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DIFFERENTIAL GEOMETRY OF $S^n \times S^n$

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To Shoshichi Kobayashi on his fortieth birthday

0. Introduction

Blair [1], $\cdots$, [5], Eum [9], Ishihara [10], Ki [9], [11], [12], Ludden [1], $\cdots$, [5], Okumura [13], [14], [15] and the present author [2], $\cdots$, [5], [7], $\cdots$, [15] started the study of the structures induced on submanifolds of codimension 2 of an almost Hermitian manifold or on hypersurfaces of an almost contact metric manifold. Okumura and the present author called these structures (f, g, u, v, λ)-structures, where f is a tensor field of type $(1, 1)$, g a Riemannian metric, u and v 1-forms, and λ a function satisfying

$$
\begin{gathered}
f^2 = -1 + u \otimes U + v \otimes V , \\
u \circ f = \lambda v , \quad v \circ f = -\lambda u , \quad fU = -\lambda V , \quad fV = \lambda U , \\
u(U) = 1 - \lambda^2 , \quad u(V) = 0 , \quad v(V) = 1 - \lambda^2 , \\
g(fX, fY) = g(X, Y) - u(X)u(Y) - v(X)v(Y)
\end{gathered}
$$

for arbitrary vector fields X and Y, where U and V are vector fields associated with 1-forms u and v respectively.

An (f, g, u, v, λ)-structure is said to be normal if it satisfies $S = 0$ where S is a tensor field of type $(1, 2)$ defined by

$$
S(X, Y) = N(X, Y) + (du)(X, Y)U + (dv)(X, Y)V
$$

for arbitrary vector fields X and Y, N being the Nijenhuis tensor formed with f.

A typical example of a differentiable manifold with a normal (f, g, u, v, λ)-structure is an even-dimensional sphere S^{2n}. Ki [11], [12], Okumura [14] and the present author [11], [12], [14] obtained some characterizations of an even-dimensional sphere from this point of view.

The product $S^n \times S^n$ of two spheres of the same radius and the same dimension is also an example of a differentiable manifold with an (f, g, u, v, λ)-structure, but the structure is not normal. Blair [3], [5], Ishihara [10], Ludden

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[3], [5] and the present author [3], [5], [10] obtained some characterizations of $S^n \times S^n$.

The main purpose of the present paper is to study the differential geometry of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ as a submanifold of codimension 2 of a $(2n+2)$-dimensional Euclidean space E^{2n+2} or as a hypersurface of a $(2n+1)$-dimensional sphere S^{2n+1}, to derive the properties of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ as a $2n$-dimensional differentiable manifold admitting an (f, g, u, v, λ)-structure, and to give some characterizations of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.

1. $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ as a submanifold of codimension 2 of E^{2n+2}

Let E^{n+1} be an $(n+1)$-dimensional Euclidean space and O the origin of a cartesian coordinate system in E^{n+1}, and denote by X the position vector of a point P in E^{n+1} with respect to the origin O.

Consider a sphere $S^n(1/\sqrt{2})$ with center at O and radius $1/\sqrt{2}$, and suppose that $S^n(1/\sqrt{2})$ is covered by a system of coordinate neighborhoods $\{U; x^a\}$. Here and in the sequel the indices a, b, c, d, e, f run over the range $\{1, \cdots, n\}$. Then the position vector X of a point P on $S^n(1/\sqrt{2})$ is a function of x^a satisfying $X \cdot X = \frac{1}{2}$ where the dot denotes the inner product of two vectors in a Euclidean space. Now we put

$$X_b = \partial_b X\,, \quad M = -\sqrt{2}X\,, \quad g_{cb} = X_c \cdot X_b\,, \tag{1.1}$$

where $\partial_b = \partial/\partial x^b$, and denote by ∇_c the operator of covariant differentiation with respect to the Christoffel symbols $\{_c{}^a{}_b\}$ formed with the metric tensor g_{cb} of $S^n(1/\sqrt{2})$. Since X_b is tangent to $S^n(1/\sqrt{2})$ and M is the unit normal to $S^n(1/\sqrt{2})$, the equations of Gauss and Weingarten are respectively of the form

$$\nabla_c X_b = \sqrt{2}g_{cb}M\,, \qquad \nabla_c M = -\sqrt{2}X_c\,. \tag{1.2}$$

We next suppose that $S^n(1/\sqrt{2})$ is covered by a system of coordinate neighborhoods $\{V; x^r\}$. Here and in the sequel the indices r, s, t, u, v, w run over the range $\{n+1, \cdots, 2n\}$. Then the position vector Y of a point Q on $S^n(1/\sqrt{2})$ is a function of x^r satisfying $Y \cdot Y = \frac{1}{2}$. We now put

$$Y_s = \partial_s Y\,, \quad N = -\sqrt{2}Y\,, \quad g_{ts} = Y_t \cdot Y_s\,, \tag{1.3}$$

where $\partial_s = \partial/\partial x^s$, and denote by ∇_t the operator of covariant differentiation with respect to the Christoffel symbols $\{_t{}^r{}_s\}$ formed with the metric tensor g_{ts} of $S^n(1/\sqrt{2})$. Since Y_s is tangent to $S^n(1/\sqrt{2})$ and N is the unit normal to $S^n(1/\sqrt{2})$, the equations of Gauss and Weingarten are respectively of the form

$$\nabla_t Y_s = \sqrt{2}g_{ts}N\,, \qquad \nabla_t N = -\sqrt{2}Y_t\,. \tag{1.4}$$

We now consider $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ and regard it as a submanifold of codimension 2 in an E^{2n+2}. Denoting by Z the position vector of a point of

$S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$, we have

$$Z(x^h) = \begin{pmatrix} X(x^a) \\ Y(x^r) \end{pmatrix} . \tag{1.5}$$

Here and in the sequel the indices h, i, j, k, l, m run over the range $\{1, \cdots, n; n+1, \cdots, 2n\}$. Since $Z \cdot Z = X \cdot X + Y \cdot Y = 1$ in E^{2n+2}, $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ is a hypersurface of $S^{2n+1}(1)$ in E^{2n+2}.

By putting

$$Z_i = \partial_i Z , \qquad G_{ji} = Z_j \cdot Z_i , \tag{1.6}$$

we see that

$$Z_b = \begin{pmatrix} X_b \\ 0 \end{pmatrix} , \qquad Z_s = \begin{pmatrix} 0 \\ Y_s \end{pmatrix} , \tag{1.7}$$

$$G_{ji} = \begin{pmatrix} g_{cb} & 0 \\ 0 & g_{ts} \end{pmatrix} , \tag{1.8}$$

and hence

$$G^{ih} = \begin{pmatrix} g^{ba} & 0 \\ 0 & g^{sr} \end{pmatrix} , \tag{1.9}$$

G^{ih}, g^{ba} and g^{sr} being elements of the inverse matrices of (G_{ji}), (g_{cb}) and (g_{ts}) respectively.

Because of (1.8) and (1.9), we shall denote G_{ji} hereafter by g_{ji}. The Christoffel symbols $\{{}_j{}^h{}_i\}$ formed with g_{ji} are all zero except $\{{}_c{}^a{}_b\}$ and $\{{}_t{}^r{}_s\}$. In the sequel, we denote by ∇_i the operator of covariant differentiation with respect to the Christoffel symbols $\{{}_j{}^h{}_i\}$.

Now putting

$$C = \begin{pmatrix} -X(x^a) \\ -Y(x^r) \end{pmatrix} , \qquad D = \begin{pmatrix} -X(x^a) \\ Y(x^r) \end{pmatrix} , \tag{1.10}$$

we see that

$$Z_i \cdot C = 0 , \quad Z_i \cdot D = 0 , \quad C \cdot C = 1 , \quad C \cdot D = 0 , \quad D \cdot D = 1 , \tag{1.11}$$

and consequently that C and D are unit normals to $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.

Denoting by h_{ji} and k_{ji} the components of the second fundamental tensors respectively with respect to the unit normals C and D, we can write the equations of Gauss for $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ as

$$\nabla_j Z_i = h_{ji} C + k_{ji} D . \tag{1.12}$$

From (1.2), (1.4), (1.10) and (1.12) it follows that h_{ji} and k_{ji} are of the form

$$h_{ji} = \begin{pmatrix} g_{cb} & 0 \\ 0 & g_{ts} \end{pmatrix}, \qquad k_{ji} = \begin{pmatrix} g_{cb} & 0 \\ 0 & -g_{ts} \end{pmatrix} \tag{1.13}$$

and hence

$$h_j{}^h = \begin{pmatrix} \delta_c^a & 0 \\ 0 & \delta_t^r \end{pmatrix}, \qquad (k_j{}^h) = \begin{pmatrix} \delta_c^a & 0 \\ 0 & -\delta_t^r \end{pmatrix} \tag{1.14}$$

respectively, where $h_j{}^h = h_{ji}g^{ih}$ and $k_j{}^h = k_{ji}g^{ih}$.

