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SECOND ORDER CONNECTIONS. II

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1. Introduction

The purpose of this paper is the development of certain implications of the second order connection, introduced previously by the present writer [1]. If M is an n-dimensional C^∞ manifold, we show that a linear second order connection on M determines a "covariant derivative" ∇' on TM, which satisfies the usual conditions over the ring $\mathfrak{F}'(TM)$, the vertical lift of the ring $\mathfrak{F}(M)$ of C^∞ functions on M. Using the properties of ∇', we obtain equations analogous to those of Gauss and Weingarten, and an analog of the second fundamental form.

If A, B, $C \in \mathfrak{X}'(TM)$, the module of C^∞ vector fields on TM over the ring $\mathfrak{F}'(TM)$, then we obtain the maps Tor (A, B) and $R(A, B)C$ which are $\mathfrak{F}'(TM)$ multilinear analogs of the torsion and curvature tensors. From the components of R we obtain equations analogous to those of Gauss and Codazzi, as well as an additional equation which defines a "vertical curvature tensor" on M. Finally, we obtain an invariant which we call the second order curvature of M; this yields as a special case the usual (first order) curvature of M.

2. Preliminary remarks

In this section we will briefly outline the main results of [1] utilized in the main body of this paper. The notation employed is essentially that of [1] and [2], with the summation convention employed on lower case Latin indices.

A second order connection on M is a connection on the bundle ${}^2_0\Pi : {}^2M \to M$ which naturally induces a (first order) connection on M. If ${}^1_0\Pi_*$ is the tangent map of ${}^1_0\Pi : TM \to M$, and $\tilde{D}$ is the connection map of the induced connection, then TTM and consequently ${}^2M$ may be given a vector bundle structure over M, such that if HTM and VTM are the horizontal and vertical subbundles of TTM determined by the vector bundle structure, then

$$ {}^1_0\Pi_* : HTM_p \to TM_{{}^1_0\Pi(p)} , \qquad \tilde{D} : VTM_p \to TM_{{}^1_0\Pi(p)} $$

are isomorphisms at each $p \in TM$.

Given a coordinate chart (U, ϕ) of M there are determined two sets of bases, relative to the induced coordinates $x^{01}, \cdots, x^{0n}; x^{11}, \cdots, x^{1n}$ on ${}^1_0\Pi^{-1}(U)$,

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$$(1) \qquad X_i^h = \partial/\partial x^{0i} - \Gamma_{ik}^j x^{1k}\partial/\partial x^{1j}\,, \qquad X_i^v = \partial/\partial x^{1i}$$

respectively spanning HTM_p and VTM_p at each $p \in {}_0^1\Pi^{-1}(U)$. Similarly, there are two sets of bases

$$X_i^h = \partial/\partial x^{0i} - \Gamma_{ii}^j\partial/\partial x^{1j}\,, \qquad X_i^v = \partial/\partial x^{1i}$$

spanning the vertical and horizontal subbundles of ${}_0^2\Pi : {}^2M \to M$ at each $x \in U$. We call these subbundles the horizontal and vertical bundles over M.

A second order connection on M determines a covariant differentiation of a section A of ${}_0^2\Pi$: ${}^2M \to M$ with respect to a vector field X on M. The local form of this differentiation in terms of local coordinates on M is

$$(2) \qquad D_X A = \xi^j\left(\frac{\partial A^{0i}}{\partial x^{0j}} + \Gamma_{jk}^{0i}A^{0k}\right)X_i^h + \xi^j\left(\frac{\partial A^{1i}}{\partial x^{0j}} + \Gamma_{j0k}^{1i}A^{0k} + \Gamma_{j1k}^{1i}A^{1k}\right)X_i^v\,,$$

where $X = \xi^i\partial/\partial x^{0j}$ and $A = A^{0i}X_i^h + A^{1i}X_i^v$.

