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# CONDITION ($C$) FOR THE ENERGY INTEGRAL ON CERTAIN PATH SPACES AND APPLICATIONS TO THE THEORY OF GEODESICS

KARSTEN GROVE

## Introduction

Let $M$ be a complete connected Riemannian manifold, and $L_1^2(I, M)$ the Hilbert manifold of absolutely continuous maps from the unite interval $I = [0, 1]$ to $M$ with square integrable derivative. See, e.g., Eells [4] for the manifold structure on $L_1^2(I, M)$, or Karcher [9] and Palais [16] for analogous spaces. There are various interesting submanifolds of $L_1^2(I, M)$ related to the study of different kinds of geodesics on $M$, which appear as critical points for the energy integral on the submanifolds.

This paper is divided into three sections. In the first two sections we point out some interesting submanifolds of $L_1^2(I, M)$ and their related geodesics on $M$, and study to which extent the energy integral satisfies Condition ($C$) of Palais and Smale (a necessary condition for making critical point theory like Morse theory and Lusternik-Schnirelmann theory on infinite dimensional manifolds). Our first result was a generalization of those obtained by McAlpin or Karcher [9] and Palais [16]. However Eliasson has recently obtained a general result on Condition ($C$) based on the notion of weak submanifolds and local coercive properties of the involved function [5]. Conversations with Eliasson made it clear that his results applies to our case, so that Theorem 2.4 now in some sense is the best possible result on Condition ($C$) for the energy function on path spaces. The author is indebted to Eliasson for pointing out this to him. Immediate applications of Theorem 2.4 are made to geodesics between submanifolds of $M$ and to geodesics invariant under an isometry without fixed points; by invariant we mean that the geodesic is mapped onto itself with the direction of speed preserved. In the last section we apply the results of the first two sectionsto get existence thorems for geodesics on a compact manifold invariant under a given isometry. Our main results in the last section are contained in

**Theorem.** *Let $M$ be a compact Riemannian manifold, and $A: M \to M$ an isometry on $M$.*

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(1) *If $A$ is homotopic to the identity map $1_M$ of $M$, then it has a nontrivial invariant geodesic.*

(2) *If $\pi_1(M) = 0$, then $A$ has a nontrivial invariant geodesic except possibly when $A$ has exactly one fixed point.*

*Finally there exist compact manifolds with isometries which have no nontrivial invariant geodesics.*

We note that V. Ozols in [15] studied the square of the displacement function $\delta_A : M \to \boldsymbol{R}$ of an isometry $A$ defined by $\delta_A(x) = d_M(x, A(x))$ for all $x \in M$. If $A$ has small displacement, i.e., if $A(x)$ is never in the cut-locus of $x$ for any $x \in M$, then the critical points for $\delta_A^2$ are the points $p \in M$ for which the continuation of the unique minimizing geodesic from $p$ to $A(p)$ is invariant under $A$. Thus, if $M$ is compact and $A : M \to M$ has small displacement, then $A$ has a nontrivial invariant geodesic. Part (1) of our theorem obviously generalizes this result.

## 1. Critical points

Let $(M, \langle\ ,\ \rangle)$ be a complete Riemannian manifold. Then $L_1^2(I, M)$ has a natural complete Riemannian structure given by

$$\langle\langle X_\sigma, Y_\sigma \rangle\rangle_\sigma = \int_0^1 \{\langle X_\sigma(t), Y_\sigma(t)\rangle_{\sigma(t)} + \langle \nabla_\sigma X_\sigma(t), \nabla_\sigma Y_\sigma(t)\rangle_{\sigma(t)}\} dt \ ,$$

where $X_\sigma$ and $Y_\sigma$ are elements of the tangent space at $\sigma \in L_1^2(I, M)$, i.e., $X_\sigma$ is an absolutely continuous vector field along $\sigma$ on $M$ with square integrable covariant derivative $\nabla_\sigma X_\sigma$ (see, for example, Flaschel [7] or Klingenberg [10]). All submanifolds of $L_1^2(I, M)$ will be given the induced Riemannian structure from $\langle\langle\ ,\ \rangle\rangle$.

It is easy to see that the map $P$: $L_1^2(I, M) \to M \times M$ defined by $P(\sigma) = (\sigma(0), \sigma(1))$ for all $\sigma \in L_1^2(I, M)$ is a submersion so that the preimage of any submanifold of $M \times M$ by $P$ is a submanifold of $L_1^2(I, M)$. If $N \subseteq M \times M$, we write $\Lambda_N(M)$ for $P^{-1}(N)$.

**Example 1.1.** Let $V$ and $V'$ be closed submanifolds of $M$. Then $\Lambda_{V \times V'}(M)$ is a complete Riemannian Hilbert manifold. The study of $\Lambda_{V \times V'}(M)$ is as we shall see related to that of geodesics from $V$ to $V'$ orthogonal to $V$ and $V'$. (Note the special cases where $V$, $V'$ or both are points of $M$.)

**Example 1.2.** Let $A : M \to M$ be an isometry on $M$, and let $G(A)$ denote the graph of $A$. Then $\Lambda_{G(A)}(M)$ is a complete Riemannian Hilbert manifold. The study of $\Lambda_{G(A)}(M)$ is related to that of $A$-invariant geodesics on $M$. (Note that in the special case $A = 1_M$, $\Lambda_{G(A)}(M) = \Lambda_\Delta(M)$ is the space of closed curves on $M$, which is denoted by $\Lambda(M)$ in Klingenberg's notation [10]).

The relation between the Hilbert manifolds $\Lambda_N(M)$ and the corresponding Banach manifolds $C_N^0(M)$ of continuous maps with the uniform (compact-open) topology is given by

**Theorem 1.3.** *Let $N \subseteq M \times M$ be a submanifold of $M \times M$. Then the inclusion $\Lambda_N(M) \to C^0_N(M)$ is a homotopy equivalence.*

*Proof.* $C^0(I, M) \to M \times M$ is a fibration by Serre [18], and $L^2_1(I, M) \to M \times M$ is a fibration by Earle and Eells [3, Proposition, p. 40]. Thus the homotopy sequences for

$$\begin{array}{ccc} \Lambda_{\{p\}\times\{q\}}(M) & \xrightarrow[i]{} & C^0_{\{p\}\times\{q\}}(M) \\ \downarrow & & \downarrow \\ L^2_1(I, M) & \xrightarrow[i]{} & C^0(I, M) \\ \downarrow & & \downarrow \\ M \times M & \xrightarrow[1_{M\times M}]{} & M \times M \end{array}$$

and the five lemma give that $\Lambda_{\{p\}\times\{q\}}(M) \to C^0_{\{p\}\times\{q\}}(M)$ induces isomorphisms on all homotopy groups for all fibers $P^{-1}(\{p\} \times \{q\})$ because the inclusion $L^2_1(I, M) \to C^0(I, M)$ is a homotopy equivalence by a general theorem of Palais [17, Theorem 13.14]. Now using this on the inclusion between the fibrations

$$\begin{array}{ccc} \Lambda_{\{p\}\times\{q\}}(M) & \longrightarrow & C^0_{\{p\}\times\{q\}}(M) \\ \downarrow & & \downarrow \\ \Lambda_N(M) & \longrightarrow & C^0_N(M) \\ \downarrow P_1 & & \downarrow P_1 \\ N & \xrightarrow[1_N]{} & N \end{array}$$

together with the five lemma yields that $\Lambda_N(M) \to C^0_N(M)$ induces isomorphisms on homotopy groups and hence is a homotopy equivalence because $\Lambda_N(M)$ and $C^0_N(M)$ are ANR's (even manifolds).

