction. The Proposition is trivial for $r = 1$, because $D_{t_1,t_2}[\Phi, \Phi] = 2[t_1, t_2] = 2K_R(t_1, t_2)$. Let $(t_1, t_2, t_3) \in H^{(1)} \times H^{(2)} \times H^{(3)}$. Then we have

$$D_{t_1,t_2,t_3}[\Psi^{(1)},\Psi^{(1)}] = 2\sum_{i=1}^{3}[D_{t_i}\Phi\circ\theta^1, D_{t_1,\hat{t}_i,t_3}\Phi\circ\theta^1] \;. \tag{4.8}$$

Now applying Lemma (4.1), the right hand side of (4.8) becomes

$$\begin{aligned}
&2\sum_{i=1}^{3}[D_{t_i}\Phi, D_{t_i,\hat{t}_i,t_3}\Phi + \theta_2^1(t_1,\hat{t}_i,t_3)] \\
&= 2K_R(t_1,t_2,t_3\,;\theta_2^1(t_2,t_3)) + D_{t_2,\theta_2^1(t_1,t_3)}[\Psi^{(0)},\Psi^{(0)}] + D_{t_3,\theta_2^1(t_1,t_2)}[\Psi^{(0)},\Psi^{(0)}] \\
&\simeq 2K_R(t_1,t_2,t_3\,;\theta_2^1(t_2,t_3))\;, \qquad \mathrm{mod}\, B^2\;.
\end{aligned}$$

Thus the proposition is proved for $r = 2$.

As we proceed further, the computations become quite complicated. For example for $(t_1,\cdots,t_4)\in H^{(1)}\times H^{(2)}\times H^{(3)}\times H^{(3)}$, we have

$$\begin{aligned}
D_{t_1,\cdots,t_4}[\Psi^{(2)},\Psi^{(2)}] &= 2K_R(t_1,\cdots,t_4\,;\theta_3^2(t_2,t_3,t_4),\theta_3^2(t_3,t_4)) \\
&\quad+\sum_{\substack{1\le i_1\le i_2\le 4\\(i_1,i_2)\ne(3,4)}} D_{t_{i_1},t_{i_2},\theta_2^2(t_1,\hat{t}_{i_1},\hat{t}_{i_2},t_4)}[\Psi^{(1)},\Psi^{(1)}] \\
&\quad+\sum_{2\le i\le 4} D_{t_{i_1},\theta_3^2(t_1,t_i,t_4)}[\Psi^{(0)},\Psi^{(0)}] \\
&\quad+\frac{1}{2}\sum_{1\le i_1\le i_2\le 4} D_{\theta_2^2(t_{i_1},t_{i_2}),\theta_2^2(t_1,\hat{t}_{i_1},\hat{t}_{i_2},t_4)}[\Psi^{(0)},\Psi^{(0)}]\;.
\end{aligned} \tag{4.9}$$

The exact expression for $D_{t_1,\cdots,t_{r+1}}[\Psi^{(r-1)},\Psi^{(r-1)}]$ turns out to be very complicated, but we do not need it.

Assume that (2.3) is proved for all $r = 1, 2, \cdots, n-1$. Then

$$\begin{aligned}
&D_{t_1,\cdots,t_{n+1}}[\Psi^{(n-1)},\Psi^{(n-1)}) \\
&\quad= 2\sum_{1\le i_1<\cdots<i_q\le n+1} D_{t_{i_1},\cdots,t_{i_q}}[\Psi^{(n-1)}, D_{t_1\cdots,\hat{t}_{I_q},\cdots,t_{n+1}}\Psi^{(n-1)}]\;,
\end{aligned} \tag{4.10}$$

where $\hat{t}_{I_q}$ means omission of $t_{i_1},\cdots,t_{i_q}$. Now we apply Lemma (4.1) to expand $D_{t_{i_1},\cdots,t_{i_q}}\Psi^{(n-1)}$ and $D_{t_1,\cdots,\hat{t}_{I_q},\cdots,t_{n+1}}\Psi^{(n-1)}$. Observing the definition of $K_R(t_1,\cdots,t_{n+1}\,;\theta_n^1(t_2,\cdots,t_{n+1}),\cdots,\theta_2^{(n-1)}(t_n,t_{n+1}))$ and applying the induction hypothesis, we have

$$\begin{aligned}
&D_{t_1,\cdots,t_{n+1}}[\Psi^{(n-1)},\Psi^{(n-1)}] \\
&\quad\cong 2K_R(t_1,\cdots,t_{n+1}\,;\theta_n^{(1)}(t_2,\cdots,t_{n+1}),\cdots,\theta_2^{(n-1)}(t_n,t_{n+1})) \\
&\qquad+\sum_{\substack{1\le i_1<i_2\le n+1\\(i_1,i_2)\ne(n,n+1)}} D_{\theta_2^{(n-1)}(t_{i_1},t_{i_2}),t_1,\cdots,\hat{t}_{i_1},\cdots,\hat{t}_{i_2},\cdots,t_{n+1}}[\Psi^{(n-2)},\Psi^{(n-2)}]\;, \\
&\qquad\qquad \mathrm{mod}\, B^{n-1}\;. \\
&\quad\cong 2K_R(t_1,\cdots,t_{n+1}\,;\theta_n^{(1)}(t_2,\cdots,t_{n+1}),\cdots,\theta_2^{(n-1)}(t_n,t_{n+1})) \\
&\qquad\qquad \mathrm{mod}\, B^n\;.
\end{aligned}$$

This completes the proof of Proposition (2.1).

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