3. The $\mathfrak{F}'$-derivative

Theorem 1. *A second order linear connection on M determines a C^∞ map ∇': $\mathfrak{X}'(TM) \times \mathfrak{X}'(TM) \to \mathfrak{X}'(TM)$ such that if $A, B, C \in \mathfrak{X}'(TM)$ and $f \in \mathfrak{F}'(TM)$, then*

1) $\nabla'_A(B + C) = \nabla'_A B + \nabla'_A C$,
2) $\nabla'_{A+B}C = \nabla'_A C + \nabla'_B C$,
3) $\nabla'_{fA}B = f\nabla'_A B$,
4) $\nabla'_A fB = (Af)B + f\nabla'_A B$.

We call ∇' an $\mathfrak{F}'$-derivative.

Proof. Form the map

$${}_0^1\Pi_* \oplus \tilde{D}\colon TTM \to TM \oplus TM\ .$$

Since ${}^2M \approx TM \oplus TM$, we may regard ${}_0^1\Pi_* \oplus \tilde{D}$ as a map of TTM onto ${}^2M$ which is an isomorphism of TM_p onto ${}^2M_{{}_0^1\Pi(p)}$ at each $p \in TM$. Suppose that $A, B \in \mathfrak{X}'(TM)$ and that $B^h \in \mathfrak{X}'(TM)$ is the vector field obtained by taking the horizontal component of B_p at each $p \in TM$. If $\sigma_p(t)$ is an integral curve of B^h through $p \in TM$, then ${}_0^1\Pi_*\, B_p^h$ is tangent to ${}_0^1\Pi\cdot\sigma_p(t)$ at ${}_0^1\Pi(p)$. Since ${}_0^1\Pi_* \oplus \tilde{D}(A)$ is a well defined section of ${}_0^2\Pi$: ${}^2M \to M$ along ${}_0^1\Pi\cdot\sigma_p(t)$,

$$D_{{}_0^1\Pi_* B_p^h}\,{}_0^1\Pi_* \oplus \tilde{D}(A)$$

is defined and we take

$$(3) \qquad (\nabla'_B A)_p = ({}_0^1\Pi_* \oplus \tilde{D})_p^{-1}(D_{{}_0^1\Pi_* B_p^h}\,{}_0^1\Pi_* \oplus \tilde{D}(A))\,,$$

where $({}^1_0\Pi_* \oplus \tilde{D})_p^{-1}$ denotes the inverse of the isomorphism ${}^1_0\Pi_* \oplus \tilde{D}: TM_p \to {}^2M_{{}^1_0\Pi(p)}$. That $\nabla'_B A$ is C^∞ on TM follows from the fact that $\sigma_p(t)$ is C^∞ as a function of p. Conditions 1)–4) follow from the fact that if $f' \in \mathfrak{F}'(TM)$ is the vertical lift of $f \in \mathfrak{F}(M)$ and $B \in \mathfrak{X}'(TM)$, then ${}^1_0\Pi_*(f'B)_p = f({}^1_0\Pi(p))\,{}^1_0\Pi_* B_p$ and $(Bf')_p = ({}^1_0\Pi_* B_p)f$, together with the local expression (2).

Lemma 1. *If $p \in TM_0$, then $(\nabla'_B A)$ depends only on the value of B^h at p and the values of A on TM_0.*

Proof. If $p \in TM_0$ and $B \in \mathfrak{X}'(TM)$, then the integral curve σ_p of B^h through p lies entirely in the zero section TM_0 of ${}^1_0\Pi: TM \to M$. Hence ${}^1_0\Pi_* \oplus \tilde{D}(A)$ depends only on the values of $A \mid TM_0$. That $(\nabla'_B A)_p$ depends only on the value of B^h at p follows from (3).

Since the restriction of ${}^1_0\Pi$ to TM_0 is a diffeomorphism ${}^1_0\Pi: TM_0 \to M$, the restriction of ${}^1_0\Pi_*$ to $T(TM_0)$ is an isomorphism ${}^1_0\Pi_*: T(TM_0) \to TM$. Because the second order connection is linear, the induced first order connection is also. This means that if we choose a point of M and a coordinate neighborhood containing it, then the induced local bases for HTM and $T(TM_0)$ coincide on TM_0. Thus we may identify the bundle ${}^1_0\Pi: TM \to M$ with the subbundle ${}^2_1\Pi: HTM_0 \to TM_0$.