The energy integral is defined by

$$E(\sigma) = \tfrac{1}{2}\int_0^1 \langle \sigma'(t), \sigma'(t) \rangle_{\sigma(t)} dt$$

for all $\sigma \in L^2_1(I, M)$. $E$ is differentiable, and its differential at $\sigma \in L^2_1(I, M)$ is given by

$$dE_\sigma(X_\sigma) = \int_0^1 \langle \nabla_\sigma X_\sigma(t), \sigma'(t) \rangle_{\sigma(t)} dt$$

for all $X_\sigma \in T_\sigma L^2_1(I, M)$ (see, for example, Flaschel [8] or Karcher [9]). Thus we have

**Proposition 1.4.** *If $\sigma \in L_1^2(I, M)$ is a geodesic on $M$, then the differential of $E$ at $\sigma$ is given by*

$$dE_\sigma(X_\sigma) = \langle X_\sigma(1), \sigma'(1)\rangle_{\sigma(1)} - \langle X_\sigma(0), \sigma'(0)\rangle_{\sigma(0)}$$

*for all $X_\sigma \in T_\sigma L_1^2(I, M)$.*

*Proof.*

$$\frac{d}{dt}\langle X_\sigma(t), Y_\sigma(t)\rangle_{\sigma(t)} = \langle \nabla_\sigma X_\sigma(t), Y_\sigma(t)\rangle_{\sigma(t)} + \langle X_\sigma(t), \nabla_\sigma Y_\sigma(t)\rangle_{\sigma(t)}$$

almost everywhere (see, for example, Karcher [9]), and

$$\sigma \text{ geodesic} \Longrightarrow \sigma' \in T_\sigma L_1^2(I, M) \wedge \nabla_\sigma \sigma' = 0 \ .$$

On the other hand, we have

**Proposition 1.5** (*regularity*). *If $X$ is a submanifold of $L_1^2(I, M)$ such that $T_\sigma X$ for $\sigma \in X$ contains all $X_\sigma \in T_\sigma L_1^2(I, M)$ with $X_\sigma(0) = 0$ and $X_\sigma(1) = 0$, and if $\sigma$ is a critical point for $E|_X$, then $\sigma$ is $C^\infty$ and $\sigma$ is a geodesic.*

*Proof.* We just remark that our condition on $X$ is sufficient for us to give a proof quite similar to a part of Karcher's proof of Theorem 8.39 in [9], which states that critical points for $E: \Lambda(M) \to \boldsymbol{R}$ are closed geodesics.

Combining Propositions 1.4 and 1.5 we get

**Theorem 1.6.** (1) *Let $V$ and $V'$ be submanifolds of $M$. Then $\sigma \in \Lambda_{V\times V'}(M)$ is a critical point for $E$: $\Lambda_{V\times V'}(M) \to \boldsymbol{R}$ iff $\sigma$ is a geodesic on $M$ starting orthogonal to $V$ and ending orthogonal to $V'$, i.e., $\sigma'(0) \in T_{\sigma(0)}V^\perp$ and $\sigma'(1) \in T_{\sigma(1)}V'^\perp$.*

(2) *Let $A: M \to M$ be an isometry on $M$. Then $\sigma \in \Lambda_{G(A)}(M)$ is a critical point for $E$: $\Lambda_{G(A)}(M) \to \boldsymbol{R}$ iff $\sigma$ is a geodesic on $M$ with the property that $A_{*\sigma(0)}(\sigma'(0)) = \sigma'(1)$, i.e., that the unique maximal geodesic on $M$ determined by $\sigma'(0) \in T_{\sigma(0)}(M)$ is invariant under $A$.*

*Proof.* First note that the manifolds $\Lambda_N(M)$ satisfy the condition on $X$ in Proposition 1.5 since the tangent spaces are given by

$$T_\sigma \Lambda_N(M) = \{X_\sigma \in T_\sigma L_1^2(I, M) \,|\, (X_\sigma(0), X_\sigma(1)) \in T_{(\sigma(0),\sigma(1))}N\} \ .$$

(1) Assume that $\sigma \in \Lambda_{V\times V'}(M)$ is a critical point for $E$: $\Lambda_{V\times V'}(M) \to \boldsymbol{R}$. By Proposition 1.5, $\sigma$ is a geodesic on $M$, so that $dE_\sigma(X_\sigma) = \langle X_\sigma(1), \sigma'(1)\rangle_{\sigma(1)} - \langle X_\sigma(0), \sigma'(0)\rangle_{\sigma(0)}$ by Proposition 1.4. Now since $\sigma$ is critical, we get from this that $\langle X_\sigma(1), \sigma'(1)\rangle_{\sigma(1)} = \langle X_\sigma(0), \sigma'(0)\rangle_{\sigma(0)}$, $\forall X_\sigma \in T_\sigma \Lambda_{V\times V'}(M)$, but this cannot happen unless both are zero.

Assume next that $\sigma$ is a geodesic orthogonal to $V$ and $V'$. By Proposition 1.4 and the assumption, $dE_\sigma(X_\sigma) = \langle X_\sigma(1), \sigma'(1)\rangle_{\sigma(1)} - \langle X_\sigma(0), \sigma'(0)\rangle_{\sigma(0)} = 0$, $\forall X_\sigma \in T_\sigma \Lambda_{V\times V'}(M)$.

(2) Assume that $\sigma \in \Lambda_{G(A)}(M)$ is a geodesic on $M$ with $A_{*(0)}(\sigma'(0)) = \sigma'(1)$. By Proposition 1.4

$$\begin{aligned} dE_\sigma(X_\sigma) &= \langle X_\sigma(1), \sigma'(1)\rangle_{\sigma(1)} - \langle X_\sigma(0), \sigma'(0)\rangle_{\sigma(0)} \\ &= \langle A_{*\sigma(0)}X_\sigma(0), A_{*\sigma(0)}\sigma'(0)\rangle_{\sigma(1)} - \langle X_\sigma(0), \sigma'(0)\rangle_{\sigma(0)} \\ &= 0 \qquad \text{since } A \text{ is an isometry.} \end{aligned}$$

Assume next that $\sigma \in \Lambda_{G(A)}(M)$ is a critical point for $E\colon \Lambda_{G(A)}(M) \to \boldsymbol{R}$. By Proposition 1.5, $\sigma$ is a geodesic on $M$, so by Proposition 1.4,

$$\begin{aligned} dE_\sigma(X_\sigma) &= \langle X_\sigma(1), \sigma'(1)\rangle_{\sigma(1)} - \langle X_\sigma(0), \sigma'(0)\rangle_{\sigma(0)} \\ &= \langle A_{*\sigma(0)}X_\sigma(0), \sigma'(1)\rangle_{\sigma(1)} - \langle A_{*\sigma(0)}X_\sigma(0), A_{*\sigma(0)}\sigma'(0)\rangle_{\sigma(1)} \end{aligned}$$

since $A$ is an isometry. This together with $dE_\sigma = 0$ gives that $\sigma'(1) = A_{*\sigma(0)}\sigma'(0)$.

That the maximal geodsic determined by $\sigma$ is $A$-invariant follows easily from $A_{*\sigma(0)}\sigma'(0) = \sigma'(1)$ either by "geometry" or by the fact that $A$ induces an isometry on $\Lambda_{G(A)}(M)$, which commutes with the energy integral, and similar for $A^{-1}$.

