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THE REFLECTION PRINCIPLE FOR MINIMAL SUBMANIFOLDS OF RIEMANNIAN SYMMETRIC SPACES

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1. Introduction

The classical Schwartz reflection invariance of minimal surfaces states that if a minimal surface N in a Euclidean 3-space E^3 contains a straight line L, then N is locally invariant under the symmetry of E^3 with respect to L [1, p. 246]. A similar reflection principle has also been proved for a minimal surface N in a space of constant curvature which contains a geodesic L of the ambient space [7]. It is the purpose of the present note to investigate to what extent this reflection principle holds for a minimal submanifold N of a Riemannian manifold M which contains a totally geodesic manifold B of M as a hypersurface, in particular, when M is a local symmetric space.

To this end, we introduce the notion of a *reflective submanifold* of a Riemannian manifold M. An imbedded submanifold B of M is said to be *locally reflective* if there exists an involutive isometry ρ_B (i.e., $\rho_B^2 = \mathrm{id}$), called the reflection with respect to B, defined at least in an open neighborhood U of B in M such that $B \cap U$ is precisely the fixed point set of ρ_B when restricted to U. B is said to be *globally reflective* or simply *reflective* if it is complete and the isometry ρ_B is defined everywhere on M with B as its fixed point set. Using a well-known fact about the fixed point set of isometries [5, p. 61], one can conclude that every *reflective submanifold is a totally geodesic submanifold.*

We will prove the following basic facts about reflective submanifolds. *Let M be an analytic Riemannian manifold, and B a locally reflective submanifold of codimension greater than one. Then every minimal submanifold N of M, which contains B as a hypersurface, is locally invariant under ρ_B. Furthermore, if B is globally reflective and N is complete, then N is invariant under ρ_B.*

For an arbitrary Riemannian manifold, there may not exist any totally geodesic submanifold of dimension greater than one. To ensure a good supply of candidates for locally reflective snbmanifolds we will restrict our attentions to local symmetric spaces in most parts of this paper.

As with many problems related to symmetric spaces, *the problem of finding reflective submanifolds in Riemannian symmetric spaces can be reduced to*

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one in Lie algebra. Using this, we will prove that *every geodesic submanifold B of a space from M is locally reflective*; *if B is complete and M is simply connected, then B is globally reflective*. Together with the basic properties of reflective submanifolds, we have *a generalization of the classical reflection invariance of minimal surfaces in E^3*. However, it is not true that every totally geodesic submanifold of an arbitrary symmetric space is reflective. In fact we will prove that *if a local symmetric space M contains a geodesic segment which is locally reflective, then M must be a space of constant curvature*. We will also *classify all reflective submanifolds in the complex projective spaces and complex hyperbolic spaces*.

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For notations and terminologies related to differential geometry we will follow [5] *closely*.

2. Reflection invariance for a minimal submanifold containing a reflective submanifold as a hypersurface

In this section we will prove the following basic facts related to reflective submanifolds of a Riemannian manifold.

Theorem 1. *Let M be an analytic Riemannian manifolds, and B a locally reflective (imbedded) submanifold of codimension greater than one. If N is a minimal submanifold of M which contains B as a hypersurface, then every point $p \in B$ has an open neighborhood W in N such that W is invariant under the reflection map ρ_B. Furthermore, if B is globally reflective and N is complete with respect to the induced metric, then N is invariant under ρ_B.*

We will begin by proving two lemmas and recalling some well-known facts. In the remainder of this paper, M will denote a real analytic Riemannian manifold unless otherwise specified.

Lemma 1. *Any C^2 minimal submanifold of M is real analytic.*

Proof. Any minimal submanifold N of M is locally the solution of a system of quasi-linear analytic partial differential equations [2, p. 178] which can be easily seen to be elliptic along the solution under consideration. It follows from Theorem 6.7.6 in [10] that N is an analytic submanifold. q.e.d.

For completeness we will give a proof of the following lemma which asserts the uniqueness of the analytic continuation of a real analytic submanifold.