The first equation of (1.13) and the second equation of (1.14) imply immediately that

$$h_{ji} = g_{ji}\ , \tag{1.15}$$

$$k_m{}^m = 0\ , \qquad k_j{}^m k_m{}^h = \delta_j^h\ . \tag{1.16}$$

Also taking account of the fact that $k_j{}^h$ has the form given by the second equation of (1.14) and the Christoffel symbols $\{_j{}^h{}_i\}$ are all zero except $\{_c{}^a{}_b\}$ and $\{_t{}^r{}_s\}$, we find

$$\nabla_j k_i{}^h = 0\ . \tag{1.17}$$

On the other hand, denoting by l_j the components of the third fundamental tensor with respect to unit normals C and D, we can write the equations of Weingarten as

$$\nabla_j C = -h_j{}^i Z_i + l_j D\ , \qquad \nabla_j D = -k_j{}^i Z_i - l_j C\ . \tag{1.18}$$

From (1.10), (1.14) and (1.18) it follows that

$$l_j = 0\ . \tag{1.19}$$

Thus the equations of Gauss and Weingarten of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ as a submanifold of codimension 2 of E^{2n+2} are respectively

$$\nabla_j Z_i = g_{ji}C + k_{ji}D\ , \tag{1.20}$$

$$\nabla_j C = -Z_j\ , \qquad \nabla_j D = -k_j{}^i Z_i\ , \tag{1.21}$$

from which we can easily derive

$$K_{kji}{}^h = \delta_k^h g_{ji} - \delta_j^h g_{ki} + k_k{}^h k_{ji} - k_j{}^h k_{ki}\ , \tag{1.22}$$

which are the equations of Gauss, $K_{kji}{}^h$ being the components of the curvature tensor of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$, that is,

$$K_{kji}{}^h = \partial_k\{{}_j{}^h{}_i\} - \partial_j\{{}_k{}^h{}_i\} + \{{}_k{}^h{}_l\}\{{}_j{}^l{}_i\} - \{{}_j{}^h{}_l\}\{{}_k{}^l{}_i\} \, . \tag{1.23}$$

From (1.17) and (1.22) it follows that

$$\nabla_l K_{kji}{}^h = 0 \, , \tag{1.24}$$

and consequently $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ is a locally symmetric Riemannian manifold. This can also be seen from the fact that the product of two locally symmetric manifolds is locally symmetric.

By (1.16) and (1.22) we have

$$K_{ji} = 2(n-1)g_{ji} \, , \tag{1.25}$$

K_{ji} being the Ricci tensor, that is, $K_{ji} = K_{lji}{}^l$. Thus $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ is an Einstein manifold with scalar curvature $4n(n-1)$. This can also be seen from the fact that the product of two Einstein manifolds of the same dimension with the same scalar curvature is also an Einstein manifold whose scalar curvature is twice as that of each factor manifold.

2. $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ as a hypersurface of $S^{2n+1}(1)$

Consider an $S^{2n+1}(1)$ in E^{2n+2} covered by a system of coordinate neighborhoods $\{W; y^\kappa\}$. Here and in the sequel the indices $\kappa, \lambda, \mu, \nu, \omega$ run over the range $\{1, \cdots, 2n+1\}$. Then the position vector Z of a point on $S^{2n+1}(1)$ in E^{2n+2} is a function of y^κ such that $Z \cdot Z = 1$. We put

$$Z_\lambda = \partial_\lambda Z \, , \quad C = -Z \, , \quad G_{\mu\lambda} = Z_\mu \cdot Z_\lambda \, , \tag{2.1}$$

where $\partial_\lambda = \partial/\partial y^\lambda$, and denote by ∇_λ the operator of covariant differentiation with respect to the Christoffel symbols $\{{}_\mu{}^\kappa{}_\lambda\}$ formed with $G_{\mu\lambda}$. Since Z_λ is tangent to $S^{2n+1}(1)$ and C is the unit normal to $S^{2n+1}(1)$, the equations of Gauss and Weingarten are respectively of the form

$$\nabla_\mu Z_\lambda = G_{\mu\lambda} C \, , \qquad \nabla_\mu C = -Z_\mu \, . \tag{2.2}$$

Since $S^n(1/\sqrt{2}) \times (S^n(1/\sqrt{2})$ is a hypersurface of $S^{2n+1}(1)$ and is covered by a system of coordinate neighborhoods $\{U \times V; x^h\}$, its equations are of the form $y^\kappa = y^\kappa(x^h)$. Denote by D^κ the components of the unit normal to $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ as a hypersurface of $S^{2n+1}(1)$, and put $D = D^\kappa Z_\kappa$. Then $Z_i = B_i{}^\kappa Z_\kappa$, where $B_i{}^\kappa = \partial_i y^\kappa$, are $2n$ vectors tangent to $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$, and C and D are mutually orthogonal unit normals to $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$. Thus from $Z_i = B_i{}^\kappa Z_\kappa$ we have

$$\nabla_j Z_i = (\nabla_j B_i{}^\kappa) Z_\kappa + B_j{}^\mu B_i{}^\lambda \nabla_\mu Z_\lambda \, ,$$

which, together with the first equation of (2.2), implies

$$\nabla_j Z_i = (\nabla_j B_i^\kappa) Z_\kappa + g_{ji} C .$$

By this equation, (1.12) and $D = D^\kappa Z_\kappa$, we have

$$h_{ji} C + k_{ji} D^\kappa Z_\kappa = (\nabla_j B_i^\kappa) Z_\kappa + g_{ji} C ,$$

from which it follows that

$$\nabla_j B_i^\kappa = k_{ji} D^\kappa , \tag{2.3}$$

which are the equations of Gauss of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ as a hypersurface of $S^{2n+1}(1)$. The equations of Weingarten are easily found to be

$$\nabla_j D^\kappa = -k_j{}^i B_i^\kappa . \tag{2.4}$$

Since $k_l{}^l = 0$, we have the well known

Proposition 2.1. *$S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ is a minimal hypersurface of $S^{2n+1}(1)$.*

3. (f, g, u, v, λ)-structure on $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$

In E^{2n+2}, there exists a natural Kählerian structure

$$F = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} , \tag{3.1}$$

E being the unit square matrix of order $n + 1$. Of course, F satisfies

$$F^2 = -1 , \qquad FU \cdot FV = U \cdot V \tag{3.2}$$

for arbitrary vectors U and V in E^{2n+2}, 1 denoting the identity transformation in E^{2n+2}.

Applying F to Z_i, C and D in § 1 gives

$$FZ_i = f_i{}^h Z_h + u_i C + v_i D , \tag{3.3}$$

$$FC = -u^i Z_i + \lambda D , \tag{3.4}$$

$$FD = -v^i Z_i - \lambda C , \tag{3.5}$$

where $f_i{}^h$ are the components of a tensor field of type (1, 1), u_i and v_i are the components of 1-forms, and λ is a function on $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$, u^i and v^i being respectively given by $u^i = u_j g^{ji}$ and $v^i = v_j g^{ji}$.

From (3.2), (3.3), (3.4) and (3.5), we find

$$
(3.6)\quad \begin{gathered}
f_j{}^i f_i{}^h = -\delta_j^h + u_j u^h + v_j v^h , \\
u_i f_j{}^i = \lambda v_j , \quad f_i{}^h u^i = -\lambda v^h , \quad v_i f_j{}^i = -\lambda u_j , \quad f_i{}^h v^i = \lambda u^h , \\
u_i u^i = v_i v^i = 1 - \lambda^2 , \qquad u_i v^i = 0 , \\
f_j{}^m f_i{}^l g_{ml} = g_{ji} - u_j u_i - v_j v_i .
\end{gathered}
$$

A set of f, g, u, v and λ satisfying these equations is called an (f, g, u, v, λ)-structure [8], [13], [14]. It is easily verified that $f_{ji} = f_j{}^l g_{li}$ is skew-symmetric in j and i.

