J. DIFFERENTIAL GEOMETRY
8 (1973) 95-102

CONGRUENCE OF HYPERSURFACES

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Perhaps the simplest type of function which measures curvature is the Frenet-Serret curvature of a plane curve. It is well-known that this curvature as a function of the arc length determines the congruence class of a curve. Its generalization in one direction is the normal curvature (or what we shall call the *bending*) of a hyperfurface. It may be defined as follows: let M be a hypersurface of dimension n in a Euclidean space $\boldsymbol{R}^{n+1}$, τ its tangent bundle, and

$$\alpha\colon \tau \times \tau \to \boldsymbol{R}$$

the second fundamental form. Let $\pi\colon G_1 \to M$ denote the Grassmann bundle of lines on M. Define the bending $K_\alpha\colon G_1 \to \boldsymbol{R}$ to be the function which assigns to each tangential direction v (at some point of M) the number

$$K_\alpha(v) = \alpha(u, v)/\|v\|^2 .$$

This function usually appears in textbooks only as an auxiliary before defining the sectional curvature. It is perhaps surprising that it has apparently not been noted before that this function essentially determines the congruence class of a hypersurface. To make this precise, we shall define two hypersurfaces $M, \overline{M}$ of $\boldsymbol{R}^{n+1}$ to be *similarly bent* if there exists a diffeomorphism $f\colon M \to \overline{M}$ such that $f^*K_{\bar\alpha} = K_\alpha$; in this case we shall call f a *bending preserving diffeomorphism*. We have

Theorem A. *Let $M, \overline{M}$ be two hypersurfaces in $\boldsymbol{R}^{n+1}$, $n \geq 2$, and $f\colon M \to \overline{M}$ a bending-preserving diffeomorphism. Suppose that*

a) *the nonumbilic points are dense in M, and*

b) *the sectional curvature of M is not identically zero.*

Then f is a congruence.

(Recall that a point $x \in M$ is *nonumbilic* if $K_{\alpha|\pi^{-1}(x)}$ is not identically constant.)

The congruence problem for hypersurfaces has a long history. The underlying analytic statement is that a diffeomorphism f, which is an isometry and preserves the second fundamental form (meaning $f^*\bar\alpha = \alpha$), is a congruence, and the point is to catch this analytic content in Frenet-Serret type, more intuitive geometric terms. A very interesting variant of this is Minkowski's

Received October 12, 1971.

theorem which asserts that an ovaloid with positive Gaussian curvature is determined by its Gaussian curvature regarded as a function of the outer normal up to translation. So also is the Cohn-Vossen's rigidity theorem: an isometry of ovaloids is a congruence. Other variants involve the third fundamental forms, etc.; cf. Chern [2, p. 29]. As late as 1943 E. Cartan [1] again explicitly posed the problem of investigating the geometry of the second fundamental form. As opposed to our $f^*K_{\bar\alpha} = K_\alpha$, he investigated $f^*\bar\alpha = \alpha$ which led him to a Cauchy problem. Because of the difficulties of the Cauchy problem at a singular point the analysis is quite complicated. This has been further investigated by Grove [5], Erard [3], Simon [9], and Gardner [4] who have proved congruence for ovaloids satisfying some additional hypothesis.

Our method of proving theorem A is similar to that in [7] where we proved

Theorem B. *Let (M, g), $(\bar M, \bar g)$ be two Riemannian manifolds of dimension ≥ 4, and $f: M \to \bar M$ a sectional curvature preserving diffeomorphism. Suppose that*

$(*)$ *nonisotropic points are dense in M.*

Then f is an isometry.

(A point $x \in M$ is *nonisotropic* if the sectional curvature $K|_{\pi^{-1}(x)}$ is not identically constant where $\pi: G_2 \to M$ is the Grassmann bundle of 2-planes.)