Lemma 2. *Let N and N' be two l-dimensional connected real analytic submanifolds of an analytic manifold M which are both maximal (i.e., not a proper subset of any other connected submanifold). If there is an open set W in M such that $W \cap N = W \cap N'$ and $W \cap N$ contains a coordinate neighborhood of N, then $N = N'$.*

Proof. Let S be the subset of N consisting of points p such that there exists a neighborhood U of p in M for which $U \cap N = U \cap N'$ contains a coordinate neighborhood of N. It follows that S is nonempty and open in N. Let q be in the closure of S, and V be a coordinate neighborhood of q in M such that $V \cap N$ and $V \cap N'$ are connected coordinate neighborhoods of N and N' respectively. Then there exist two finite sets of analytic functions $\{f_i\}$ and $\{g_i\}$ defined on V such that $V \cap N' =$ zeros of $\{f_i\}$ and $V \cap N' =$ zeros of $\{g_i\}$. $A = S \cap V \cap N = S \cap V \cap N'$ is an open subset of $V \cap N$ and $V \cap N'$. The restriction of $\{f_i\}$ to $V \cap N'$ defines analytic functions on it. Since $\{f_i\}$ vanish on the open subset $A \subset V \subset N'$, they vanish identically on $V \cap N'$. Therefore $V \cap N' \subset V \cap N$. Similarly, we can prove $V \cap N' \supseteq V \cap N$. It follows that $V \cap N = V \cap N'$ and $S = N$. q.e.d.

A proof of the following theorem can be found for example in [8, Theorem 6.1].

Theorem 2. *Let G_{n-1} be an imbedded $(n-1)$-dimensional submanifold of M ($\dim M > n$) and P be an n-dimensional distribution along G_{n-1} such that $T_q(G_{n-1}) \subset P(q)$ for all $q \in G_{n-1}$. Then assuming the data are real analytic, for each $q \in G_{n-1}$ there exists in every sufficiently small neighborhood U of q a unique imbedded analytic minimal submanifold N of dimension n such that*

1. $U \supset N \supset G_{n-1} \cap U$,
2. $T_q(N) = P(q)$ *for all* $q \in G_{n-1} \cap U$.

Note that Theorem 2 implies in particular that if N' is another n-dimensional minimal submanifolds which satisfies conditions 1 and 2 above, then $N = N'$.

We are now ready to prove Theorem 1. By Lemma 1, we can assume N and B to be real analytic. Let $p \in B$. Then there exists an open neighborhood W of p in N such that W and $W \cap B$ are imbedded submanifolds of M and ρ_B is defined on a neighborhood of W in M. Since ρ_B is an isometry and $B \cap W$ is its fixed point set when restricted to W, it follows that $\rho_B(W)$ is also a minimal submanifold of M which contains $W \cap B$. For $q \in B \cap W$, $T_q(W)$ considered as a subspace of $T_q(M)$ is spanned by $T_q(B)$ and a nonzero vector $x \in T_q(W) \cap T_q(B)^\perp$, $T_q(B)^\perp$ being the orthogonal complement of $T_q(B)$ in $T_q(M)$. ρ_{B*} leaves $T_q(B)$ fixed. Since $\rho_B^2 = \mathrm{id}$, we must have $\rho_{B*}(x) = -x$. In other words we have

$$T_q(W) = T_q(\rho_B(W)) \qquad \text{for all } q \in W \cap B .$$

Applying Theorem 2, we can conclude that there exists an open neighborhood U of W in M such that $W \cap U = \rho_B(W) \cap U$. This proves the first part of the theorem. The second part of the theorem now follows ready from Lemm 2.

3. Reflective submanifolds in Riemannian symmetric spaces

For notations and terminologies related to symmetric spaces we will follow

[5] closely. Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be an orthogonal symmetric Lie algebra, and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ the canonical decomposition. We also assume that there is an ad (H) invariant inner product on $\mathfrak{g}$. A linear subspace $\mathfrak{b}$ of $\mathfrak{m}$ is said to be *reflective* if both $\mathfrak{b}$ and its orthogonal complement $\mathfrak{b}^\perp$ are Lie triple systems such that

$$[[\mathfrak{b}, \mathfrak{b}], \mathfrak{b}^\perp] \subset \mathfrak{b}^\perp , \qquad [[\mathfrak{b}^\perp, \mathfrak{b}^\perp], \mathfrak{b}] \subset \mathfrak{b} ,$$
$$[[\mathfrak{b}, \mathfrak{b}^\perp], \mathfrak{b}] \subset \mathfrak{b}^\perp , \qquad [[\mathfrak{b}, \mathfrak{b}^\perp], \mathfrak{b}^\perp] \subset \mathfrak{b} .$$

Clearly, $\mathfrak{b}^\perp$ is also a reflective subspace. Let $M = G/H$ be a Riemannian global symmetric space, where G is the largest connected group of isometries of M, and H the isotropy subgroup of G at a point 0 (called the origin) of M. We will also let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be its canonical decomposition. Upon identifying $\mathfrak{m}$ with $T_0(M)$, there is an ad (H) invariant inner product defined on $\mathfrak{m}$.

