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THE AXIOM OF 2-SPHERES IN KAEHLER GEOMETRY

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1. Introduction

Let M be an almost complex manifold of complex dimension >1. A subspace of the tangent space M_m at $m \in M$ is called a *holomorphic plane* if it is spanned by a tangent vector at m and its transform by the almost complex structure tensor J of M. A Kaehler manifold satisfies the *axiom of holomorphic planes* if for each $m \in M$ and holomorphic plane $\Pi \in M_m$ there is a totally geodesic submanifold N such that $m \in N$ and $N_m = \Pi$. This notion was introduced by Yano and Mogi [3] who proved that a manifold with this property has constant holomorphic curvature.

A Riemannian manifold M of (real) dimension ≥ 3 is said to satisfy the *axiom of 2-spheres* if for each $m \in M$ and plane $\Pi \in M_m$ there exists a 2-dimensional umbilical submanifold N with parallel mean curvature vector field such that $m \in N$ and $N_m = \Pi$. This notion was introduced by Leung and Nomizu [2] who proved that a manifold with this property has constant sectional curvature. This suggests the following concept for hermitian manifolds.

Axiom of holomorphic 2-spheres. *For each $m \in M$ and holomorphic plane $\Pi \in M_m$ there exists a 2-dimensional umbilical submanifold N with parallel mean curvature vector field such that $m \in N$ and $N_m = \Pi$. (If N is a complex, i.e., invariant submanifold, it is totally geodesic.)*

This yields the following generalization of the theorem of Yano and Mogi.

Theorem. *A Kaehler manifold satisfying the axiom of holomorphic 2-spheres has constant holomorphic curvature.*

2. Proof of theorem

A Kaehler manifold $(M, \langle\ ,\ \rangle)$ is considered as a Riemannian manifold with metric $\langle\ ,\ \rangle$ admitting a parallel skew-symmetric linear transformation field J (the almost complex structure). Let R denote the curvature tensor. Then, for any $m \in M$ and $X, Y \in M_m$,

$$\text{(i)} \qquad R(JX, Y) = -R(X, JY) ,$$

$$\text{(ii)} \qquad K(JX, Y) = K(X, JY) ,$$

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where $K(X, Y)$ is the sectional curvature determined by the plane of X and Y.

The Riemannian connections of M and N will be denoted by $\tilde{\nabla}$ and ∇, respectively, and the connection in the normal bundle of N in M by $\nabla^{\perp}$. The second fundamental form h is defined by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) ,$$

where X and Y are vector fields tangent to N. Associated with any vector field ξ normal to N there is a linear transformation field A_ξ given by

$$\tilde{\nabla}_X \xi = \nabla^{\perp}_X \xi - A_\xi X ,$$

where X is tangent to N. The tensor fields h and A_ξ are related by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle .$$

The *mean curvature normal* H of N in M is defined by the relation

$$\operatorname{trace} A_\xi = 2\langle \xi, H \rangle$$

for all ξ normal to N. H is said to be *parallel* (in the normal bundle) if $\nabla^{\perp} H = 0$. The surface N is *umbilical* in M if

$$h(X, Y) = \langle X, Y \rangle H ,$$

i.e., if

$$A_\xi = \langle \xi, H \rangle I = \tfrac{1}{2} \operatorname{trace} A_\xi \cdot I ,$$

where I is the identity transformation. An umbilical submanifold is *totally geodesic* if H vanishes.

For any $m \in M$, let X, JX and ζ be three orthonormal vectors in M_m, and let Π denote the holomorphic plane determined by X. Then there is an umbilical surface N with parallel mean curvature normal H such that $m \in N$ and $N_m = \Pi$. Let U be a normal neighborhood of m in N, and for each $n \in U$ let ξ_n be the normal to N at n parallel (with respect to $\nabla^{\perp}$) to ζ along the geodesic in U from m to n. Along each such geodesic, $\langle \xi, H \rangle$ is a constant c, i.e., $A_\xi = cI$ at every point of U. Thus

$$\nabla_X A_\xi = \nabla_{JX} A_\xi = 0 , \qquad \nabla^{\perp}_X \xi = \nabla^{\perp}_{JX} \xi = 0$$

at m. Applying Codazzi's equation

$$(R(X, Y)\xi)_t = (\nabla_Y A_\xi)X - (\nabla_X A_\xi)Y + A_{\nabla^{\perp}_X \xi} Y - A_{\nabla^{\perp}_Y \xi} X ,$$

valid for any X, Y tangent to N and vector field ξ in the normal direction, where the subscript t denotes the tangential component, it follows that

$(R(X, JX)\zeta)_t = 0$. In particular, $\langle R(X, JX)\zeta, X\rangle = 0$, so that by putting $Y' = (JX + \zeta)/\sqrt{2}$ and $Z' = (JX - \zeta)/\sqrt{2}$, and then making use of the special symmetry properties (i) and (ii) of R, it is easily seen that $K(Y', JY') = K(Z', JZ')$. Consequently, M has constant holomorphic curvature (see [1, p. 201]).

Note that a 2-dimensional umbilical submanifold of a space of constant holomorphic curvature has parallel mean curvature vector field. For, if X and ξ are any vector fields tangent and normal to N, respectively, $\langle R(X, JX)\xi, JX\rangle = 0$, so that $\langle \xi, \nabla^{\perp}_X H\rangle = -\langle \nabla^{\perp}_X \xi, H\rangle = 0$.

Bibliography

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