**Remark.** If $A = 1_M$ in Theorem 1.6, then we get that $\sigma \in \Lambda(M)$ is a critical point for $E\colon \Lambda(M) \to \boldsymbol{R}$ iff $\sigma$ is a closed (periodic) geodesic on $M$. From Theorem 1.6 it is interesting to know under which conditions on $V$, $V'$, $M$ and $A$ the energy integrals on $\Lambda_{V\times V'}(M)$ and $\Lambda_{G(A)}(M)$ respectively satisfy Condition $(C)$ of Palais and Smale. The next section will be concerned with this by first looking at $\Lambda_N(M)$ in general as a closed submanifold $N$ of $M \times M$.

## 2. Condition $(C)$

Using the metric $\langle\langle\ ,\ \rangle\rangle$ of § 1 on the various submanifolds of $L_1^2(I, M)$ we define the corresponding vector field $-\operatorname{grad} E$. To establish Condition $(C)$ of Palais and Smale we must show that whenever $\{\sigma_n\}_{n\in N}$ is a sequence on which $E$ is bounded and for which $|||\operatorname{grad} E_{\sigma_n}|||_{\sigma_n} \to 0$ when $n \to \infty$, $\sigma_n$ has a convergent subsequence, where $|||\cdot|||_\sigma$ denote the norm in $T_\sigma X$ corresponding to $\langle\langle\ ,\ \rangle\rangle_\sigma$.

Let us first examine what the boundedness of $E$ on $\{\sigma_n\}_{n\in N} \subseteq L_1^2(I, M)$ implies

**Lemma 2.1.** *Let $S \subseteq L_1^2(I, M)$ be a subset of $L_1^2(I, M)$ on which $E$ is bounded. Then $S$ is an equi-continuous family of curves on $M$ with uniformly bounded length.*

*Proof.* Write $d_M$ for the distance on $M$, and $L_M$ for the length of a curve on $M$. For $\sigma \in L_1^2(I, M)$ we have

$$\begin{aligned} d_M(\sigma(t_1), \sigma(t_2)) \leq L_M(\sigma|_{[t_1,t_2]}) &= \int_{t_1}^{t_2} \langle\sigma'(t), \sigma'(t)\rangle_{\sigma(t)} dt \\ &\leq |t_1 - t_2|^{1/2} \cdot (2E(\sigma))^{1/2} \qquad \text{by Cauchy-Schwartz,} \end{aligned}$$

from which it follows that $S$ is an equi-continuous family of curves on $M$ and furthermore that $L_M(\sigma) \leq 2E(\sigma)$ for all $\sigma$ in $L_1^2(I, M)$, where equality holds iff $\sigma$ is parametrised proportional to arc length.

**Proposition 2.2.** *Let $N \subset M \times M$ be a closed submanifold of $M \times M$ with compact $P_1(N) \subset M$ or $P_2(N) \subset M$. Then any sequence $\{\sigma_n\}$ in $\Lambda_N(M)$, on which $E$ is bounded, has a subsequence converging uniformly to a continuous path $h \in C_N^0(M)$ on $M$.*

*Proof.* Assume without loss of generality that $P_1(N) \subset M$ is compact. From Lemma 2.1 we have that $\{\sigma_n\}_{n \in N}$ is an equicontinuous family of curves on $M$ of bounded length, i.e., there exists a closed and bounded set $K \subset M$ such that $\sigma_n(I) \subset K$ for all $n \in N$ since $\sigma_n(0) \in P_1(M)$ for all $n \in N$. Since $M$ is a complete Riemannian manifold, $K$ is compact by the Hopf-Rinow theorem and hence we can apply Ascoli's theorem to obtain the statement of the proposition.

**Remark.** It is easy to see that the conclusion of Proposition 2.2 fails in general when we omit the condition that $P_1(N)$ or $P_2(N)$ is compact. Since the manifold topology on $\Lambda_N(M)$ is stronger than the uniform topology ($d_\infty(f, g) = \max_{t \in I} d_M(f(t), g(t))$), it becomes more difficult to obtain convergence in $\Lambda_N(M)$.

**Definition 2.3.** By a "natural chart" around $\sigma$ in $L_1^2(I, M)$ we understand a chart defined by means of the exponential map for $\langle\, ,\, \rangle$ in the following way:

$$\exp_\sigma\colon T_\sigma L_1^2(I, M) \to L_1^2(I, M)$$

is given by $\exp_\sigma(X_\sigma)(t) = \exp_{\sigma(t)} X_\sigma(t)$ for all $X_\sigma \in T_\sigma L_1^2(I, M)$ and all $t \in I$; $\exp_\sigma$ is a diffeomorphism of a neighborhood around $0_\sigma$ in $T_\sigma L_1^2(I, M)$ on a neighborhood around $\sigma$ in $L_1^2(I, M)$, i.e., a chart on $L_1^2(I, M)$, (Eells [4] and Karcher [9]). We can as well consider $T_\sigma L_1^2(I, M)$ as $L_1^2$-sections in the pullback $\sigma^*TM$ of $TM$ by $\sigma$.

**Theorem 2.4.** *Let $M$ be a complete Riemannian manifold, and $N \subset M \times M$ be a closed submamifold of $M \times M$ such that $P_1(N) \subset M$ or $P_2(N) \subset M$ is compact. Then $E\colon \Lambda_N(M) \to \boldsymbol{R}$ satisfies condition (C) of Palais and Smale.*

*Proof.* Let $\{\sigma_n\}_{n \in N}$ be a sequence in $\Lambda_n(M)$, on which $E$ is bounded (say $E(\sigma_n) \leq k \in \boldsymbol{R}_+ \forall n \in N$) and for which $|||\operatorname{grad} E(\sigma_n)|||_{\sigma_n} \to 0$, or, equivalently, $|||dE(\sigma_n)|||_{\sigma_n} \to 0$. We want to show that $\{\sigma_n\}$ has a convergent subsequence. Now by Proposition 2.2 we can assume that $\sigma_n$ converges uniformly (in the $d_\infty$-topology) to a continuous map $h \in C_N^0(M)$. From this follows that all $\sigma_n$ from a certain step, say $n_0 \in N$, is in the domain of a "natural chart" on $L_1^2(I, M)$ without loss of generality centered at a $C^\infty$ curve say $a \in C_N^\infty(M)$.

From now on we work locally in a natural chart around $a$. Let $\mathcal{O}_a \subset a^*TM$ be an open neighborhood of the zero-section such that $\exp_a\colon L_1^2(\mathcal{O}_a) \to L_1^2(I, M)$ is a natural chart around $a$ in $L_1^2(I, M)$, where $L_1^2(\mathcal{O}_a)$ is $L_1^2$-sections of $a^*TM$ belonging to $\mathcal{O}_a$, i.e., an open neighborhood of $0_a$ in $T_a L_1^2(I, M) \equiv L_1^2(a^*TM)$.

Now by the orthogonal decomposition $L_1^2(a^*TM) = L_1^2(a^*TM)_0 + V$ where $L_1^2(a^*TM)_0$ is the subspace of $L_1^2(a^*TM)$ consisting of sections being zero at the endpoints, we have

$$V = \{X \in L_1^2(a^*TM) \mid \langle\langle X, Y\rangle\rangle = 0 \ \forall Y \in L_1^2(a^*TM)_0\}$$
$$= \left\{X \,\middle|\, \int_0^1 \langle X(t) - \nabla^2 X(t), Y(t)\rangle dt = 0 \ \forall Y \in L_1^2(a^*TM)_0\right\} ,$$
$$V' = \{X \mid \nabla^2 X = X\} = \{X \mid X(t) = c_1(t)X_1(t) + c_2(t)X_2(t)\} ,$$

where $c_i'' = c_i$, $c_1(0) = 1$, $c_1(1) = 0$, $c_2(0) = 0$ and $c_2(1) = 1$, or precisely,

$$c_1(t) = -\frac{e^{-1}}{e - e^{-1}}e^t + \frac{e}{e - e^{-1}}e^{-t}, \qquad c_2(t) = \frac{1}{e - e^{-1}}e^t - \frac{1}{e - e^{-1}}e^{-t} ,$$

and $X_1$ and $X_2$ are parallel fields along $a$.