By putting $i = b$ in (3.3), we obtain

$$
(3.7)\qquad f_b{}^a = 0 , \qquad u_b + v_b = 0 ,
$$

$$
(3.8)\qquad X_b = f_b{}^r Y_r - 2u_b Y .
$$

Similarly, by putting $i = s$ in (3.3), we find

$$
(3.9)\qquad f_s{}^r = 0 , \qquad u_s = v_s ,
$$

$$
(3.10)\qquad Y_s = -f_s{}^a X_a - 2u_s X .
$$

Thus $f_i{}^h, u_i, u^h, v_i$ and v^h are respectively of the form

$$
(3.11)\qquad f_i{}^h = \begin{pmatrix} 0 & f_s{}^a \\ f_b{}^r & 0 \end{pmatrix} ,
$$

$$
(3.12)\qquad u_i = (u_b, u_s), \qquad u^h = \begin{pmatrix} u^a \\ u^r \end{pmatrix} ,
$$

where $u^a = u_b g^{ba}$, $u^r = u_s g^{sr}$ and

$$
(3.13)\qquad v_i = (-u_b, u_s), \qquad v^h = \begin{pmatrix} -u^a \\ u^r \end{pmatrix} .
$$

From the second equations of (1.14) and (3.11) it follows that

$$
(3.14)\qquad k_l{}^h f_i{}^l + f_l{}^h k_i{}^l = 0 ,
$$

that is, $k_i{}^h$ and $f_i{}^h$ anti-commute with each other. From the scond equations of (1.14), (3.12) and (3.13); (3.4) or (3.5); the first equations of (3.6) and (3.13) and the second equations of (3.6) and (3.13) we obtain, respectively,

$$
(3.15)\qquad k_i{}^h u^i = -v^h , \qquad k_i{}^h v^i = -u^h ,
$$

$$
(3.16)\qquad X = u^r Y_r - \lambda Y , \qquad Y = -u^a X_a - \lambda X ,
$$

$$
(3.17)\qquad f_c{}^r f_r{}^a = -\delta_c^a + 2u_c u^a , \qquad f_t{}^a f_a{}^r = -\delta_t^r + 2u_t u^r ,
$$

$$u_r f_c{}^r = -\lambda u_c\ , \quad f_v{}^a u^r = \lambda u^a\ , \quad u_a f_r{}^a = \lambda u_r\ , \quad f_c{}^r u^c = -\lambda u^r\ . \tag{3.18}$$

Moreover, from $u_i u^i = 1 - \lambda^2$ or $v_i v^i = 1 - \lambda^2$, $u_i v^i = 0$, and the last equations of (3.6) and (3.13), we have, respectively,

$$2u_a u^a = 1 - \lambda^2\ , \tag{3.19}$$

$$u_a u^a = u_r u^r\ , \tag{3.20}$$

$$f_c{}^t f_b{}^s g_{ts} = g_{cb} - 2u_c u_b\ , \qquad f_t{}^c f_s{}^b g_{cb} = g_{ts} - 2u_t u_s\ . \tag{3.21}$$

Now applying the operator ∇_j of covariant differentiation to (3.3), (3.4) and (3.5) and taking account of $\nabla_j F = 0$, we find

$$\begin{gathered} \nabla_j f_i{}^h = -g_{ji} u^h + \delta_j^h u_i - k_{ji} v^h + k_j{}^h v_i\ , \\ \nabla_j u_i = f_{ji} - \lambda k_{ji}\ , \quad \nabla_j v_i = -k_{jl} f_i{}^l + \lambda g_{ji}\ , \quad \nabla_j \lambda = -2v_j\ . \end{gathered} \tag{3.22}$$

From the first and the second equations of (3.22) we obtain, respectively,

$$\begin{aligned} &\partial_c f_s{}^a + \{{}_c{}^a{}_b\} f_s{}^b = 2\delta_c^a u_s\ , && \partial_t f_s{}^a - \{{}_t{}^r{}_s\} f_r{}^a = -2g_{ts} u^a\ , \\ &\partial_c f_b{}^r - \{{}_c{}^a{}_b\} f_a{}^r = -2g_{cb} u^r\ , && \partial_t f_b{}^r + \{{}_t{}^r{}_s\} f_b{}^s = 2\delta_t^r u_b\ , \end{aligned} \tag{3.23}$$

$$\begin{aligned} &\partial_c u_b - \{{}_c{}^a{}_b\} u_a = -\lambda g_{cb}\ , && \partial_c u_s = f_{cs}\ , \\ &\partial_t u_s - \{{}_t{}^r{}_s\} u_r = +\lambda g_{ts}\ , && \partial_t u_b = f_{tb}\ . \end{aligned} \tag{3.24}$$

From the third equation of (3.22) we find the same equations as those in (3.24). From the last equation of (3.22) we find

$$\nabla_b \lambda = 2u_b\ , \qquad \nabla_s \lambda = -2u_s\ . \tag{3.25}$$

From the first equation of (3.24), which can also be written as $\nabla_c u_b = -\lambda g_{cb}$, and the first equation of (3.25), it follows that

$$\nabla_c \nabla_b \lambda = -2\lambda g_{cb}\ . \tag{3.26}$$

Similarly we have

$$\nabla_t \nabla_s \lambda = -2\lambda g_{ts}\ . \tag{3.27}$$

By putting

$$\begin{aligned} S_{ji}{}^h = f_j{}^m \nabla_m f_i{}^h - f_i{}^m \nabla_m f_j{}^h - (\nabla_j f_i{}^m - \nabla_i f_j{}^m) f_m{}^h \\ + (\nabla_j u_i - \nabla_i u_j) u^h + (\nabla_j v_i - \nabla_i v_j) v^h\ , \end{aligned} \tag{3.28}$$

we obtain, in consequence of (3.14) and (3.22),

$$S_{ji}{}^h = -2(k_j{}^m f_m{}^h v_i - k_i{}^m f_m{}^h v_j)\ , \tag{3.29}$$

which becomes, due to the third equations of (3.22) written as $\nabla_i v^h = k_i{}^m f_m{}^h + \lambda\delta_i^h$,

$$S_{ji}{}^h = 2v_j(\nabla_i v^h - \lambda\delta_i^h) - 2v_i(\nabla_j v^h - \lambda\delta_j^h) . \tag{3.30}$$

Taking account of (3.15), we have, from (3.22),

$$\begin{gathered} u^j\nabla_j f_i{}^h = 0 ; \qquad v^j\nabla_j f_i{}^h = 2(u_i v^h - v_i u^h) , \\ u^j\nabla_j u_i = 0 , \quad v^j\nabla_j u_i = 2\lambda u_i , \quad u^i\nabla_j v_i = 0 , \quad v^j\nabla_j v_i = 2\lambda v_i , \\ u^j\nabla_j\lambda = 0 , \quad v^j\nabla_i\lambda = -2(1-\lambda^2) . \end{gathered} \tag{3.31}$$

Since the first equation of (3.22) can be written as

$$\nabla_j f_{ih} = -g_{ji}u_h + g_{jh}u_i - k_{ji}v_h + k_{jh}v_i ,$$

by applying the operator ∇_k of the covariant differentiation to the second equation of (3.22) we find, by using (1.17) and the last equation of (3.22),

$$\nabla_k\nabla_j u_i = -g_{kj}u_i + g_{ki}u_j - k_{kj}v_i + k_{ki}v_j + 2v_k k_{ji} . \tag{3.32}$$

Differentiating covariantly the third equation of (3.22) written as $\nabla_j v_i = -k_j{}^l f_{il} + \lambda g_{ji}$ gives

$$\nabla_k\nabla_j v_i = -k_{kj}u_i - k_{ki}u_j - g_{kj}v_i - g_{ki}v_j - 2v_k g_{ji} . \tag{3.33}$$

To compute the sectional curvature $K(\gamma)$ with respect to the section γ spanned by u^h and v^h, assume that $1-\lambda^2$ is not zero at the point under consideration. Since the covariant components K_{kjih} of the curvature tensor and $K(\gamma)$ are given by

$$K_{kjih} = g_{kh}g_{ji} - g_{jh}g_{ki} + k_{kh}k_{ji} - k_{jh}k_{ki} , \tag{3.34}$$

$$K(\gamma) = -K_{kjih}u^k v^j u^i v^h/(u_j u^j v_i v^i) ,$$

$$K(\gamma) = 0 . \tag{3.35}$$

To close this section, we sum up all the results obtained up to here on $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ as a hypersurface of $S^{2n+1}(1) \subset E^{2n+2}$ admitting an (f, g, u, v, λ)-structure.

The second fundamental tensor k_{ji} appearing in the equations (1.12) and (1.18) of Gauss and Weingarten and the curvature tensor $K_{kji}{}^h$ satisfy (1.16), (1.17), (1.22), (1.24), (1.25) and

$$K = 4n(n-1) . \tag{I}$$

The (f, g, u, v, λ)-structure induced on $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ satisfies (3.14), (3.15), (3.22), (3.30), (3.31), (3.32), (3.33) and

$$\nabla_j\lambda = k_{ji}u^i - v_j \, , \tag{3.36}$$

$$k_{jm}f_i{}^m - k_{im}f_j{}^m = 0 \, , \tag{II}$$

$$K_{kjih}u^k v^j u^i v^h = 0 \, . \tag{III}$$

For an orientable $2n$-dimensional differentiable manifold M^{2n} immersed in $S^{2n+1}(1)$ as a hypersurface by the immersion $i: M^{2n} \to S^{2n+1}(1) \subset E^{2n+2}$, we choose the first unit normal C in the direction opposite to that of the radius vector of $S^{2n+1}(1)$, and the second unit normal D in the direction normal to M^{2n} and tangent to $S^{2n+1}(1)$. Then we have (2.3) and (2.4) as the equations of Gauss and Weingarten, and the first three equations of (3.22) and (3.36) as the equations satisfied by the (f, g, u, v, λ)-structure induced on M^{2n}.