Theorem 3. *Let $M = G/H$ be a simply connected global Riemannian symmetric space. There is a one-to-one correspondence between the set of reflective linear subspaces $\mathfrak{b}$ of $\mathfrak{m}$ and the set of complete globally reflective submanifolds B through the origin* 0 *of M, the correspondence being given by $\mathfrak{b} = T_0(B) \subset T_0(M)$ under the identification $\mathfrak{m} = T_0(M)$.*

Proof. Let B be a reflective submanifold of M, and ρ the involutive isometry whose fixed point set is B. Put $\mathfrak{b} = T_0(B)$, and denote by $\mathfrak{b}^\perp$ its orthogonal complement in $\mathfrak{m} = T_0(M)$. It follows directly from the definition of ρ that $\mathfrak{b}$ and $\mathfrak{b}^\perp$ are respectively the $+1$ and -1 eigenspaces of ρ_*. Let s_0 be the symmetry of M at 0. Then $s_{0*} = -\mathrm{id}$. Since $\rho \circ s_0$ and $s_0 \circ \rho$ both leave 0 fixed and $\rho_* \circ s_{0*} = s_{0*} \circ \rho_*$, we can conclude that $\rho \circ s_0 = s_0 \circ \rho$. The involutive isomorphism σ of the symmetric space (G, H, σ) is given by [5, p. 244]

$$\sigma(g) = s_0 \circ g \circ s_0^{-1} \qquad \text{for } g \in G .$$

Define an automorphism $\rho\colon G \to G$ of G by

$$\rho(g) = \rho \circ g \circ \rho^{-1} \qquad \text{for } g \in G .$$

It is obvious that $\rho(H) = H$. It also follows readily from the fact $\rho \circ s_0 = s_0 \circ \rho$ that $\rho \circ \sigma = \sigma \circ \rho$. Therefore ρ is in fact an involutive automorphism of (G, H, σ). If we denote also by ρ the induced map on $\mathfrak{g}$, then we have $\rho(\mathfrak{h}) = \mathfrak{h}$ and $\rho(\mathfrak{m}) = \mathfrak{m}$, since the eigenspaces of σ are invariant under ρ. For $x, y, z \in \mathfrak{m}$ we have $\rho([[x, y], z]) = [[\rho(x), \rho(y)], \rho(z)]$. Using the fact that $\mathfrak{b}$ and $\mathfrak{b}^\perp$ are respectively the $+1$ and -1 eigenspaces of ρ restricted to $\mathfrak{m}$, it is easy to see that $\mathfrak{b}^\perp$ is also a Lie triple system and is a reflective subspace of $\mathfrak{m}$.

Conversely, suppose $\mathfrak{b}$ is a reflexive subspace of $\mathfrak{m} = T_0(M)$, $\mathfrak{b}^\perp$ its orthogonal complement, and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ the canonical decomposition. First of all we consider the case where M is a Euclidean n-space with the usual flat metric. Then $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is of Euclidean type. Since $[\mathfrak{m}, \mathfrak{m}] = 0$, every subspace $\mathfrak{b}$ of $\mathfrak{m}$ is reflective. The corresponding submanifold is the linear subspace N