$V \supset V'$ is obvious, and $V = V'$ then follows since both have dimension $2n$.

Let $N_a \subset \mathcal{O}_{a(0)} \times \mathcal{O}_{a(1)}$ be the submanifold of $\mathcal{O}_{a(0)} \times \mathcal{O}_{a(1)}$ which by $\exp_{a(0)} \times \exp_{a(1)}$ is mapped into $N \subset M \times M$. Then $P_a^{-1}(N_a) = L_{1N_a}^2(\mathcal{O}_a)$ is mapped diffeomorphically by $\exp_a$ onto an open neighborhood of $a$ in $\Lambda_N(M)$, where $P_a \colon L_1^2(\mathcal{O}_a) \to \mathcal{O}_{a(0)} \times \mathcal{O}_{a(1)}$ is of course the map defined by $P_a(X) = (X(0), X(1))$. From $L_1^2(a^*TM)_0 \subset T_a\Lambda_N(M)$ and the orthogonal decomposition $L_1^2(a^*TM)_0 + V = L_1^2(a^*TM)$ it then follows that $L_{1N_a}^2(\mathcal{O}_a) = (L_1^2(a^*TM)_0 + V_N) \cap L_1^2(\mathcal{O}_a)$ where $V_N = \{X \in V \mid (X(0), X(1)) = (X_1(0), X_2(1)) \in N_a\}$ is a submanifold of the $2n$-dimensional vector space $V$. Thus $L_{1N_a}^2(\mathcal{O}_a)$ is diffeomorphic to the product of a finite dimensional manifold and an open subset of Hilbert space.

Let $X_n = \exp_a^{-1}(\sigma_n)$ so that there exists an $X_\infty \in C_{N_a}^0(\mathcal{O}_a)$ such that $\|X_n - X_\infty\|_\infty \to 0$ when $n \to \infty$. Using the local expression for the energy (see Eliasson [6]) it follows from the boundedness of $E$ that $|||X_n|||$ is bounded. Furthermore, the estimates in [6] also show that $E$ is locally coercive (see also Eliasson [5]), i.e., there exist constants $\lambda > 0$ and $C$ such that

$$(*) \qquad (dE(X) - dE(Y))(X - Y) \geq \lambda\, |||X - Y|||^2 - C\,\|X - Y\|_\infty^2$$

for sufficiently small $\|X\|_\infty$ and $\|Y\|_\infty$; we have used $E$ for $E \circ \exp_a$ which should cause no confusion.

Write now $X_n = X_n^0 + Y_n$ where $X_n^0 \in L_1^2(\ )_0$ and $Y_n \in V_N$. Then $(*)$ gives

$$\lambda\, |||X_i - X_j|||^2 \leq C\,\|X_i - X_j\|_\infty + (dE_N(X_i) - dE_N(X_j))(X_i^0 - X_j^0)$$
$$+ (dE(X_i) - dE(X_j))(Y_i - Y_j),$$

or

$$\begin{aligned}\lambda |||X_i - X_j|||^2 \le C \|X_i - X_j\|_\infty \\ + (|||dE_N(X_i)||| + |||dE_N(X_j)|||)\cdot|||X_i^0 - X_j^0||| \\ + |(dE(X_i) - dE(X_j))(Y_i - Y_j)| ,\end{aligned}$$

where $E_N$ denotes $E|_{L^2_1 N_a(\mathcal{O}_a)}$. From $\|X_n - X_\infty\|_\infty \to 0$, $|||dE_N(X_n)||| \to 0$ and the boundedness of $|||X_n|||$ it follows that the first two terms on the right side of the above ineqality tend to zero, and that the last term also tends to zero by using the local expression for $dE$ in [6] and the fact that $Y_n$ converges, since $X_n$ converges uniformly and the $C^0$-norm dominates the $V_N$-norm. Thus $\{X_n\}_{n\in N}$ is a Cauchy sequence and therefore convergent. Hence $\{\sigma_n\}_{n\in N}$ is convergent.

**Corollary 2.5.** *Let $M$ be a complete Riemannian manifold, $V$ and $V'$ be closed submanifolds of $M$, and $A: M \to M$ be an isometry on $M$. Then the following hold:*

(1) *$E: \Lambda_{V\times V'}(M) \to \boldsymbol{R}$ satisfies Condition (C) if $V$ or $V'$ is compact.*

(2) *$E: \Lambda_{G(A)}(M) \to \boldsymbol{R}$ satisfies Condition (C) if $M$ is compact.*

By using Condition (C) (for its consequences see Palais [16]) we get immediately:

**Theorem 2.6.** *Let $M$ be a complete Riemannian manifold, and let $V$ and $V'$ be closed submanifolds of $M$ with say $V$ compact. Then in any homotopy class of curves from $V$ to $V'$ there exists a geodesic orthogonal to $V$ and $V'$ with length smaller than that of any other curve in that class. Furthermore, there exists a geodesic orthogonal to $V$ and $V'$ with length equal to $d(V, V')$, and there are at least* cat $(\Lambda_{V\times V'}(M))$ *geodesics joining $V$ and $V'$ orthogonal.*

*Proof.* Condition (C) implies that the energy integral on $\Lambda_{V\times V'}(M)$ attains its infimum on any component of $\Lambda_{V\times V'}(M)$ and its lower bound (see Palais [16, § 15]). The inf points are of course critical points of the energy. Now we only have to apply Theorem 1.6, and note that an inf of the energy is an inf of the length by using the proof of Lemma 2.1 and the fact that a change of parameter does not affect the homotopy class of the curve. The cat ( ) statement is a consequence of Lusternik-Schnirelmann theory.

By using Theorem 1.6 and Corollary 2.5 similar arguments prove the

**Proposition 2.7.** *If $M$ is a compact Riemannian manifold, and $A: M \to M$ an isometry without fixed points, then $A$ has a nontrivial invariant geodesic.*

**Remark.** Proposition 2.7 and part of Theorem 2.6 are also easy to prove by geometrical arguments.

In § 3 we shall study the case where the fixed point set of $A$ is nonempty.

## 3. Invariant geodesics

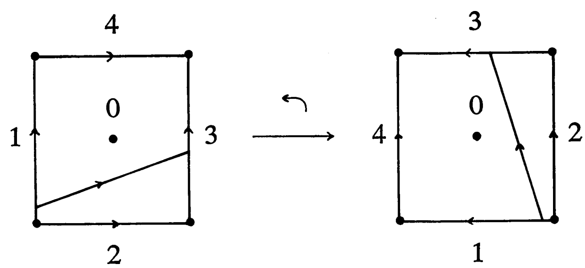
Throughout this section $M$ will be a compact connected Riemannian manifold, and $A: M \to M$ an isometry on $M$. We ask the following question: under what conditions on $M$ and $A$ there is a nontrivial maximal geodesic on $M$, which is mapped onto itself by $A$? One way to investigate this question as we have seen from §§ 1, 2 is to study the space $\Lambda_{G(A)}(M)$ introduced there (Theorem 1.6); a main reason why this becomes successful is that the energy integral $E\colon \Lambda_{G(A)}(M) \to \boldsymbol{R}$ satisfies Condition (*C*) when $M$ is compact (Corollary 2.5).

We have seen that if $A$ has no fixed points it has a nontrivial invariant geodesic. Let us therefore assume that the fixed point set of $A$ is nonempty, i.e., Fix $(A) \neq \emptyset$.