In a sense Theorem A is simpler and more primitive than Theorem B. Using the proof of Theorem A it is possible to simplify the proof of Theorem B as well as to improve it by setting some cases when $\dim M = 3$. One can ask similar questions for hypersurfaces of submanifolds of space forms and for Gauss Kronecker curvatures. In the past these problems have been treated separately but in our formulation with appropriate hypothesis Theorems A and B can be generalized to cover these cases. As usual "generic" cases are settled while the "nongeneric" cases present some interesting problems. It is possible to give a uniform treatment of these various curvatures and the tensors which appear in their definitions. This effectively systematizes computations involveing the Bianchi identities, Ricci, Weyl, Gauss Kronecker curvatures, etc. We shall present it elsewhere.

It is a pleasant duty to thank Professor Klingenberg for his interest in this work. Thanks are also to Professors Pohl and Takens for some useful comments, and to the NSF GP-20289 and the Sonderforschungsbereich Theoretische Mathematik for their financial support.

All manifolds in this paper are assumed to be connected. Also "smooth" means C^2.

1. Proof of Theorem A

Intuitively the proof may be described as follows: consider an infinitesimal intelligent being standing on the mountain top. What he feels is not the second fundamental form α but the bending K_α. If the point is umbilic he is lost and cannot distinguish between directions. If however the point is nonumbilic then by measurements of K_α, in principle, he would discover angles between different directions. After this he cannot make any further progress unless he knows the differential of bending—such as the equation of Codazzi. Of course even with this he may still miss the absolute measure of distance.

We proceed to a rigorous proof. Let $\alpha_\circledcirc$ (resp. $\bar{\alpha}_\circledcirc$) denote the operator corresponding to α (resp. $\bar{\alpha}$). If v is a nonzero tangent vector, let (v) denote the corresponding point on the Grassmann bundle of lines.

Lemma 1. *Under the hypothesis* a) *of the theorem, f is conformal.*

At first we deduced this lemma from a more general theorem to appear in [8]. This short proof in this special case was pointed out to us by Professor P. Hartman.

Proof. Because of continuity it suffices to show that the differential f_* is a homothety at a nonumbilic point $x \in M$.

First consider the case of a 2-dimensional hypersurface. Let $\{e_1, e_2\}$ be the unit principal direction vectors at x so that

$$\alpha_\circledcirc e_1 = \lambda_1 e_1 \ , \quad \alpha_\circledcirc e_2 = \lambda_2 e_2 \qquad (\lambda_1 > \lambda_2) \ .$$

λ_1, λ_2 are characterized by the fact that they are respectiely maximum and minimum of $K_{\alpha|\pi^{-1}(x)}$, and must be preserved by a bending preserving map. It follows that $f_* e_1 = \bar{e}_1$, $f_* e_2 = \bar{e}_2$ are the principal directions at x which are necessarily orthogonal since $\lambda_1 \neq \lambda_2$. Let $\|\bar{e}_i\|^2 = a_i$, $i = 1, 2$. Now the bending preserving condition $K_\alpha(xe_1 + ye_2) = K_{\bar{\alpha}}(x\bar{e}_1 + y\bar{e}_2)$ leads to

$$\frac{\lambda_1 x^2 + \lambda_2 y^2}{x^2 + y^2} = \frac{\lambda_1 a_1 x^2 + \lambda_2 a_2 y^2}{a_1 x^2 + a_2 y^2} \ , \qquad (x, y) \neq (0, 0) \ ,$$

which implies

$$(\lambda_1 - \lambda_1)(a_1 - a_2) = 0 \ .$$

Since $\lambda_1 \neq \lambda_2$, we have $a_1 = a_2$, i.e., f_* is a homothety at x.

When $\dim M \geq 2$, it is easy to see that v is an eigenvector of $\alpha_\circledcirc$ if and only if (v) is a critical point of $K_{\alpha|\pi^{-1}(x)}$. Since a critical point is carried into a critical point by f_* it follows that an eigenvector of $\alpha_\circledcirc$ is carried into an eigenvector of $\bar{\alpha}$.

The argument in the 2-dimensional case shows that if v, w are two unit eigenvectors of $\alpha_\circledcirc$ with *distinct* eigenvalues, then $f_* v$ and $f_* w$ have the same length. Since x is nonumbilic, it easily follows that f_* is a homothety on each eigenspace of $\alpha_\circledcirc$ with the same homothety factor, and hence is a homothety. q.e.d.