through 0 such that $T_0(N) = \mathfrak{b}$, and the usual reflection map ρ with respect to the linear subspace N (cf. [6, p. 195]) will have N as its fixed point set. N is therefore a reflective submanifold. Furthermore ρ induces an involutive automorphism ρ of $(\mathfrak{g}, \mathfrak{h}, \sigma)$ such that when restricted to $\mathfrak{m}, \rho$ has $\mathfrak{b}$ and $\mathfrak{b}^\perp$ as its $+1$ and -1 eigenspaces respectively. Next we assume that $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is irreducible so that $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$. We can now define a linear mapping ρ on $\mathfrak{g}$ by setting $\rho = \mathrm{id}$ on $\mathfrak{b} + [\mathfrak{b}, \mathfrak{b}] + [\mathfrak{b}^\perp, \mathfrak{b}^\perp]$ and $\rho = -\mathrm{id}$ on $\mathfrak{b}^\perp + [\mathfrak{b}, \mathfrak{b}^\perp]$. We claim that ρ is an involutive automorphism of the symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$. To see this let $x_1, \cdots, x_4 \in \mathfrak{b}, y_1, \cdots, y_4 \in \mathfrak{b}^\perp$, and denote respectively by $\mathfrak{g}^+, \mathfrak{g}^-$ the $+1$, -1 eigenspaces of ρ. Using the properties of a reflective subspace, we have

$$\begin{aligned}
[[x_1, x_2], [x_3, x_4]] &= -[x_3, [x_4, [x_1, x_2]]] - [x_4, [[x_1, x_2], x_3]] \in \mathfrak{g}^+ \ , \\
[[y_1, y_2], [y_3, y_4]] &= -[y_3, [y_4, [y_1, y_2]]] - [y_4, [[y_1, y_2], y_3]] \in \mathfrak{g}^+ \ , \\
[[x_1, y_1], [x_2, y_2]] &= -[x_2, [y_2, [x_1, y_1]]] - [y_2, [[x_1, y_1], x_2]] \in \mathfrak{g}^+ \ , \\
[[x_1, y_1], [x_2, x_3]] &= -[x_2, [x_3, [x_1, y_1]]] - [x_3, [[x_1, y_1], x_2]] \in \mathfrak{g}^- \ , \\
[[x_1, y_1], [y_2, y_3]] &= -[y_2, [y_3, [x_1, y_1]]] - [y_3, [[x_1, y_1], y_2]] \in \mathfrak{g}^- \ .
\end{aligned}$$

It is now easy to check that ρ is an automorphism of the Lie algebra $\mathfrak{g}$. Obviously, we have $\rho(\mathfrak{h}) = \mathfrak{h}$ and $\rho \circ \sigma = \sigma \circ \rho$. Therefore ρ is an involutive automorphism of $(\mathfrak{g}, \mathfrak{h}, \sigma)$. In the general case, we have $(\mathfrak{g}, \mathfrak{h}, \sigma) = \sum (\mathfrak{g}_k, \mathfrak{h}_k, \sigma_k)$, a direct sum of orthogonal symmetric Lie algebras $(\mathfrak{g}_k, \mathfrak{h}_k, \sigma_k)$, which are either Euclidean or irreducible. We also have $\mathfrak{m} = \sum \mathfrak{m}_k$, where $\mathfrak{g}_k = \mathfrak{h}_k + \mathfrak{m}_k$ is the canonical decomposition of $(\mathfrak{g}_k, \mathfrak{h}_k, \sigma_k)$. It is easy to see that $\mathfrak{b} \cap \mathfrak{m}_k$ is also reflective. By previous considerations, there is a unique involutive automorphism defined on $(\mathfrak{g}_k, \mathfrak{h}_k, \sigma_k)$ whose fixed point set on $\mathfrak{m}_k$ is $\mathfrak{b} \cap \mathfrak{m}_k$. Therefore we have a uniquely defined involutive automorphism ρ of $(\mathfrak{g}, \mathfrak{h}, \sigma)$ whose fixed point set on $\mathfrak{m}$ is precisely $\mathfrak{b}$. Furthermore, ρ is an isometry of $\mathfrak{m} = T_0(M)$ and also preserves the curvature tensor of M restricted to $T_0(M)$. Therefore ρ can be extended to a unique isometry ρ of the Riemannian symmetric space $M = G/H$, [11, p. 61]. Let B be the unique totally geodesic submanifold of M through the origin 0 of M (cf. [5, p. 237]) such that $T_0(B) = \mathfrak{b}$. Since the fixed point set of ρ on $\mathfrak{m}$ is $\mathfrak{b}$ and ρ also leaves $[\mathfrak{b}, \mathfrak{b}]$ fixed, it is easy to see that the fixed point set of ρ as an isometry of M is precisely B. Obviously $\rho^2 = \mathrm{id}$.