First we give examples to show that not all isometries on a compact manifold in general has nontrivial invariant geodesics.

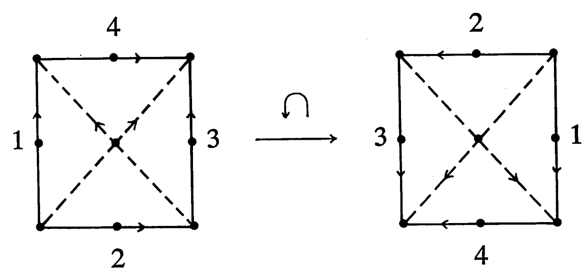
**Examples 3.1.** Let $M = T^2 = S^1 \times S^1$ with flat metric.

(1) Let $A\colon T^2 \to T^2$ be induced from a rotation through 90° in $\boldsymbol{R}^2$:




Then $A$ is an isometry on the flat torus with two fixed points and no nontrivial invariant geodesics.

(2) Let $A\colon T^2 \to T^2$ be the map induced from a rotation through 180° in $\boldsymbol{R}^2$:




Then $A$ is an isometry with four fixed points and no nontrivial invariant geodesics. If we did not require that $A$ should preserve the direction of the geodesic, $A$ would have infinitely many "invariant" geodesics.

By Theorem 1.6 we see that the fixed points of $A$ occur as critical points of the energy integral on $\Lambda_{G(A)}(M)$ (trivial invariant geodesics) with $E$-value zero. Thus in order to prove the existence of nontrivial invariant geodesics we shall prove the existence of positive critical values. Let us therefore first study how the fixed point set Fix $(A)$ of $A$ behaves by itself and inside of $\Lambda_{G(A)}(M)$.

**Proposition 3.2.** *Let $A: M \to M$ be an isometry on $M$. Then the set of fixed points for $A$ is a disjoint union of totally geodesic submanifolds each of which is a nondegenerate submanifold of $\Lambda_{G(A)}(M)$ with index* 0.

*Proof.* Using $\exp_{x_0} \circ A_{*x_0} = A \circ \exp_{x_0}$ when $A(x_0) = x_0$, it is easy to see that $\operatorname{Fix}(A) = \bigcup_{i=1}^k F_i(A)$, where $F_i(A) \cap F_j(A) \neq \emptyset \Rightarrow F_i(A) = F_j(A)$ and $F_i(A)$ is a totally geodesic submanifold of $M$ (see also, e.g., Kobayashi [11]). Furthermore, $F_i(A) \subset \Lambda_{G(A)}(M)$ is a critical submanifold of $\Lambda_{G(A)}(M)$. Thus we now have to prove that $F_i(A)$ is nondegenerate and of index 0 (see Meyer [14]).

Since $A: M \to M$ is an isometry, by a computation of the Hessian of $E$ on the manifold $\Lambda_{G(A)}(M)$, which is quite similar to that of $H(E)_\sigma$ in the space of closed curves (see Flaschel [8]), we get

$$H(E)_\sigma(X_\sigma, Y_\sigma) = \int_0^1 \langle \nabla_\sigma X_\sigma(t), \nabla_\sigma Y_\sigma(t)\rangle_{\sigma(t)} dt - \int_0^1 \langle R(X_\sigma(t), \sigma'(t))\sigma'(t), Y_\sigma(t)\rangle_{\sigma(t)} dt ,$$

when $\sigma \in \Lambda_{G(A)}(M)$ is a critical point of $E$.

Now let $\sigma \in F_i(A)$. Then $\sigma(t) = \sigma(0)\ \forall t \in I$, and therefore $\sigma'(t) = 0\ \forall t \in I$. Thus

$$H(E)_\sigma(X_\sigma, Y_\sigma) = \int_0^1 \langle \nabla_\sigma X_\sigma(t), \nabla_\sigma Y_\sigma(t)\rangle_{\sigma(t)} dt , \qquad \forall X_\sigma, Y_\sigma \in T_\sigma \Lambda_{G(A)}(M) .$$

The adjoint map $h(E)_\sigma$ is given by $\langle\langle h(E)_\sigma X_\sigma, Y_\sigma\rangle\rangle_\sigma = H(E)_\sigma(X_\sigma, Y_\sigma)$, from which we see that $\ker h(E)_\sigma = \{X_\sigma \in T_\sigma\Lambda_{G(A)}(M) \mid \nabla_\sigma X_\sigma = 0$ almost everywhere$\}$. Since $\nabla_\sigma X_\sigma$ as a curve in $T_{\sigma(0)}M$ is just $X'_\sigma(t)$, $X_\sigma(t)$ is constant in $T_{\sigma(0)}M$ when $X_\sigma \in \ker h(E)_\sigma$. Furthermore, since $A_{*\sigma(0)}(X_\sigma(0)) = X_\sigma(1)$, we have that $X_\sigma(t) \in T_{\sigma(0)}F_i(A)$, so that $\ker h(E)_\sigma = T_{\sigma(0)}F_i(A)$.

Using that $h(E)_\sigma$ is self-adjoint it is now easy to show that $h(E)_{\sigma|}: T_\sigma F_i(A)^\perp \to T_\sigma F_i(A)^\perp$ is bijective, i.e., $F_i(A)$ is a nondegenerate critical submanifold of $\Lambda_{G(A)}(M)$. From the fact that $H(E)_\sigma$ is semi-positive definite for any $\sigma \in \Lambda^0_{G(A)}(M) \equiv E^{-1}(0)$, it follows that the index of $F_i(A)$ is 0.

**Remark.** We note that if $A: M \to M$ is an orientation-preserving isometry on an oriented Riemannian manifold $M$, then $\operatorname{codim}(F_i(A))$ is even (quite similar to Kobayashi [11]) for each $F_i(A) \subset \operatorname{Fix}(A)$. Thus, if $M$ is compact and odd-dimensional, then any such isometry has of course a nontrivial invariant geodesic.

Proposition 3.2 has the important

**Corollary 3.3.** *There exists an $e > 0$ such that* $\operatorname{Fix}(A)$ *is a strong deformation retract of $\Lambda^e_{GA)}(M) \equiv E^{-1}([0, e])$.*

*Proof.* We first look at a fixed component $F_i(A) \subset \operatorname{Fix}(A)$ of the fixed point set of $A$.

Let $N(F_i(A))$ denote the total space of the normal bundle $\nu_i = (N(F_i(A)), \pi, F_i(A))$ to $F_i(A)$ in $\Lambda_{G(A)}(M)$. Then $\nu_i$ is a Hilbert bundle with metric induced from $\Lambda_{G(A)}(M)$. Since $M$ is compact and $\operatorname{Fix}(A)$ is closed, $F_i(A)$ is compact, so that there exists an $r > 0$ such that $N_r(F_i(A)) = \{V_\sigma \in N(F_i) \mid |||V_\sigma|||_\sigma < r\}$ is diffeomorphic to an open neighborhood (a tubular neighborhood, Lang [13]) of $F_i(A)$ in $\Lambda_{G(A)}(M)$, say $\varphi_i\colon N_r(F_i(A)) \to \varphi_i(N_r(F_i(A))) \equiv U_i$, a diffeomorphism with $\varphi_i$ (0-section) $= F_i(A)$. Now $\bar{E} = E \circ \varphi_i\colon N_{r_i}(F_i) \to \boldsymbol{R}$ has the 0-section as nondegenerate critical submanifold. Since $F_i(A)$ is compact, there exist an $\varepsilon_i > 0$ (without loss of generality, $\varepsilon_i < r_i$), a fiber-preserving 0-section-preserving diffeomorphism $\psi\colon \overline{N_{\varepsilon_i}(F_i(A))} \to \psi(\overline{N_{\varepsilon_i}(F_i(A))})$ ($\subset N_{r_i}(F_i(A))$ without loss of generality), and an orthogonal bundle-projection $P_i\colon N(F_i) \to N(F_i)$ such that

$$\bar{E} \circ \psi_i(V) = |||P_{i\sigma}(V_\sigma)|||_\sigma^2 - |||(I_i - P_i)_\sigma V_\sigma|||_\sigma^2$$

for all $V_\sigma \in \overline{N_{\varepsilon_i}(F_i)}$ (see Meyer [14]). But from index $F_i(A) = 0$ we conclude that $I_i - P_i = 0$, i.e.,

$$(*) \qquad \bar{E} \circ \psi_i(V) = |||V|||^2 \qquad \text{for all } V \in \overline{N_{\varepsilon_i}(F_i)} \ .$$