4. Hypersurfaces λ = constant of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$

In this section, we study the submanifold of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ defined by

$$\lambda = \text{constant} \, , \qquad \lambda^2 < 1 \, . \tag{4.1}$$

Since $v_i v^i = 1 - \lambda^2 \neq 0$, we have

$$\nabla_i\lambda = -2v_i \neq 0 \, , \tag{4.2}$$

so that λ = constant ($\lambda^2 < 1$) defines a hypersurface M^{2n-1} of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$. Thus we can cover M^{2n-1} by a system of coordinate neighborhoods $\{W; y^a\}$, and represent M^{2n-1} by

$$x^h = x^h(y^a) \, . \tag{4.3}$$

Here and throughout this section the indices a, b, c, d, e run over the range $\{1, \cdots, 2n-1\}$. Put

$$B_b{}^h = \partial_b x^h \qquad (\partial_b = \partial/\partial y^b) \, . \tag{4.4}$$

Then $B_b{}^h$ are $2n-1$ linearly independent vectors tangent to M^{2n-1} and

$$v_i B_b{}^i = 0 \, . \tag{4.5}$$

The unit normal to M^{2n-1} is represented by

$$N^h = v^h/\sqrt{1-\lambda^2} \, . \tag{4.6}$$

Since u^h is orthogonal to v^h and consequently tangent to M^{2n-1}, we can put

$$u^h = u^a B_a{}^h \ . \tag{4.7}$$

Represent the transform $f_i{}^h B_b{}^i$ of $B_b{}^i$ by $f_i{}^h$ as a linear combination of $B_a{}^h$ and N^h:

$$f_i{}^h B_b{}^i = f_b{}^a B_a{}^h + f_b N^h \ , \tag{4.8}$$

where $f_b{}^a$ is a tensor field of type (1, 1), and f_b a 1-form in M^{2n-1}. Then the transform $f_i{}^h N^i$ of N^i by $f_i{}^h$ can be written as

$$f_i{}^h N^i = -f^a B_a{}^h \ , \tag{4.9}$$

where f^a is the vector field of M^{2n-1} associated with the 1-form f_b with respect to the induced metric $g_{cb} = g_{ji} B_c{}^j B_b{}^i$ on M^{2n-1}. From $f_i{}^h v^i = \lambda u^h$, (4.6), (4.7) and (4.9) we obtain

$$f^a = -\lambda u^a / \sqrt{1-\lambda^2} \ , \qquad f_b = -\lambda u_b / \sqrt{1-\lambda^2} \ , \tag{4.10}$$

where $u_b = g_{ba} u^a$. Putting

$$\eta^a = u^a / \sqrt{1-\lambda^2} \ , \qquad \eta_b = u_b / \sqrt{1-\lambda^2} \ , \tag{4.11}$$

we have

$$f^a = -\lambda \eta^a \ , \qquad f_b = -\lambda \eta_b \ , \tag{4.12}$$

$$\eta_a \eta^a = 1 \ . \tag{4.13}$$

Thus (4.8) and (4.9) can be written respectively as

$$f_i{}^h B_b{}^i = f_b{}^a B_a{}^h - \lambda \eta_b N^h \ , \tag{4.14}$$

$$f_i{}^h N^i = \lambda \eta^a B_a{}^h \ . \tag{4.15}$$

If the transform $f_i{}^h B_b{}^i$ of $B_b{}^i$ by $f_i{}^h$ is tangent to the hypersurface, the hypersurface is said to be invariant. Thus we have

Theorem 4.1. *The hypersurface $\lambda =$ constant ($\lambda^2 < 1$) of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ is invariant if and only if $\lambda = 0$.*

Transvecting $f_h{}^k$ to (4.14) and taking account of the first equation of (3.6), (4.14), (4.15) and $u_b u^a = (1-\lambda^2)\eta_b \eta^a$, we can easily obtain

$$f_b{}^c f_c{}^a = -\delta_b^a + \eta_b \eta^a \ . \tag{4.16}$$

Transvecting u_h to (4.14) we find $\lambda v_i B_b{}^i = f_b{}^a u_a$, which implies

$$f_b{}^a \eta_a = 0 \ . \tag{4.17}$$

Transvecting $B_c{}^k B_b{}^h$ to $f_k{}^j f_h{}^i g_{ji} = g_{kh} - u_k u_h - v_k v_h$ and taking account of (4.14) and $u_c u_b = (1 - \lambda^2)\eta_c\eta_b$, we find

$$f_c{}^e f_b{}^d g_{ed} = g_{cb} - \eta_c\eta_b \ . \tag{4.18}$$

From (4.13), (4.16), (4.17) and (4.18) we thus have

Theorem 4.2. *The hypersurface $\lambda = constant$ ($\lambda^2 < 1$) of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ admits an almost contact metric structure.*

Represent the transform $k_i{}^h B_b^i$ of $B_b{}^i$ by $k_i{}^h$ as a linear combination of $B_a{}^h$ and N^h:

$$k_i{}^h B_b{}^i = k_b{}^a B_a{}^h + k_b N^h \ , \tag{4.19}$$

where $k_b{}^a$ is a tensor field of type (1, 1), and k_b a 1-form in M^{2n-1}. As to the transform $k_i{}^h N^i$ of N^i by $k_i{}^h$, by (3.15), (4.6), (4.7), (4.11) we obtain

$$k_i{}^h N^i = -\eta^a B_a{}^h \ . \tag{4.20}$$

Transvecting u_h to (4.19) and remembering $u_h k_i{}^h = -v_i$, we find $k_b{}^a u_a = 0$, which and (4.11) imply

$$k_b{}^a \eta_a = 0 \ . \tag{4.21}$$

Transvecting v_h to (4.19) and remembering $v_h k_i{}^h = -u_i$, we find $-u_b = k_b v_h N^h$, from which follows

$$k_b = -\eta_b \ . \tag{4.22}$$

Thus (4.19) can be written as

$$k_i{}^h B_b{}^i = k_b{}^a B_a{}^h - \eta_b N^h \ . \tag{4.23}$$

Transvecting $k_h{}^k$ to (4.23) and using $k_h{}^k k_i{}^h = \delta_i^k$ and (4.23), we find

$$k_b{}^c k_c{}^a = \delta_b^a - \eta_b\eta^a \ . \tag{4.24}$$

Now we write down the equations of Gauss and Weingarten, respectively,

$$\nabla_c B_b{}^h = h_{cb} N^h \ , \tag{4.25}$$

$$\nabla_c N^h = -h_c{}^a B_a{}^h \ , \tag{4.26}$$

where ∇_c denotes the operator of covariant differentiation along M^{2n-1} in the sense of van der Waerden-Bortolotti, h_{cb} is the second fundamental tensor of M^{2n-1}, and $h_c{}^a = h_{cb} g^{ba}$.

Differentiating $u_b = u_i B_b{}^i$ covariantly along M^{2n-1} gives $\nabla_c u_b = (f_{ji} - \lambda k_{ji}) B_c{}^j B_b{}^i + u_i h_{cb} N^i$, which implies

$$\nabla_c u_b = f_{cb} - \lambda k_{cb} \ , \tag{4.27}$$

or

$$\nabla_c \eta_b = f_{cb}/\sqrt{1-\lambda^2} - \lambda k_{cb}/\sqrt{1-\lambda^2} . \tag{4.28}$$

Next, differentiating (4.5) covariantly along M^{2n-1} and using the third equation of (3.22), (4.6), (4.14), (4.19), (4.20) and (4.25) we can easily obtain

$$-k_{ca}f_b{}^a - \lambda \eta_c \eta_b + \lambda g_{cb} + \sqrt{1-\lambda^2}\, h_{cb} = 0 , \tag{4.29}$$

which, together with $f_b{}^a \eta^b = 0$, implies

$$h_{cb}\eta^b = 0 . \tag{4.30}$$

Transvecting $f_d{}^b$ to (4.29) and using (4.16), (4.17), (4.21) we find $k_{dc} + \lambda f_{dc} + \sqrt{1-\lambda^2}\, h_{cb} f_d{}^b = 0$, which implies

$$2\lambda f_{cb} = \sqrt{1-\lambda^2}(h_{ca}f_b{}^a - h_{ba}f_c{}^a) . \tag{4.31}$$