If $X, Y \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{X}^v(M)$, the module of vertical vector fields on $TM_0 \approx M$, then we may utilize the above identification and the fact that Lemma 1 implies that ∇' may be restricted to $TM_0 \approx M$ to decompose $\nabla'_X Y$ and $\nabla'_X \xi$ into horizontal and vertical components. Thus using (2) we have, on $TM_0 \approx M$,

$$\nabla'_X Y = D_X Y + \alpha(X, Y) , \qquad \nabla'_X \xi = \nabla_X \xi . \tag{4}$$

Theorem 2. *If $X, Y \in \mathfrak{X}(M)$, and $\xi \in \mathfrak{X}^v(M)$, then*

1) *the horizontal component $D_X Y$ of $\nabla'_X Y$ is the usual covariant derivative of the induced connection,*

2) *the vertical component $\alpha(X, Y)$ of $\nabla'_X Y$ is bilinear over $\mathfrak{F}(M)$,*

3) *$\nabla: \mathfrak{X}(M) \times \mathfrak{X}^v(M) \to \mathfrak{X}^v(M)$ satisfies the usual conditions* (1)–4) *of* ∇') *of a covariant derivative over $\mathfrak{F}(M)$.*

Proof. 1) D is obtained by taking the horizontal component of the restriction of ∇' to horizontal vector fields on TM_0. The induced (first order) covariant derivative may be obtained by taking the horizontal component of the second order covariant derivative restricted to horizontal vector fields of ${}^2_0\Pi: {}^2M \to M$. Since ${}^1_0\Pi_* \oplus \tilde{D}$ may be viewed as an identification of $TTM \mid TM_0$ with ${}^2M$ we see that D is the induced (first order) covariant derivative on M.

2) That α is bilinear over $\mathfrak{F}(M)$ may be seen by noting that since

$$\nabla'_X fY - D_X fY = \alpha(X, fY)$$

and

$$f\nabla'_X Y + (Xf)Y - fD_X Y - (Xf)Y = f(\alpha(X,Y)) ,$$

we have

$$\alpha(X, fY) = f\alpha(X,Y) .$$

Similarly, we see that

$$\alpha(fX, Y) = f\alpha(X,Y) .$$

3) The fact that ∇ satisfies conditions 1)–4) over $\mathfrak{F}(M)$ follows from the fact that ∇' satisfies these conditions over $\mathfrak{F}'(TM)$, and the fact that if $f' \in \mathfrak{F}'(TM)$ then it is the vertical lift of some function $f \in \mathfrak{F}(M)$ with $f' | TM_0 = f$ (since $TM_0 \approx M$).

In analogy with the equations of Gauss and Weingarten we call α the second fundamental form of the second order connection, and note that ∇ represents covariant differentiation with respect to a connection in the vertical bundle over M, and that the Weingarten map vanishes identically.

If $A, B \in \mathfrak{X}'(TM)$, we define

$$\mathrm{Tor}\,(A,B) = \nabla'_A B - \nabla'_B A - [A,B] . \tag{5}$$

Theorem 3. *The map* $\mathrm{Tor}\colon \mathfrak{X}'(TM) \times \mathfrak{X}'(TM) \to \mathfrak{X}'(TM)$ *is skew-symmetric and bilinear over* $\mathfrak{F}'(TM)$.

Since Tor is bilinear over $\mathfrak{F}'(TM)$ but not over $\mathfrak{F}(TM)$, $\mathrm{Tor}\,(A,B)_p$ depends in general on the behavior of A and B in a neighborhood of p; however, we may localize Tor on TM_0.

Lemma 2. *If* $A, B \in \mathfrak{X}'(TM)$ *are horizontal, and* $p \in TM_0$, *then* $[A,B]_p$ *depends only upon the values of* A *and* B *on* TM_0.