By defining $\varphi = \bigcup_i \varphi_i$ and $\psi = \bigcup_i \psi_i$, we obtain $\varphi(\psi(\bigcup_i \overline{N_\varepsilon(F_i(A))})) = \Lambda^e_{G(A)}(M)$ with $e = \min_i \varepsilon_i^2$.

1) $\varphi(\psi(\bar{N}_\varepsilon)) \subset \Lambda^e_{G(A)}$ is trivial by $(*)$, so assume that

2) there exists a $\sigma_0 \in \complement\varphi(\psi(\bar{N}_\varepsilon))$ with $E(\sigma_0) \leq e$.

Having chosen $e$ small enough we can assume that the only critical value less than or equal to $e$ is zero, otherwise we could pick a sequence of critical points of $E$ with decreasing $E$-values. By Condition $(C)$ this sequence would have a convergent subsequence necessarily converging to a critical point with $E$-value zero, i.e., to an element of $F_i(A) \subset \operatorname{Fix}(A)$ say. Thus there would exists critical points in $N_\varepsilon(F_i(A))$ besides $F_i(A)$ contradicting $(*)$.

A contradiction to 2) is now obtained as follows.

Since the energy is decreasing along integral lines for $-\operatorname{grad} E$ and any integral curve has a critical point as limit point (by Condition $(C)$ see Palais [16, § 15]), i.e., a point of $\operatorname{Fix}(A)$ by our assumption on $e$, the integral curve through $\sigma_0$ starts from the outside of $\varphi(\psi(\bar{N}_\varepsilon))$ and eventually gets into $\varphi(\psi(\bar{N}_\varepsilon))$, a contraction by $(*)$ again. Since $\bigcup_i$ {0-section of $N(F_i(A))$} is a strong deformation retract of $\bar{N}_\varepsilon(\operatorname{Fix}(A))$, this finishes the proof of the corollary. q.e.d.

For any compact subset $\Phi \subset \Lambda_{G(A)}(M)$ the function $\max_{\sigma\in\Phi} E(\varphi_t(\sigma))$, where $\varphi_t$ is the flow for $-\operatorname{grad} E$, is continuous and decreasing. Putting $C(\Phi) = \lim_{t\to\infty} \max_{\sigma\in\Phi} E(\varphi_t(\sigma))$ and using Condition $(C)$ it is then easy to see that

$C(\Phi) \in \boldsymbol{R}$ is a critical value of the energy integral. Furthermore, a proposition analogous to that on pp. 77–78 in Klingerberg [10] holds. We could define the critical values of homology classes of $\Lambda_{G(A)}(M)$ mod Fix $(A)$, study the subordinated homology classes, etc. However, we will not go into those topics and turn our attention instead to our main

**Lemma 3.4.** *Let $M$ be a compact Riemannian manifold, and $A: M \to M$ an isometry with* Fix $(A) \neq \emptyset$. *If $A$ has no nontrivial invariant geodesics, the inclusion $i$*: Fix $(A) \to \Lambda_{G(A)}(A)$ *is a homotopy equivalence.*

*In particular, if the number of components of* Fix $(A)$ *is different from that of $\Lambda_{G(A)}(M)$, then $A$ has a nontrivial invariant geodesic. By Proposition* 2.7, *it is also true if* Fix $(A) = \emptyset$.

*Proof.* Choose a base point in Fix $(A)$, and let the corresponding constant curve be the base point for $\Lambda_{G(A)}(M)$.

1) $i_q: \pi_q(\text{Fix}\,(A)) \to \pi_q(\Lambda_{G(A)}(M))$ is $1-1$.

Let $[f] \in \pi_q(\text{Fix}\,(A))$ be represented by $f: S^q \to \text{Fix}\,(A)$ such that $i_q[f] = 0$ in $\pi_q(\Lambda_{G(A)}(M))$, thus $i \circ f: S^q \to \Lambda_{G(A)}(M)$ is null homotopic in $\Lambda_{G(A)}(M)$. Let $H: S^q \times I \to \Lambda_{G(A)}(M)$ be a homotopy between $i \circ f$ and the zero-map (map into the base point). Then $H(S^q \times I) \subset \Lambda_{G(A)}(M)$ is compact, and the assumption gives that $C(H(S^q \times I)) = 0$ (see above Lemma 3.4). Choose $e$ as in Corollary 3.3 and $t_0 \in \boldsymbol{R}_+$ so that $\varphi_{t_0}(H(S^q \times I)) \subset \Lambda^e_{G(A)}(M)$, where $\varphi_t$ is the flow for $-\text{grad}\,E$. Let $D: \Lambda^e_{G(A)}(M) \to \text{Fix}\,(A)$ be a deformation retraction. Then $D \circ \varphi_{t_0} \circ H$ is a homotopy between $i \circ f$ and the constant map inside of Fix $(A)$, and hence $[f] = 0$.

2) $i_q: \pi_q(\text{Fix}\,(A)) \to \pi_q(\Lambda_{G(A)}(M))$ is onto.

Let $[F] \in \pi_q(\Lambda_{G(A)}(M))$ be represented by $F: S^q \to \Lambda_{G(A)}(M)$. Proceed as under 1) and obtain thereby that $D \circ \varphi_{t_0} \circ F: S^q \to \text{Fix}\,(A)$ is homotopic to $F$. Thus $i_q([D \circ \varphi_{t_0} \circ F]) = [F]$. Since $i_q: \pi_q(\text{Fix}\,(A)) \to \pi_q(\Lambda_{G(A)}(M))$ is an isomorphism for all $q$ in $\boldsymbol{N} \cup \{0\}$, and the spaces Fix $(A)$ and $\Lambda_{G(A)}(M)$ are ANR's, $i$ is a homotopy equivalence.

Before we go on with the study of isometries in general, let us apply Lemma 3.4 to prove the well-known

**Theorem 3.5.** *Let $M$ be a compact Riemannian manifold. Then there exists a nontrivial closed geodesic on $M$.*

*Proof.* With $A = 1_M$, $\Lambda_{G(1_M)}(M) = \Lambda_\Delta(M) = \Lambda(M)$ is the space of closed curves on $M$, and critical points of $E: \Lambda(M) \to \boldsymbol{R}$ are closed geodesics (as we have seen). Now Fix $(1_M) = M$. When $\pi_1(M) \neq 0$, by Theorem 1.3 we get $0 \neq \pi_0(\Omega(M)) \cong \pi_0(\Lambda(M))$, where $\Omega(M) = C^0(S^1, M)$ are the free loops on $M$. Thus the "particular part" of Lemma 3.4 applies.