Differentiating (4.14) covariantly along M^{2n-1} and using the first equation of (3.22), (4.25), (4.26) we find

$$\begin{aligned} -g_{cb}u^a B_a{}^h + B_c{}^h u_b - \sqrt{1-\lambda^2}\, k_{cb}N^h + \lambda h_{cb}\eta^a B_a{}^h \\ = (\nabla_c f_b{}^a)B_a{}^h + h_{ca}f_b{}^a N^h - \lambda(\nabla_c \eta_b)N^h + \lambda h_c{}^a \eta_b B_a{}^h , \end{aligned}$$

which, together with (4.11), implies

$$\nabla_c f_b{}^a = -(\sqrt{1-\lambda^2}\, g_{cb} - \lambda h_{cb})\eta^a + (\sqrt{1-\lambda^2}\,\delta_c^a - \lambda h_c{}^a)\eta_b . \tag{4.32}$$

Now by putting

$$\begin{aligned} S_{cb}{}^a = f_c{}^e \nabla_e f_b{}^a - f_b{}^e \nabla_e f_c{}^a - (\nabla_c f_b{}^e - \nabla_b f_c{}^e) f_e{}^a \\ + (\nabla_c \eta_b - \nabla_b \eta_c)\eta^a , \end{aligned} \tag{4.33}$$

and using (4.28), (4.31) and (4.32), we can easily obtain

$$\begin{aligned} S_{cb}{}^a = \frac{4\lambda^2}{\sqrt{1-\lambda^2}} f_{cb}\eta^a + \lambda(h_c{}^e f_e{}^a - f_c{}^e h_e{}^a)\eta_b \\ - \lambda(h_b{}^e f_e{}^a - f_b{}^e h_e{}^a)\eta_c . \end{aligned} \tag{4.34}$$

If $S_{cb}{}^a$ vanishes, the almost contact metric structure is said to be normal. In this case, since $f_e{}^a \eta_a = 0$ and $h_e{}^a \eta_a = 0$, from $S_{cb}{}^a \eta_a = 0$ it follows immediately that $\lambda = 0$. Thus we have

Theorem 4.3. *In order for the almost contact metric structure induced on the hypersurface $\lambda = constant$ $(\lambda^2 < 1)$ of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ to be normal, it is necessary and sufficient that $\lambda = 0$.*

If $\lambda = 0$, then (4.28) and (4.32) become, respectively,

$$\nabla_c \eta_b = f_{cb} , \tag{4.35}$$

$$\nabla_c f_b{}^a = -g_{cb}\eta^a + \delta_c^a \eta_b . \tag{4.36}$$

Hence we have

Theorem 4.4. *The almost contact metric structure induced on the hypersurface $\lambda = 0$ of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ is Sasakian.*

In the remainder of this paper, we study which of the conditions (1.16), (1.17), (1.22), (1.24), (1.25), (3.14), (3.15), the fourth equation of (3.22), (3.30), (3.31), (I), (II), (III) mentioned at the end of § 3 can characterize M^{2n} as $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.

5. The case in which $\nabla_i \lambda = -2v_i$

In this section, we prove

Theorem 5.1. *Suppose that a complete orientable 2n-dimensional Riemannian manifold M^{2n} is immersed in $S^{2n+1}(1)$ as a hypersurface. If the (f, g, u, v, λ)-structure induced on M^{2n} satisfies $\nabla_i \lambda = -2v_i$ in such a way that $\lambda(1-\lambda^2)$ is almost everywhere nonzero, then M^{2n} is isometric to $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.*

Let M^{2n} be a complete orientable differentiable manifold immersed in $S^{2n+1}(1)$ as a hypersurface by the immersion i: $M^{2n} \to S^{2n+1}(1) \subset E^{2n+2}$. Then the equations of Gauss and Godazzi are given respectively by

$$K_{kji}{}^h = \delta_k^h g_{ji} - \delta_j^h g_{ki} + k_k{}^h k_{ji} - k_j{}^h k_{ki} , \tag{5.1}$$

$$\nabla_k k_{ji} - \nabla_j k_{ki} = 0 , \tag{5.2}$$

and the second fundamental tensor k_{ji} and the (f, g, u, v, λ)-structure satisfy

$$\nabla_j f_i{}^h = -g_{ji}u^h + \delta_j^h u_i - k_{ji}v^h + k_j{}^h v_i , \tag{5.3}$$

$$\nabla_j u_i = f_{ji} - \lambda k_{ji} , \tag{5.4}$$

$$\nabla_j v_i = -k_{jm} f_i{}^m + \lambda g_{ji} , \tag{5.5}$$

$$\nabla_j \lambda = k_{ji}u^i - v_j . \tag{5.6}$$

Assume that $\lambda(1-\lambda^2)$ is almost everywhere nonzero and

$$\nabla_i \lambda = -2v_i . \tag{5.7}$$

Since $\nabla_j v_i$ is symmetric, from (5.5) we have

$$k_{jm} f_i{}^m - k_{im} f_j{}^m = 0 , \tag{5.8}$$

which implies

$$k_m{}^h f_i{}^m + f_m{}^h k_i{}^m = 0 , \tag{5.9}$$

that is, $k_m{}^h$ and $f_i{}^m$ anti-commute with each other. From (5.6) and (5.7), it follows that

$$k_{ji} u^i = -v_j . \tag{5.10}$$

Transvecting u^i to (5.8) and using (3.6), (5.10) we obtain

$$k_{ji} v^i = -u_j . \tag{5.11}$$

Transvecting $f_h{}^j$ to (5.8) and taking account of (3.6), (5.10), (5.11) and $u_i v^i = 0$, we find

$$k_{kj} f_i{}^k f_h{}^j + k_{ih} + u_i v_h + u_h v_i = 0 . \tag{5.12}$$

Transvecting g^{ih} to (5.12) and using $f_i{}^k f_h{}^j g^{ih} = g^{kj} - u^k u^j - v^k v^j$, we obtain

$$k_m{}^m = 0 , \tag{5.13}$$

so that the hypersurface $i(M^{2n})$ is minimal in $S^{2n+1}(1)$.

Differentiating (5.11), written in the form $k_i{}^m v_m = -u_i$, covariantly and taking account of (5.2), (5.4) and (5.5), we find

$$f_{ml} k_j{}^m k_i{}^l + f_{ji} = 0 , \tag{5.14}$$

which, according to (5.8), can also be written as

$$f_{mj} k_l{}^m k_i{}^l + f_{ji} = 0 . \tag{5.15}$$

Transvecting $f_h{}^j$ to (5.15) and using the first equation of (3.6), we obtain

$$k_i{}^l k_{lh} = g_{ih} , \tag{5.16}$$

or

$$k_m{}^h k_i{}^m = \delta_i^h . \tag{5.17}$$

Now from (5.1), (5.13) and (5.17), by contraction we find

$$K_{ji} = 2(n-1) g_{ji} , \tag{5.18}$$

so that M^{2n} is an Einstein manifold. Thus we have

$$\nabla_l K_{kji}{}^l = 0 \tag{5.19}$$

by (5.18) and the second Bianchi identify

$$\nabla_l K_{kji}{}^h + \nabla_k K_{jli}{}^h + \nabla_j K_{lki}{}^h = 0 .$$

From (5.1), (5.2), (5.13), and (5.19) it follows immediately

$$0 = \nabla_l K_{kji}{}^l = k_k{}^l(\nabla_l k_{ji}) - k_j{}^l(\nabla_l k_{ki}) .$$

By this equation and (5.2), (5.16) we can easily obtain $k_k{}^l(\nabla_l k_{ji}) = 0$, which and (5.17) give

(5.20) $$\nabla_k k_{ji} = 0 .$$

(5.17) implies that

(5.21) $$\tfrac{1}{2}(\delta_i^h + k_i{}^h) \quad \text{and} \quad \tfrac{1}{2}(\delta_i^h - k_i{}^h)$$

are projection tensors defining two distributions of the same dimension n, and (5.20) implies that they are integrable. Since the Riemannian manifold M^{2n} is complete, this shows that M^{2n} is a product of two n-dimensional manifolds M^n and M'^n. Thus we cover M^n by a system of coordinate neighborhoods $\{U; x^a\}$ and M'^n by $\{V; x^r\}$, so that the components of the first fundamental tensor g_{ji} and the second fundamental tensor k_{ji} are of the forms

(5.22) $$g_{ji} = \begin{pmatrix} g_{cb}(x^a) & 0 \\ 0 & g_{ts}(x^r) \end{pmatrix} ,$$

(5.23) $$k_{ji} = \begin{pmatrix} g_{cb}(x^a) & 0 \\ 0 & -g_{ts}(x^r) \end{pmatrix} ,$$

which implies

(5.24) $$k_i{}^h = \begin{pmatrix} \delta_b^a & 0 \\ 0 & -\delta_s^r \end{pmatrix} .$$

Thus from (5.15) we see

$$f_{cb} = 0 , \qquad f_{ts} = 0 ,$$

that is, the tensor f_{ji} has components of the form

(5.25) $$f_{ji} = \begin{pmatrix} 0 & f_{sa} \\ f_{cr} & 0 \end{pmatrix} ,$$

which implies

(5.26) $$f_i{}^h = \begin{pmatrix} 0 & f_s{}^a \\ f_c{}^r & 0 \end{pmatrix} .$$

Now from (5.5) and (5.6) we have

$$\nabla_j \nabla_i \lambda = 2(k_{jm} f_i{}^m - \lambda g_{ji}) , \tag{5.27}$$

which, together with (5.23) and (5.26), implies

$$\nabla_c \nabla_b \lambda = -2\lambda g_{cb} , \tag{5.28}$$

$$\nabla_t \nabla_s \lambda = -2\lambda g_{ts} . \tag{5.29}$$

Thus by a theorem of Obata [6], M^n and M'^n are both isometric to $S^n(1/\sqrt{2})$. This completes the proof.