Remark. Using Theorem 3, we also have a characterization of the locally reflective submanifolds. Indeed, *every imbedded geodesic submanifold B of a complete local symmetric space M can be considered as an open submanifold of a complete totally geodesic submanifold $\tilde{B}$ of $\tilde{M}$, the universal covering manifold of M. Then B is locally reflective if and only if $\tilde{B}$ is globally reflective. The reflection map of B is a suitable restriction of that of $\tilde{B}$.*

Proof. Obviously, if $\tilde{B}$ is reflective, then B is a also locally reflective. Conversely, if B is locally reflective, then B is the fixed point set of a local isometry ρ_B. By 2.3.14 in [11], it is easy to see that ρ_B can be extended to a unique global isometry of $\tilde{M}$ whose fixed point set is $\tilde{B}$. q.e.d.

We will prove that every totally geodesic submanifold of a space of constant curvature is reflective. More precisely we have the following theorem.

Theorem 4. *Let M be a complete Riemannian manifold of constant curvature. Then every totally geodesic submanifold B of M is locally reflective. Furthermore, if M is simply connected and B is complete, then B is globally reflective.*

Proof. Instead of giving a Lie algebra theoretic proof of this theorem by using Theorem 3, we will construct the reflection maps explicitly. By the previous remark we can assume M to be simply connected. When M is E^n, the reflection map is simply the reflection map with respect to the linear subspace under consideration. For simply connected Riemannian manifolds M of nonzero constant curvatures, one can use the standard models as hypersurfaces of Euclidean spaces E^{n+1} or pseudo-Euclidean spaces E_1^{n+1}, [11, p. 66]. Totally geodesic submanifolds are merely intersections of M with linear subspaces through the origin of E^n or E_1^n; the reflection maps are induced by the reflection with respect to the corresponding linear subspaces (cf. [7, p. 340]). q.e.d.

Combining Theorems 1 and 4, we have the following reflection principle for minimal submanifolds in spaces of constant curvatures, which generalizes the Schwartz reflection invariance of minimal surfaces.

Theorem 5. *Let N be a minimal submanifold of a complete Riemannian manifold M of constant curvature. If N contains an imbedded totally geodesic submanifold B of M as a hypersurface, then M is locally invariant under the reflection ρ_B with respect to B. Furthermore, if M is simply connected, and N, B are both complete, then $\rho_B(N) = N$.*

Proof. We only need to observe that any local symmetric space is an analytic manifold [3, p. 18]. q.e.d.

For an arbitrary local symmetric space it is not true that every imbedded geodesic submanifold is locally reflective. In fact, we have the following theorem.

Theorem 6. *Let M be a complete Riemannian local symmetric space. If there exists an imbedded geodesic L in M such that L is reflective, then L must be a space of constant curvature.*

Proof. By the remark after Theorem 3 we can assume M to be simply connected and hence globally symmetric, and assume L to be a complete geodesic. Without loss of generality we can also assume that L goes through the origin 0 of M. Let $\mathfrak{b}$ be the subspace of $\mathfrak{m}$ corresponding to L. From Theorem 3 and definitions it follows that the orthogonal complement $\mathfrak{b}^\perp$ of $\mathfrak{b}$ is also a reflective subspace. Since the reflective submanifold $L^\perp$ corresponding to $\mathfrak{b}^\perp$ is of codimension 1 and the reflection map of $\mathfrak{b}^\perp$ is a "reflection" in the sense of [4], by Theorem 1 in [4] M must be a space of constant curvature. q.e.d.

We will classify all reflective submanifolds of the complex projective space

CP^n, with the Fubini study metric, and its dual, the complex hyperbolic space [5, p. 282].

Let M be a complex manifold with the almost complex structure J. For a point $x \in M$, a subspace S of $T_x(M)$ is said to be holomorphic if $J(S) = S$, and to be totally real if $J(S) \perp S$.