When $\pi_1(M) = 0$, assume that there are no nontrivial closed geodesics on $M$. Then by Lemma 3.4, $i_q: \pi_q(M) \to \pi_q(\Lambda(M))$ is an isomorphism for all $q$. Since the fibration $\Lambda(M) \to \Delta \equiv M$ with fiber $\Lambda_{p_0}(M) = \Lambda_{\{p_0\} \times \{p_0\}}(M)$ has a section, the exact homotopy sequence for $\Lambda_{p_0}(M) \to \Lambda(M) \underset{i}{\rightleftarrows} M$ splits:

$$(**) \qquad 0 \longrightarrow \pi_q(\Lambda_{p_0}(M)) \longrightarrow \pi_q(\Lambda(M)) \underset{i_q}{\overset{\longrightarrow}{\longleftarrow}} \pi_q(M) \longrightarrow 0 \quad \forall q .$$

We start the induction with $\pi_0(M) = \pi_1(M) = 0$, and assume that $\pi_q(M) = 0$. Then $\pi_q(\Lambda(M)) = 0$, and thus

$$0 = \pi_q(\Lambda_{p_0}(M)) \cong \pi_q(\Omega_{p_0}(M)) \cong \pi_{q+1}(M) = 0$$

by $(**)$ and Theorem 1.3. Hence $\pi_q(M) = 0 \ \forall q \in N \cup \{0\}$, which is a contradiction (Hurewicz). q.e.d.

It is obvious that it is rather restricted what we can say in general about the space $\Lambda_{G(A)}(M)$ and therefore about the existence of invariant geodesics, not knowing much about the manifold and the isometry $A$. Let us first see what we can say if $A$ is homotopic to $1_M$.

**Lemma 3.6.** *Let $A: M \to M$ and $B: M \to M$ be homotopic. Then $\Lambda_{G(A)}(M)$ and $\Lambda_{G(B)}(M)$ have the same homotopy type.*

*Proof.* By Theorem 1.3 it is sufficient to prove that $G^0_{G(A)}(M)$ and $C^0_{G(B)}(M)$ have the same homotopy type. Let $H: M \times I \to M$ be a homotopy with $H_0 = A$ and $H_1 = B$. Then

$$F_1: C^0_{G(A)}(M) \to C^0_{G(B)}(M) , \qquad F_2: C^0_{G(B)} \to C^0_{G(A)}(M)$$

defined by

$$F_1(h)(t) = \begin{cases} h(2t) & \text{if } 0 \le t \le \frac{1}{2} , \\ H(h(0), 2t-1) & \text{if } \frac{1}{2} \le t \le 1 , \end{cases}$$

$$F_2(f)(t) = \begin{cases} f(2t) & \text{if } 0 \le t \le \frac{1}{2} , \\ H(f(0), 2(1-t)) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

are continuous. It is easy to check that $F_1 \circ F_2 \sim 1_{C^0_{G(B)}(M)}$ and $F_2 \circ F_1 \sim 1_{C^0_{G(A)}(M)}$.

**Theorem 3.7.** *Any isometry homotopic to the identity on a compact Riemannian manifold has a nontrivial invariant geodesic.*

*Proof.* Let $A \sim 1_M$ be an isometry on $M$. By Lemma 3.6, $\Lambda_{G(A)}(M) \cong \Lambda(M)$, so that

$$\pi_q(\Lambda_{G(A)}(M)) \cong \pi_q(\Lambda(M)) \qquad \text{for all } q \in N \cup \{0\} .$$

If $\dim F_i(A) \geq 1$ for some component $F_i(A)$ of Fix $(A)$, then $A$ obviously has a nontrivial invariant geodesic (Proposition 3.2), so assume that $A$ has only isolated fixed points. Thus $\pi_q(\text{Fix}\,(A)) = 0$ for $q \geq 1$, and from Lemma 3.4 it follows that if $A$ has no invariant geodesics, then $\pi_q(\Lambda_{G(A)}(M)) \cong \pi_q(\text{Fix}\,(A)) = 0$ for $q \geq 1$, i.e., $\pi_q(\Lambda(M)) = 0$ for $q \geq 1$. Using $(**)$ we get that $\pi_q(M) = 0$ for $q \geq 1$ and hence for all $q$, a contradiction.

**Corollary 3.8.** *Let $M$ be a compact Riemannian manifold, and $A: M \to M$*

*an isometry on $M$. Then there exists an $n \in N$ such that the $n$'th power $A^n = A \circ \cdots \circ A$ of $A$ has a nontrivial invariant geodesic.*

*Proof.* Since the isometry group $\mathscr{I}(M)$ of $M$ is a compact Lie group (Theorem 3.4, Chapter VI in Kobayashi and Nomizu [12]), the quontient $\mathscr{I}(M)/\mathscr{I}_0(M)$ by the identity component $\mathscr{I}_0(M)$ is a finite group. Thus some power of $A$, say $n$, satisfies $A^n \in \mathscr{I}_0(M)$, in particular, $A^n \sim 1_M$, and hence by Theorem 3.7 has a nontrivial invariant geodesic.

In the following we do not assume that $A \sim 1_M$, but instead we make restrictions on the fixed point set of $A$ together with the topology of $M$.

**Proposition 3.9.** *Let $M$ be a compact connected and simply connected Riemannian manifold. Then any isometry $A$ on $M$ without or with at least two fixed points has a nontrivial invariant geodesic.*

*Proof.* When $\operatorname{Fix}(A) = \emptyset$, the proposition is proved by using Proposition 2.7. Since $\# \operatorname{Fix}(A) = \infty$ means that $\dim F_i(A) \geq 1$ for some component $F_i(A)$ of $\operatorname{Fix}(A)$, as in the proof of Theorem 3.7 we only need to consider the case where $\operatorname{Fix}(A)$ is a finite set of points.

By Lemma 3.4, $A$ has a nontrivial invariant geodesic if $\# \pi_0(\Lambda_{G(A)}(M)) \neq \# \operatorname{Fix}(A)$, so the statement of Proposition 3.9 follows from $\pi_0(\Lambda_{G(A)}(M)) = 0$. Since the fibration $\Lambda_{G(A)}(M) \to G(A) \equiv M$ has fiber $\Lambda_{p_0}(M)$ where $A(p_0) = p_0$ is the base point in $M$, the homotopy sequence

$$\begin{array}{ccccc} \to \pi_0(\Lambda_{p_0}(M)) & \to & \pi_0(\Lambda_{G(A)}(M)) & \to & \pi_0(G(M)) \\ \wr\| & & & & \wr\| \\ \pi_0(\Omega_{p_0}(M)) & & & & \pi_0(M) \\ \| & & & & \| \\ \pi_1(M) & & & & 0 \\ \| & & & & \\ 0 & & & & \end{array}$$

gives immediately that $\pi_0(\Lambda_{G(A)}(M)) = 0$.

**Corollary 3.10.** *Let $M$ be a compact Riemannian manifold with finite fundamental group $\pi_1(M)$, and let $A: M \to M$ be an isometry.*

(1) *If $A$ has prime power order $m = p^e$ with $p$ odd or $A$ has order two, then $A$ has a nontrivial invariant geodesic.*

(2) *If the universal covering space $\tilde{M}$ of $M$ is homeomorphic to an $n$-sphere $S^n$, then $A$ has a nontrivial invariant geodesic.*

*Proof.* (1) Let $\tilde{M}$ be the universal covering space of $M$. Then $\tilde{M}$ is compact and $\pi_1(\tilde{M}) = 0$. Assume further without loss of generality that $\operatorname{Fix}(A) \neq \emptyset$ is finite. From the properties of the universal covering space, $A$ can be covered by an isometry $\tilde{A}: \tilde{M} \to \tilde{M}$ with $\operatorname{Fix}(\tilde{A}) \neq \emptyset$ and order $(\tilde{A}) =$ order $(A)$. Now Theorem 7.1 in Atiyah and Bott [1] gives that $\tilde{A}$ has more than one fixed point and hence, by Proposition 3.9, has a nontrivial invariant geodesic which then projects to a closed nontrivial invariant geodesic for $A$ on $M$ since $A^m =$

$1_M$. If $\tilde{A}$ is an involution, it cannot have exactly an odd number of fixpoints (see, e.g., Conner and Floyd [2, p. 66]).