6. The case in which $\nabla_i \lambda = cv_i$

In this section, we assume that λ is not a constant, $\lambda(1-\lambda^2)$ is almost everywhere nonzero, and

$$\nabla_i \lambda = cv_i , \tag{6.1}$$

c being a constant. Since $\nabla_j v_i$ is symmetric, we have (5.8) and (5.9). Furthermore, from (5.6) and (6.1) we have

$$k_{ji} u^i = (c+1) v_j . \tag{6.2}$$

Transvecting u^i to (5.8) and taking account of (6.2), we find

$$k_{ji} v^i = (c+1) u_j , \tag{6.3}$$

which can also be written as

$$k_i{}^m v_m = (c+1) u_i . \tag{6.4}$$

Differentiating (6.4) covariantly and taking account of (5.2), (5.4) and (5.5), we can easily see that

$$f_{ml} k_j{}^m k_i{}^l = (c+1) f_{ji} , \tag{6.5}$$

or

$$f_{mj} k_l{}^m k_i{}^l = (c+1) f_{ji} \tag{6.6}$$

because of (5.8). Transvecting u^i to (6.5) and using (6.2), (6.3) and the third equation of (3.6) we obtain

$$c = -1 \quad \text{or} \quad c = -2 . \tag{6.7}$$

Transvecting $f_h{}^j$ to (6.6) and using the last and first equations of (3.6) we find

$$k_i{}^l k_{lh} = -(c+1)g_{ih} + (c+1)(c+2)(u_i u_h + v_i v_h) .$$

If $c = -1$, then $k_i{}^l k_{lh} = 0$ which implies

$$k_{ji} = 0 . \tag{6.8}$$

Thus from (5.5) and (5.6) we have

$$\nabla_j \nabla_i \lambda = -\lambda g_{ji} , \tag{6.9}$$

which, by a theorem of Obata [6], shows that M^{2n} is isometric to $S^{2n}(1)$. If $c = -2$, then by Theorem 5.1, M^{2n} is isometric to $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.

Hence we arrive at

Theorem 6.1. *Suppose that a complete orientable 2n-dimensional differentiable manifold M^{2n} is immersed in $S^{2n+1}(1)$ as a hypersurface. If (f, g, u, v, λ)-structure induced on M^{2n} satisfies $\nabla_i \lambda = c v_i$, c being a nonzero constant, in such a way that $\lambda \neq$ constant and $\lambda(1-\lambda^2)$ is almost everywhere nonzero, then M^{2n} is isometric to $S^{2n+1}(1)$ or $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.*

As a direct consequence of Theorem 6.1, we have

Theorem 6.2. *Suppose that a complete orientable 2n-dimensional differentiable manifold M^{2n} is immersed in $S^{2n+1}(1)$ as a hypersurface. If (f, g, u, v, λ)-structure induced on M^{2n} satisfies $k_i{}^h u^i = \beta v^h$, $k_i{}^h$ being the second fundamental tensor and β being a constant not equal to 1, in such a way that $\lambda \neq$ constant and $\lambda(1-\lambda^2)$ is almost everywhere nonzero, then M^{2n} is isometric to $S^{2n}(1)$ or $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.*

For, (5.6) and $k_i{}^h u^i = \beta v^h$ give $\nabla_i \lambda = (\beta - 1) v_i$, and the theorem follows immediately from Theorem 6.1.

7. The case in which $k_m{}^h f_i{}^m + f_m{}^h k_i{}^m = 0$

Blair, Ludden and the present author [3] proved

Theorem 7.1. *If M^{2n} is a complete orientable submanifold of $S^{2n+1}(1)$ of constant scalar curvature satisfying $k_m{}^h f_i{}^m + f_m{}^h k_i{}^m = 0$ and $\lambda \neq$ constant, where k_{ji} is the second fundamental tensor of M^{2n}, and $f_i{}^h$ and λ are respectively the tensor field of type (1, 1) and a scalar field defining the (f, g, u, v, λ)-structure on M^{2n}, $\lambda(1-\lambda^2)$ being almost everywhere nonzero, then M^{2n} is a natural sphere $S^{2n}(1)$ or $M^{2n} = S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.*

The main purpose of the present section is to show that we can reduce this theorem to Theorem 6.2.

Using Theorem 6.2, we first prove

Theorem 7.2. *If M^{2n} is a complete orientable submanifold of $S^{2n+1}(1)$ satisfying $k_m{}^h f_i{}^m + f_m{}^h k_i{}^m = 0$ and $K(\gamma) =$ constant, where k_{ji} is the second fundamental tensor of M^{2n}, $f_i{}^h$ the tensor field of type (1, 1) defining the (f, g, u, v, λ)-structure on M^{2n}, $\lambda(1-\lambda^2)$ being almost everywhere nonzero,* *and $K(\gamma)$ is the sectional curvature with respect to the section γ spanned by u^h and v^h, then M^{2n} is isometric to a natural sphere $S^{2n}(1)$ or $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.*

Proof. Transvecting u^i and v^i to

$$k_m{}^h f_i{}^m + f_m{}^h k_i{}^m = 0 \tag{7.1}$$

gives respectively

$$-\lambda k_m{}^h v^m + f_m{}^h k_i{}^m u^i = 0 \ , \tag{7.2}$$

$$\lambda k_m{}^h u^m + f_m{}^h k_i{}^m v^i = 0 \ . \tag{7.3}$$

Transvecting v_h and $f_h{}^k$ to (7.2) and using (3.6), (7.3) we obtain, respectively,

$$k_{ji}u^j u^i + k_{ji}v^j v^i = 0 \ , \tag{7.4}$$

$$(1 - \lambda^2)k_i{}^h u^i = (k_{ji}u^j u^i)u^h + (k_{ji}u^j v^i)v^h \ .$$

Similarly, we find

$$(1 - \lambda^2)k_i{}^h v^i = (k_{ji}u^j v^i)u^h + (k_{ji}v^j v^i)v^h \ .$$

Thus, at a point where $1 - \lambda^2 \neq 0$, by (7.4) we can put

$$k_i{}^h u^i = \alpha u^h + \beta v^h \ , \tag{7.5}$$

$$h_j{}^h v^i = \beta u^h - \alpha v^h \ . \tag{7.6}$$

Applying ∇_j to (7.5) written in the form $k_i{}^m u_m = \alpha u_i + \beta v_i$, using (3.22), and in the resulting equation taking the skew-symmetric part with respect to j and i and taking account of (5.2), we obtain

$$\alpha_j u_i - \alpha_i u_j + 2\alpha f_{ji} + \beta_j v_i - \beta_i v_j = 0 \ , \tag{7.7}$$

because of

$$k_{jm} f_i{}^m - k_{im} f_j{}^m = 0 \tag{7.8}$$

obtained from (7.1), where $\alpha_j = \nabla_j \alpha$ and $\beta_j = \nabla_j \beta$.