It is well-known ([12, Theorem 1] and [5, pp. 277, 285]) that the totally geodesic submanifolds of CP^n are CP^k, $1 \leq k < n$, and RP^l, $1 \leq l \leq n$, naturally imbedded in CP^n as well as their images by holomorphic isometrics of CP^n. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the canonical decomposition of CP^n as a Riemannian symmetric space. It is easy to see that the totally geodesic submanifolds of CP^n described above correspond to the holomorphic subspaces and totally real subspaces of $\mathfrak{m}$ respectively (cf. [9]). If $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is the canonical decomposition of the complex hyperbolic space as a symmetric space, then by duality or by classifying its Lie triple systems one can obtain that the totally geodesic submanifolds through its origin correspond to the holomorphic subspaces and totally real subspaces of $\mathfrak{m}$ (cf. [9]). The complex projective space and complex hyperbolic space are both Kähler manifolds of constant holomorphic sectional curvatures. Now let J denote the almost complex structure, and $\langle \ , \ \rangle$ the inner product arising from the Kählarian metric of the complex projective spaces or complex hyperbolic space. Then their curvature tensors are given by (cf. [5, p. 165])

$$R(\xi, \eta) = \frac{c}{4}\{\xi \wedge \eta + J\xi \wedge J\eta + 2\langle \xi, J\eta \rangle J\} ,$$

where $c > 0$ for the complex projective space, $c < 0$ for the complex hyperbolic space, and $\xi \wedge \eta$ is the endomorphism such that $(\xi \wedge \eta)\cdot\zeta = \langle \zeta, \eta \rangle \xi - \langle \zeta, \xi \rangle \eta$. With these preparations we are now ready to prove the following theorem.

Theorem 7. *The reflective submanifolds of CP^n are CP^k, $1 \leq k < n$, and RP^n, naturally imbedded in CP^n and their images by holomorphic isometries of CP^n. The reflective submanifolds of the complex hyperbolic space are the geodesic submanifolds which correspond to the reflective submanifolds of CP^n under duality.*

Proof. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the canonical decomposition of the complex projective space or complex hyperbolic space as Riemannian symmetric spaces. By our previous considerations it suffices to show that the reflective subspaces of $\mathfrak{m}$ are the holomorphic subspaces and the totally real subspace of maximal dimension. For $\xi, \eta, \zeta \in \mathfrak{m}$, it is well-known (cf. [5, p. 231]) that $[[\xi, \eta], \zeta] = -R(\xi, \eta)\cdot\zeta$. Therefore we have

$$(*) \quad [[\xi, \eta], \zeta] = -\frac{c}{4}\{\langle \eta, \zeta \rangle \xi - \langle \xi, \zeta \rangle \eta + \langle J\eta, \zeta \rangle J\xi - \langle J\xi, \zeta \rangle J\eta + 2\langle \xi, J\eta \rangle J\zeta\} .$$

Let e_A, $1 \leq A \leq n$ be a set $\mathfrak{m}$ such that $\{e_A, J(e_A); 1 \leq A \leq n\}$ is an orthonormal basis of $\mathfrak{m}$. A typical holomorphic subspace $\mathfrak{b}_1$ of $\mathfrak{m}$ is the one generated by the vectors $\{e_i, J(e_i); 1 \leq i \leq k < n\}$, and $\mathfrak{b}_1^{\perp}$ is generated by $\{e_\alpha; J(e_\alpha); k + 1 \leq \alpha \leq n\}$. A typical totally real subspace of maximal dimension $\mathfrak{b}_2$ of $\mathfrak{m}$ is the one generated by $\{e_A; 1 \leq A \leq n\}$, and $\mathfrak{b}_2^{\perp}$ is generated by $\{Je_A; 1 \leq A \leq n\}$. Using $(*)$ it is straightforward to check that $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are both reflective subspaces of $\mathfrak{m}$. Obviously, any totally real subspace of $\mathfrak{m}$ of dimension less than n cannot be reflective, since for that subspace there is no Lie triple system of the complementary type. q.e.d.

Finally we will mention some more examples of reflective submanifold in symmetric spaces.

1) *Let $M = M_1 \times M_2$, where M_1 and M_2 are simply connected Riemannian symmetric spaces. Then M_1 and M_2 are both reflective submanifolds of M.*

2) *Let $G_{p,q}(\boldsymbol{R})$ be the oriented grassmann manifold. For $p' \leq p, q' \leq q$, $G_{p',q'}(\boldsymbol{R})$ naturally imbedded in $G_{p,q}(\boldsymbol{R})$ are totally geodesic submanifolds. Among these, $G_{p,q'}(\boldsymbol{R})$ and $G_{p',q}(\boldsymbol{R})$ are reflective submanifolds.*

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