(2) follows from a consequence of Brouwer's fixed point theorem that any isometry on a manifold homeomorphic to $S^n$ has at least two fixed points.

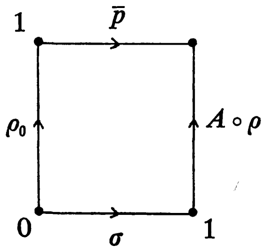
Let us now examine what we can say in the nonsimply connected case.

**Sublemma 3.11.** *Let $A: M \to M$ be an isometry on $M$ with $A(p) = p$ for some $p \in M$. If $\sigma \in \Lambda_{G(A)}(M)$ with $\sigma(0) = p$ is in the same component as $\bar{p} \in \Lambda_{G(A)}(M)$ defined by $\bar{p}(t) = p$ for all $t \in I$, then there exists a loop $\rho \in \Lambda_{G(A)}(M)$ in $p$ such that*

$$[\sigma] = [\rho] - A_*[\rho] \quad in \quad \pi_1(M) ,$$

*where $A_* \in \mathrm{Iso}\,(\pi_1(M))$ is the map induced from $A$.*

*Proof.* Let $c: I \to \Lambda_{G(A)}(M)$ be a path connecting $\sigma$ and $\bar{p}$, i.e., $c(0) = \sigma$ and $c(1) = \bar{p}$. By evaluation $c$ induces a map $\bar{c}: I \times I \to M$. Putting $\rho = \bar{c}(\cdot, 0)$ we get $A \circ \rho = \bar{c}(\cdot, 1)$ both being loops at $p$, i.e.,




$$\begin{aligned}[\sigma] &= [\rho] + [\bar{p}] + [-A \circ \rho] \\ &= [\rho] - A_*[\rho] \qquad \text{in } \pi_1(M) .\end{aligned}$$

**Proposition 3.12.** *Let $M$ be a compact Riemannian manifold with $\pi_1(M) \neq 0$, and let $A: M \to M$ be an isometry with exactly one fixed point. If the map*

$$\pi_1(M) \xrightarrow{\Delta} \pi_1(M) \times \pi_1(M) \xrightarrow{1_{\pi_1} \times (-A_*)} \pi_1(M) \times \pi_1(M) \xrightarrow{+} \pi_1(M)$$

*is not onto, then $A$ has a nontrivial invariant geodesic.*

*Proof.* By Lemma 3.4 we only need to prove that $\pi_0(\Lambda_{G(A)}(M)) \neq 0$. From the assumption we have

$$\exists[\sigma] \in \pi_1(M) : [\sigma] \neq [\rho] - A_*[\rho] \; \forall[\rho] \in \pi_1(M) .$$

Sublemma 3.11 shows that $\sigma \in \Lambda_{G(A)}(M)$ (without loss of generality) is not in the same component as $\overline{\sigma(0)} = \bar{p} \in \Lambda_{G(A)}(M)$; as remarked this finishes the proof of Proposition 3.12.

**Corollary 3.13.** *If $M$ is a compact Riemannian manifold such that $H_1(M; Z)$ has an odd number of $Z$-components or a single $Z_{2\cdot q}$ term for some $q \in N$, then any isometry on $M$ with exactly one fixed point has an invariant geodesic.*

*Proof*. An isometry with one fixed point induces an isomorphism of $\pi_1(M)$, so from Proposition 3.12 we see that if the map $\pi_1(M) \to \pi_1(M)$ defined by $x \mapsto x - \mathfrak{A}(x)$ is not onto for $\mathfrak{A} \in \mathrm{Iso}\,(\pi_1(M))$, any isometry with exactly one fixed point has a nontrivial invariant geodesic. However, if the lower map in the commutative diagram

$$\begin{array}{ccc} \pi_1(M) & \longrightarrow & \pi_1(M) \\ \downarrow & & \downarrow \\ \pi_1(M)/[\pi_1, \pi_1] & \longrightarrow & \pi_1(M)/[\pi_1, \pi_1] \end{array}$$

is not onto, the upper one is not onto either. Since $H_1(M, Z) \cong \pi_1(M)/[\pi_1, \pi_1]$, the statement of the corollary follows.

**Remark.** Example 3.1 shows that our result in Proposition 3.12 in some way is best possible. The general case where $A$ has more than one but finitely many fixed points is more difficult to handle. It is, however, easy to see from Lemma 3.4 that if there exist a path $\alpha: I \to M$ on $M$ with $A(\alpha(0)) = \alpha(0)$ and $A(\alpha(1)) = \alpha(1)$ such that $\alpha$ is homotopic to $A \circ \alpha$ rerative to the end points, then $A$ has a nontrivial invariant geodesic. Also, if there exists a component of $\Lambda_{G(A)}(M)$ without fixed points, then $A$ has a nontrivial invariant geodesic; in terms of curves on $M$ this means that there exists a path $\beta: I \to M$ on $M$ such that any loop determined by paths $\gamma: I \to M$, $\beta$ and $-A \circ \gamma$ with $A(\gamma(0)) = \gamma(0)$ and $\gamma(1) = \beta(0)$ is not null-homotopic relative to $\gamma(0)$.

Combining Proposition 2.7, Proposition 3.2, Theorem 3.7, Proposition 3.9, and Corollary 3.13, we get

**Theorem 3.14.** *Let $M$ be a compact connected Riemannian manifold.*

(1) *Any isometry homotopic to the identity map $1_M$ of $M$ has a nontrivial invariant geodesic.*

(2) *Any isometry with not exactly one fixed point has a nontrivial invariant geodesic if the fundamental group $\pi_1(M)$ is zero.*

(3) *If $M$ is not simply connected but $H_1(M; Z)$ has a single $Z_{2q}$ torsion term for some $q \in N$ or an odd number of $Z$-summands, then any isometry with exactly one fixed point has a nontrivial invariant geodesic.*

(4) *Any isometry without or with infinitely many fixed points has a nontrivial invariant geodesic.*

**Remark.** Given a diffeomorphism on a compact manifold we can deduce the existence of invariant 1-dimensional immersed submanifolds under conditions like those in Theorem 3.14 if the diffeomorphism is an element in a compact subgroup of the diffeomorphism group Diff $(M)$ of $M$.

There might be other ways to get existence theorems for geodesics invariant under an isometry, e.g., by theory of dynamical systems used on the geodesic spray:

If $A: M \to M$ is an isometry on a complete Riemannian manifold $M$, then the differential $A_*: TM \to TM$ of $A$ induces a map on the sphere bundle $A_*: STM \to STM$ which commutes with the action of the real line on $STM$ (the flow for the geodesic spray restricted to the sphere bundle of the tangent bundle), and therefore we get a continuous map $\tilde{A}_*: STM/\boldsymbol{R} \to STM/\boldsymbol{R}$. The space $STM/\boldsymbol{R}$ is in 1—1 correspondence with the set of oriented maximal geodesics on $M$, and the fixed points for $\tilde{A}_*$ are in 1—1 correspondence with nontrivial $A$-invariant geodesics on $M$, so the question is: Under what conditions on $(M, \langle\cdot, \cdot\rangle)$ and $A$, does $\tilde{A}_*$ have fixed points?

## References

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AARHUS UNIVERSITY, DENMARK