Transvecting $u^j v^i$, u^i and v^i to (7.7), we find, respectively,

$$\alpha_i v^i - \beta_i u^i + 2\lambda\alpha = 0 \ , \tag{7.9}$$

$$(1 - \lambda^2)\alpha_j = (\alpha_i u^i)u_j + (\alpha_i v^i)v_j \ , \tag{7.10}$$

$$(1 - \lambda^2)\beta_j = (\beta_i u^i)u_j + (\beta_i v^i)v_j \ . \tag{7.11}$$

Thus multiplying (7.7) by $1 - \lambda^2$ and substituting (7.10) and (7.11) into the resulting equation give

$$(7.12) \qquad 2\alpha(1-\lambda^2)f_{ji} = (\alpha_m v^m - \beta_m u^m)(u_j v_i - u_i v_j) \ .$$

Since the rank of f_{ji} is greater than or equal to $2n-2$, we have, if $n > 1$,

$$(7.13) \qquad \alpha = 0 \ , \qquad \beta_i u^i = 0 \ .$$

Transvecting v^i to (7.7) and using (3.6), (7.13) yield

$$(7.14) \qquad (1-\lambda^2)\beta_j = (\beta_i v^i)v_j \ .$$

Applying ∇_j to $k_i{}^m v_m = \beta u_i$, obtained from (7.6) and (7.13), using (3.22) and taking the skew-symmetric part with respect to j and i, we have

$$(7.15) \qquad 2f_{ml}k_j{}^m k_i{}^l = \beta_j u_i - \beta_i u_j + 2\beta f_{ji} \ .$$

Transvecting u^i to (7.15) and taking account of (7.13) and (7.14), we find

$$(7.16) \qquad 2\lambda\beta^2 + 2\lambda\beta + \beta_i v^i = 0 \ ,$$

which shows that if β is a constant, then $\beta = 0$ or $\beta = -1$.

Since the covariant components of the curvature tensor of the M^{2n} is given by

$$(7.17) \qquad K_{kjih} = g_{kh}g_{ji} - g_{jh}g_{ki} + k_{kh}k_{ji} - k_{jh}k_{ki} \ ,$$

at a point at which $1-\lambda^2 \neq 0$ the sectional curvature $K(\gamma)$ with respect to the section spanned by u^h and v^h is given by

$$(7.18) \qquad K(\gamma) = -K_{kjih}u^k v^j u^i v^h / [(u_j u^j)(v_i v^i)] = 1 - \beta^2 \ ,$$

which shows that if $K(\gamma)$ is constant, then β is constant and $\beta = 0$ or $\beta = -1$. Thus applying Theorem 6.2 we have Theorem 7.2.

Now, transvecting f^{ji} to (7.8), and using $k_{ji}u^i = \beta v_j$, $k_{ji}v^i = \beta u_j$, we find

$$(7.19) \qquad k_{ji}g^{ji} = 0 \ .$$

Multiplying (7.15) by $1-\lambda^2$ and using (7.5), (7.14) give

$$2(1-\lambda^2)f_{mj}k_l{}^m k_i{}^l = -\beta_m v^m (u_j v_i - u_i v_j) + 2\beta(1-\lambda^2)f_{ji} \ .$$

By transvecting $f_h{}^j$ to the above equation and using (3.6) we obtain

$$(7.20) \qquad (1-\lambda^2)k_i{}^l k_{lh} = \beta(\beta+1)(u_i u_h + v_i v_h) - \beta(1-\lambda^2)g_{ih} \ ,$$

which implies

$$(7.21) \qquad k_h{}^i k_i{}^h = 2\beta[(\beta+1) - n] \ .$$

Thus from (7.17), (7.19) and (7.21) we find

$$K = 2n(2n - 1) - 2\beta[(\beta + 1) - n] ,$$

which shows that if the scalar curvature K is constant, then β is constant. This proves Theorem 7.1.

8. A lemma

We prove

Lemma 8.1. *Let M^{2n} be a complete $2n$-dimensional differentiable manifold admitting an (f, g, u, v, λ)-structure, and assume that there exists in M^{2n} a tensor field k_{ji} satisfying*

$$k_m{}^m = 0 , \tag{8.1}$$

$$k_{jm}k_i{}^m = g_{ji} , \tag{8.2}$$

$$\nabla_k k_{ji} = 0 , \tag{8.3}$$

$$k_{jm}f_i{}^m - k_{im}f_j{}^m = 0 , \tag{8.4}$$

$$\nabla_j \nabla_i \lambda = 2k_{jm}f_i{}^m - 2\lambda g_{ji} . \tag{8.5}$$

Then M^{2n} is globally isometric to $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.

Proof. Assumptions (8.1), (8.2) and (8.3) show that M^{2n} is a product $M^n \times M'^n$ of M^n and M'^n both of which are of the same dimension n. Thus we cover M^n by a system of coordinate neighborhoods $\{U; x^a\}$, M'^n by a system of coordinate neighborhoods $\{V; x^r\}$ and consequently $M^n \times M'^n$ by $\{U \times V; x^h\}$. Then the metric tensor g_{ji} and the tensor $k_i{}^h$ have components of the form:

$$g_{ji} = \begin{pmatrix} g_{cb}(x^a) & 0 \\ 0 & g_{ts}(x^r) \end{pmatrix} , \tag{8.6}$$

$$k_i{}^h = \begin{pmatrix} \delta_b^a & 0 \\ 0 & -\delta_s^r \end{pmatrix} . \tag{8.7}$$

Thus from (8.4), $f_i{}^h$ has components of the form

$$f_i{}^h = \begin{pmatrix} 0 & f_s{}^a \\ f_b{}^r & 0 \end{pmatrix} , \tag{8.8}$$

and from (8.5) we have

$$\nabla_c \nabla_b \lambda = -2\lambda g_{cb} , \tag{8.9}$$

$$\nabla_t \nabla_s \lambda = -2\lambda g_{ts} . \tag{8.10}$$

Since the submanifolds M^n and M'^n are both complete, by a theorem of Obata [6], (8.9) and (8.10) show that M^n is isometric to $S^n(1/\sqrt{2})$ and M'^n is also isometric to $S^n(1/\sqrt{2})$. Hence M^{2n} is isometric to $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.

9. Intrinsic geometry of $S^n \times S^n$

In this section, we first prove

Theorem 9.1. *Assume that a complete 2n-dimensional differentiable manifold M^{2n} admits an (f, g, u, v, λ)-structure such that $\lambda(1-\lambda^2)$ is almost everywhere nonzero, and*

$$\nabla_j u_i - \nabla_i u_j = 2f_{ji} , \tag{9.1}$$

$$\nabla_i \lambda = -2v_i . \tag{9.2}$$

At a point where $\lambda \neq 0$, we define a tensor field k_{ji} of type $(0, 2)$ by

$$\nabla_j u_i + \nabla_i u_j = -2\lambda k_{ji} , \tag{9.3}$$

and assume that u_i satisfies

$$\nabla_k \nabla_j u_i = -g_{kj}u_i + g_{ki}u_j - k_{kj}v_i + k_{ki}v_j + 2v_k k_{ji} . \tag{9.4}$$

Then M^{2n} is isometric to $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.

Proof. We find, from (9.1) and (9.3),

$$\nabla_j u_i = f_{ji} - \lambda k_{ji} . \tag{9.5}$$

and, from (9.2), (9.3) and (9.4),

$$\nabla_k k_{ji} = 0 . \tag{9.6}$$

Thus by (9.4), (9.5) and (9.6) we have

$$\nabla_k f_{ji} = -g_{kj}u_i + g_{ki}u_j - k_{kj}v_i + k_{ki}v_j . \tag{9.7}$$

On the other hand, transvecting u^j to (9.1) and using $u_j u^j = 1 - \lambda^2$ and (9.2), (3.6) we obtain

$$u^j \nabla_j u_i = 0 . \tag{9.8}$$

Thus from (9.3) it follows

$$k_{ji}u^i = -v_j . \tag{9.9}$$

Differentiating (9.9) covariantly and taking account of (9.5), we obtain

$$\nabla_j v_i = -k_{jm}f_i{}^m + \lambda k_{jm}k_i{}^m , \tag{9.10}$$

which implies

$$k_{jm}f_i{}^m - k_{im}f_j{}^m = 0 . \tag{9.11}$$

Transvecting u^i to (9.11) and using (9.9), we find

$$k_{ji}v^i = -u_j . \tag{9.12}$$

Transvecting f^{ji} to (9.11) and using (3.6), (9.9) and (9.12), we find

$$k_m{}^m = 0 . \tag{9.13}$$

By differentiating (9.11) covariantly and taking account of (9.6), (9.7), (9.9) and (9.12), we obtain

$$k_{jm}k_i{}^m = g_{ji} , \tag{9.14}$$

and consequently (9.10) becomes

$$\nabla_j v_i = -k_{jm}f_i{}^m + \lambda g_{ji} , \tag{9.15}$$

or

$$\nabla_j \nabla_i \lambda = 2k_{jm}f_i{}^m - 2\lambda g_{ji} . \tag{9.16}$$

Thus using Lemma 8.1 we have Theorem 9.1.