Using Lemma 1 we can write $f^*\bar{g} = e^{2\varphi}g$ where $\varphi\colon M \to \boldsymbol{R}$ is a smooth function. The condition $f^*K_{\bar{\alpha}} = K_\alpha$ then implies

$$f^*\bar{\alpha} = e^{2\varphi}\alpha \ .$$

To simplify the notation we identify M with $\bar{M}$ via f, and omit f^* from the formulas.

We shall need the following fact about the connections ∇, $\bar{\nabla}$ defined by the conformally related metrics g, $\bar{g}$ respectively: Set

$$S = \bar{\nabla} - \nabla \quad \text{and} \quad G = \operatorname{grad} \varphi \qquad \text{(with respect to } g\text{)},$$

and let x be a vector field on M. Considered as a derivation of the full tensor algebra, S_x is determined as follows:

a) If f is a smooth function, then $S_x f = 0$.

b) If y is a vector field, then

$$S_x y = (x\varphi)y + (y\varphi)x - \langle x, y\rangle G \ .$$

c) If θ is a 1-form, then for any vector field y

$$(S_x\theta)y = -\theta(S_x y) \ .$$

(For a proof of b) see [7, Proposition 2.1]; a) and c) easily follow.)

Using this fact we have

$$\nabla_x\bar{\alpha} = \bar{\nabla}_x(e^{2\varphi}\alpha) = e^{2\varphi}(2x\varphi\alpha + \nabla_x\alpha + S_x\alpha) \ .$$

Substituting this in the Codazzi's equation

$$(\bar{\nabla}_{x_1}\bar{\alpha})(x_2, y) - (\bar{\nabla}_{x_2}\bar{\alpha})(x_1, y) = 0$$

and using the Codazzi's equation for α (with respect to g), we get the following basic differential equation satisfied by φ:

$$(*) \qquad 2(x_1\varphi)\alpha(x_2, y) - 2(x_2\varphi)\alpha(x_1, y) + \alpha(x_1, S_{x_2}y) - \alpha(x_2, S_{x_1}y) = 0 \ ,$$

for any vector fields x_1, x_2, y.

Lemma 2. *Under the hypothesis* a) *of the theorem, f is a homothety.*

Proof. Suppose at a point $m \in M$, $G_m = \operatorname{grad}\varphi|_m \neq 0$. In the system $(*)$ set

$$x_1 = h = G_m/\|G_m\| \ , \qquad x_2 = x \ , \quad y_2 = y \ .$$

Suppose that x is orthogonal to h, and y is arbitrary. Since

$$S_h y = \|G_m\| \, y \quad \text{and} \quad S_x y = (y\varphi)x - \langle x, y\rangle \|G_m\| \, h \ ,$$

the system $(*)$ becomes

$$\begin{aligned} 0 &= 2\,\|G_m\|\,\alpha(x,y) + \alpha(h, S_x y) - \alpha(x, S_h y) \\ &= \|G_m\|\,\alpha(x,y) + (y\varphi)\alpha(h,x) - \langle x,y\rangle \|G_m\|\,\alpha(h,h)\ . \end{aligned}$$

In this equation settting $y = x$ and $y = h$ in turn we get

$$\alpha(h, l) = \alpha(x, x)\ , \qquad \alpha(h, x) = 0\ .$$

It clearly follows that $K_\alpha =$ constant at the point m, i.e., m is umbilic. But then the set $\{m \mid G_m \neq 0\}$ is an open subset consisting of umbilics which contradicts the hypothesis. Hence $G \equiv 0$. In other words f is a homothety. q.e.d.

The proof of Theorem A can be completed as follows. By Lemma 2 there exists a constant $c > 0$ such that

i) $$f^*\bar{g} = cg\ .$$

Thus if K, $\bar{K}$ are the respective sectional curvature functions of M, $\bar{M}$, then

ii) $$f^*\bar{K} = K/c\ .$$

But since bending is preserved, we also have

iii) $$f^*\bar{\alpha} = c\alpha\ ,$$

and i), iii) together with Gauss equations imply

iv) $$f^*\bar{K} = K\ .$$

From ii) and iv) it follows that $c = 1$ since $K \not\equiv 0$. So f is an isometry. Congruence is now classical and the proof is finished.