Blair, Ludden and the present author have proved [5]

Theorem 9.2. *Suppose that a complete 2n-dimensional Riemannian manifold M^{2n} admits a vector field u^h satisfying*

$$u_i u^i = 1 - \lambda^2 , \qquad u^j \nabla_j u^h = 0 ,$$

where λ is a nonconstant function such that $\lambda(1 - \lambda^2)$ is almost everywhere nonzero. Let tensor fields f_{ji}, k_{ji} and a covector field v_i be defined by, respectively,

$$\nabla_j u_i - \nabla_i u_j = 2f_{ji} , \quad \nabla_j u_i + \nabla_i u_j = -2\lambda k_{ji} , \quad \nabla_i \lambda = -2v_i .$$

If the vectors u^h and v^h satisfy

$$\nabla_j v_i = -k_{jm}f_i{}^m + \lambda g_{ji} ,$$
$$\nabla_j \nabla_i u_h = -g_{ji}u_h + g_{jh}u_i - k_{ji}v_h + k_{jh}v_i + 2v_k k_{ji} ,$$

then M^{2n} is globally isometric to $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.

To conclude this paper we establish

Theorem 9.3. *Suppose that a complete 2n-dimensional Riemannian manifold M^{2n} with metric tensor g_{ji} admits a vector field u^h satisfying*

$$(9.17) \qquad u_i u^i = 1 - \lambda^2 ,$$

$$(9.18) \qquad u^j \nabla_j u_i = 0 ,$$

$$(9.19) \qquad v^j \nabla_j u_i = 2\lambda u_i ,$$

where λ is a nonconstant function such that $\lambda(1 - \lambda^2)$ is almost everywhere nonzero, and v_i is defined by

$$(9.20) \qquad \nabla_i \lambda = -2v_i .$$

Let tensors f_{ji} and k_{ji} be defined by

$$(9.21) \qquad \nabla_j u_i - \nabla_i u_j = 2f_{ji} ,$$

$$(9.22) \qquad \nabla_j u_i + \nabla_i u_j = -2\lambda k_{ji}$$

respectively, and assume that u_i satisfies

$$(9.23) \qquad \nabla_j \nabla_i u_h = -g_{ji} u_h + g_{jh} u_i - k_{ji} v_h + k_{jh} v_i + 2v_j k_{ih} .$$

Then $f_i{}^h, g_{ji}, u^h, v^h$ and λ define an (f, g, u, v, λ)-structure on $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.

Proof. First of all, we prove that $f_i{}^h$, g_{ji}, u^h, v^h and λ define an (f, g, u, v, λ)-structure. From (9.17) and (9.20) it follows that

$$(9.24) \qquad (\nabla_j u_i) u^i = 2\lambda v_j .$$

Transvecting u^j and v^j to (9.24) and using (9.18), (9.19) we obtain, respectively,

$$(9.25) \qquad u_j v^j = 0 ,$$

$$(9.26) \qquad v_j v^j = 1 - \lambda^2 .$$

Transvecting u^i to (9.21) and using (9.18), (9.24) give

$$(9.27) \qquad f_{ji} u^i = \lambda v_j .$$

From (9.21) and (9.22) it follows that

$$(9.28) \qquad \nabla_j u_i = f_{ji} - \lambda k_{ji} .$$

Transvecting u^i to (9.28) and using (9.24) and (9.27) we thus find

$$(9.29) \qquad k_{ji} u^i = -v_j .$$

Now we have, from (9.21) and (9.23),

$$\nabla_j f_{ih} = -g_{ji}u_h + g_{jh}u_i - k_{ji}v_h + k_{jh}v_i \,, \tag{9.30}$$

and, from (9.20), (9.22) and (9.23),

$$\nabla_j k_{ih} = 0 \,. \tag{9.31}$$

Transvecting v^i to (9.22) and using (9.19), and substituting (9.28) in the resulting equation we obtain

$$f_{ji}v^i + \lambda k_{ji}v^i = -2\lambda u_j \,. \tag{9.32}$$

Differentiating (9.29) covariantly and taking account of (9.31) yield

$$\nabla_j v_i = -k_{im}f_j{}^m + \lambda k_{jm}k_i{}^m \,, \tag{9.33}$$

which implies, due to the symmetry of $\nabla_j v_i$,

$$k_{jm}f_i{}^m - k_{im}f_j{}^m = 0 \,. \tag{9.34}$$

Transvecting u^i to (9.34) and using (9.27) and (9.29), we find

$$f_{ji}v^i - \lambda k_{ji}v^i = 0 \,. \tag{9.35}$$

Thus from (9.32) and (9.35) follow

$$f_{ji}v^i = -\lambda u_j \,, \tag{9.36}$$

$$k_{ji}v^i = -u_j \,. \tag{9.37}$$

By differentiating (9.34) covariantly, taking account of (3.22), (9.30), (9.31), (9.29) and (9.37), and transvecting v^j to the resulting equation, we easily obtain

$$k_{jm}k_i{}^m = g_{ji} \,, \tag{9.38}$$

so that (9.33) becomes

$$\nabla_j v_i = -k_{jm}f_i{}^m + \lambda g_{ji} \,. \tag{9.39}$$

Now differentiating (9.18) covariantly gives

$$(\nabla_j u^m)(\nabla_m u_i) + u^m \nabla_j \nabla_m u_i = 0 \,. \tag{9.40}$$

On the other hand due to (9.23), (9.40) becomes

$$(\nabla_j u^m)(\nabla_m u_i) = -(1-\lambda^2)g_{ji} + u_j u_i + v_j v_i \,. \tag{9.41}$$

Since from (9.28),

$$f_j{}^m = \nabla_j u^m + \lambda k_j{}^m \,, \qquad f_{mi} = \nabla_m u_i + \lambda k_{mi} \,,$$

by using (9.14), (9.28), (9.29), (9.31), (9.39) we can easily obtain

$$f_j{}^m f_{mi} = (\nabla_j u^m)(\nabla_i u_m) - \lambda^2 g_{ji} ,$$

which becomes, in consequence of (9.41),

$$f_j{}^m f_{mi} = -g_{ji} + u_j u_i + v_j v_i , \tag{9.42}$$

showing that

$$f_j{}^m f_m{}^h = -\delta_j^h + u_j u^h + v_j v^h , \tag{9.43}$$

$$g_{ml} f_j{}^m f_i{}^l = g_{ji} - u_j u_i - v_j v_i . \tag{9.44}$$

(9.17), (9.25), (9.26), (9.27), (9.36), (9.43) and (9.44) show that $f_i{}^h$, g_{ji}, u^h, v^h and λ define an (f, g, u, v, λ)-structure, and hence from Theorem 9.1 it follows that M^{2n} is globablly isometric to $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.

Bibliography

[1] D. E. Blair & G. D. Ludden, *On intrinsic structures similar to those on S^{2n}*, Kōdai Math. Sem. Rep. **23** (1971) 262–266.

[2] D. E. Blair, G. D. Ludden & K. Yano, *Induced structures on submanifolds,* Kōdai Math. Sem. Rep. **22** (1970) 188–198.

[3] ——, *Hypersurfaces of an odd-dimensional sphere,* J. Differential Geometry **5** (1971) 479–486.

[4] ——, *Hypersurfaces with a parallel vector field in an odd-dimensional sphere,* Differential Geometry, in Honor of K. Yano, Kinokuniya, Tokyo, 1972, 33–39.

[5] ——, *On the intrinsic geometry of $S^n \times S^n$*, Math. Ann. **194** (1971) 68–77.

[6] M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere,* J. Math. Soc. Japan **14** (1962) 333–340.

[7] K. Yano, *On a special* f-*structure with complemented frames,* Tensor **23** (1972) 35–40.

[8] ——, On (f, g, u, v, λ)-*structures induced on a hypersurface of an odd-dimensional sphere,* Tôhoku Math. J. (2) **23** (1971) 671–679.

[9] K. Yano, S. S. Eum & U-Hang Ki, *On transversal hypersurfaces of an almost contact manifold,* Kōdai Math. Sem. Rep. **24** (1972) 459–470.

[10] K. Yano & S. Ishihara, *Notes on hypersurfaces of an odd-dimensional sphere,* Kōdai Math. Sem. Rep. **24** (1972) 422–429.

[11] K. Yano & U-Hang Ki, *On quasi-normal* (f, g, u, v, λ)-*structures,* Kōdai Math. Sem. Rep. **24** (1972) 106–120.

[12] ——, *Submanifolds of codimension* 2 *in an even-dimensional Euclidean space,* Kōdai Math. Sem. Rep. **24** (1972) 315–330.

[13] K. Yano & M. Okumura, *On* (f. g, u, v, λ)-*structures,* Kōdai Math. Sem. Rep. **22** (1970) 401–423.

[14] ——, On normal (f, g, u, v, λ)-*structures on submanifolds of codimension* 2 *in an even-dimensional Euclidean space,* Kōdai Math. Sem. Rep. **23** (1971) 172–197.

[15] ——, *Invariant hypersurfaces of a manifold with* (f, g, u, v, λ)-*structure,* Kōdai Math. Sem. Rep. **23** (1971) 290–304.

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