2. Discussions

If hypothesis a) or b) of the Theorem A is omitted, some interesting pathologies appear. We discuss the various possibilities.

1) *Suppose that* (M, g) *is analytic*. Then either all points of M (and so of $\bar{M}$) are umbilics or nonumbilic points are dense. In the first case M, $\bar{M}$ are parts of either hyperplanes or hypersheres with the same radius, and so *any* diffeomorphism of M onto $\bar{M}$ is bending preserving. Thus M, $\bar{M}$ need not be globally congruent although they are locally congruent. However, if we *assume that* M, $\bar{M}$ *are also complete*, then they are either whole hyperplanes or whole hypersheres with the same radius. So they are congruent.

2) Suppose that nonumbilic points are dense in M and the sectional curvature is identically zero. Then Lemma 2 still applies, and the map f is a *homothety*. But f need not be a congruence, and indeed M, $\bar{M}$ need not even be locally congruent. It is not difficult to construct local examples of this phenomenon.

In the following we construct one for *two complete, analytic, similarly bent noncongruent flat hypersurfaces.*

Let

$$\psi : \boldsymbol{R} \to (-\pi/2, \pi/2)$$

be any analytic function. Let $0 < c < 1$, and consider the function $\bar{\psi}$ defined by

$$\bar{\psi}(s) = c\psi(s/c) \ .$$

Let C, $\bar{C}$ be the complete, analytic plane curves whose intrinsic equations are, respectively,

$$\psi = \psi(s) \ , \qquad \bar{\psi} = \bar{\psi}(s) \ .$$

The map $f \colon C \to \bar{C}$ defined by

$$f(\psi, s) = (c\psi, cs)$$

is easily seen to be homothety *which preserves the Frenet-Serret curvature.*

Let M, $\bar{M}$ be cylinders in $\boldsymbol{R}^{n+1}$ on C, $\bar{C}$ respectively such that the generators of M, $\bar{M}$ are orthogonal to the planes of C, $\bar{C}$ respectively. We can easily extend f to a diffeomorphism from M to $\bar{M}$ which is a bending preserving homothety. However, it is plain that in general M, $\bar{M}$ need not even be locally congruent.

3) Suppose that nonumbilic points are dense and M, $\bar{M}$ are *complete*, flat hypersurfaces. By Lemma 2, $f^*\bar{g} = cg$ where c is a positive constant. By a theorem of Hartman and Nirenberg [6], we have

$$M = C \times \boldsymbol{R}^{n-1} \ , \qquad \bar{M} = \bar{C} \times \boldsymbol{R}^{n-1} \ ,$$

i.e., M, $\bar{M}$ are cylinders built on plane curves C, $\bar{C}$ such that the generators are orthogonal to the planes of C, $\bar{C}$ respectively. It is clear that f induces a homothety of C onto $\bar{C}$.

If C happens to be *closed*, then that f is necessarily an isometry follows easily from the well-known integral formula

$$\oint k(s) ds = 2\pi \ ,$$

where $k(s)$ is the Frenet-Serret curvature. In this case f *is a congruence.*

If C is not closed, then M, $\bar{M}$ need not even be locally congruent as was pointed out in 2).

4) Suppose that nonumbilic points are dense. The following is a local condition which ensures the congruence.

(A) Suppose that there exists a nongeodesic curve C such that f preserves its first Frenet-Serret curvature (either with respect to either M or $\boldsymbol{R}^{n+1}$). Then f is a congruence.

Proof. Indeed, let x be the unit tangent vector field to C, and let

$$f(C) = \bar{C}\ , \quad f_* x = \bar{x}\ , \quad f^* \bar{g} = cg\ .$$

Then $\bar{x}/\sqrt{c}$ is a unit tangent vector field to $\bar{C}$, and also

$$f_* \nabla_x x = \bar{\nabla}_{\bar{x}} \bar{x}\ .$$

So

$$\overline{\|\bar{\nabla}_{x/\sqrt{c}} \bar{x}/\sqrt{c}\,\|} = \|\nabla_x x\|/\sqrt{c}\ .$$

The first Frenet-Serret curvature of C with respect to M is $\|\nabla_x x\|$, and with respect to $\boldsymbol{R}^{n+1}$ it is

$$\|\tilde{\nabla}_x x\| = (\|\nabla_x x\|^2 + |\alpha(x, x)|^2)^{1/2}\ ,$$

where $\tilde{\nabla}$ is the connection in $\boldsymbol{R}^{n+1}$. In either case since $\nabla_x x \neq 0$, it follows from our hypothesis that $c = 1$.

5) Let $S^{n+1}(c)$ denote the $(n+1)$-dimensional complete simply connected space of constant curvature c. *Theorem A* can be *generalized to hypersurfaces of* $S^{n+1}(c)$; this is basically due to the fact that the second fundamental form of a hypersurface in $S^{n+1}(c)$ again satisfies the Codazzi's equation. Now suppose that $c \neq 0$ *and* $n \geq 3$. In this case the Gauss equation implies that if $c > 0$, then a hypersurface cannot be flat; and if $c < 0$, then a flat hypersurface is totally umbilical. Hence in this case we *may omit the hypothesis* "the sectional curvature of M is not identically zero" from Theorem A. This hypothesis seems to be necessary in case $n = 2$. The author does not know whether one can replace it by a global hypothesis, e.g., M, $\bar{M}$ are *complete*. Indeed it would also be interesting to determine the shape of a complete flat hypersurface in $S^3(1)$ or $S^3(-1)$.

6) Finally we consider the congruence of n-dimensional submanifolds of $S^N(c)$. In this case the second fundamental form α must be interpreted as a bilinear form on the tangent bundle of a submanifold with values in its normal bundle. The bending K_α defined by the same formula as before also has values in the normal bundle. An umbilic point is defined as before. We define two submanifolds M, $\bar{M}$ with normal bundles ν, $\bar{\nu}$ and second fundamental forms α, $\bar{\alpha}$ respectively to be *similarly bent* if there exists a diffeomorphism $f\colon M \to \bar{M}$ *which is covered by an isometry* $\tilde{f}\colon \nu \to \bar{\nu}$ *of normal bundles such that* $K_{\bar{\alpha}} \circ f_* = \tilde{f} \circ K_\alpha$. However to get a congruence statement we shall have to take into account the connection in the normal bundle also. We define M, $\bar{M}$ (as above) to be *similarly twisted* if $\tilde{f}$ *is connection-preserving*, and ask:

are similarly bent and similarly twisted submanifolds congruent?

The answer to this question is also affirmative if the conditions a), b) of Theorem A formally hold. Lemmas 1 and 2 are still valid although the proof of Lemma 1 requires a major modification (cf. [8]). At the final step one has to invoke the following analytic fact:

Let M, $\bar{M}$ be submanifolds of $S^N(c)$. Suppose that there exists an isometry $f\colon M \to \bar{M}$ covered by a connection preserving isometry $\hat{f}\colon \nu - \bar{\nu}$ such that $\hat{f} \circ \alpha = \bar{\alpha} \circ f_*$. Then f is a congruence.

This fact is essentially well-known; see, e.g., Sternberg [10, p. 245]. The essential points in this fact are (i) the characteristic property of $S^N(c)$ that given two points p, q in $S^N(c)$ and orthogonal frames b_p, b_q at p, q respectively there exists a unique isometry of $S^N(c)$ carrying p into q and b_p into b_q, (ii) the uniqueness of solution for the initial value problem of a system of first order partial differential equations (in which form the Codazzi-Mainardi equations may be expressed). We need not elaborate this further.

Of course, the condition that $\tilde{f}$ be connection-preserving is very strong. But to replace it by "torsion functions" (as in the theory of space curves) although possible seems very artificial to us.

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