Proof. If (U, ϕ) is a coordinate chart at ${}^0_1\Pi(p)$, then $A = a^i X^h_i$, $B = b^j X^h_j$, a^i, $b^j \in \mathfrak{F}(TM)$. Thus $[A,B]_p = (a^i(X^h_i b^j)X^h_j + a^i b^j X^h_i X^h_j - b^j(X^h_j a^i)X^h_i - b^j a^i X^h_j X^h_i)_p$ Since $p \in TM_0$, $(X^h_i)_p = (\partial/\partial x^{0i})(p)$ and thus we have

$$[A,B]_p = [A | TM_0, B | TM_0]_p . \tag{6}$$

Theorem 4. *If* $p \in TM_0$ *and* $A, B \in \mathfrak{X}'(TM)$ *are horizontal, then*

$$\mathrm{Tor}\,(A_p, B_p) = \mathrm{Tor}\,(A,B)_p .$$

Proof. If A and B are horizontal vector fields on TM, and (U, ϕ) is a coordinate chart at ${}^1_0\Pi(p)$, then $A | TM_0 = a^i X^h_i | TM_0$, $B | TM_0 = b^j X^h_j | TM_0$ where $a^i, b^j \in \mathfrak{F}(M)$. Extend these to the vector fields

$$\bar{A} = a^{i'} X^h_i , \qquad \bar{B} = b^{j'} X^h_j ,$$

where $a^{i'}$ and $b^{j'}$ are the vertical lifts of a^i and b^j respectively. For $p \in TM_0$ we have by Lemmas 1 and 2

$$\mathrm{Tor}\,(A, B)_p = \mathrm{Tor}\,(\bar{A}, \bar{B})_p = \mathrm{Tor}\,(a^{i'}X_i^h, b^{j'}X_j^h)_p = a^{i'}b^{j'}\,\mathrm{Tor}\,(X_i^h, X_j^h)_p ,$$

and we see that if A or B vanishes at a point $p \in TM_0$, then $\mathrm{Tor}\,(A, B)_p = 0$. Hence we may take

$$\mathrm{Tor}\,(A_p, B_p) = \mathrm{Tor}\,(A, B)_p . \tag{7}$$

Remark. This implies that Tor induces a tensor on $TM_0 \approx M$, since the restriction of $\mathfrak{F}'(TM)$ to TM_0 may be identified with $\mathfrak{F}(M)$.

Theorem 5. *If ∇' is torsion free* (Tor $\equiv 0$ *on* TM_0), *then the induced* (*first order*) *covariant derivative is torsion free and α is symmetric.*

Proof. Suppose that $X, Y \in \mathfrak{X}(M)$. Since Tor may be restricted to $TM_0 \approx M$, it follows that if Tor $\equiv 0$ and $p \in TM_0$, then

$$\mathrm{Tor}\,(X, Y)_p = (\nabla'_X Y)_p - (\nabla'_Y X)_p - [X, Y]_p ,$$

so that

$$(D_X Y)_p + \alpha(X, Y)_p - (D_Y X)_p - \alpha(Y, X)_p - [X, Y]_p = 0 .$$

Thus we see that $\mathrm{Tor}_D\,(X, Y) = 0$ and $\alpha(X, Y) = \alpha(Y, X)$.

Definition. An $\mathfrak{F}'$-metric on TM is a map $G\colon \mathfrak{X}'(TM) \times \mathfrak{X}'(TM) \to \mathfrak{F}'(TM)$ which is C^∞, symmetric, positive definite, bilinear over $\mathfrak{F}'(TM)$, and has the additional property that if $A, B \in \mathfrak{X}'(TM)$ and $p \in TM_0$, then $G(A_p, B_p) = G(A, B)_p$.

We will say that ∇' is Riemannian with respect to the $\mathfrak{F}'$-metric G if on TM_0

$$\mathrm{Tor}\,(A, B) = 0 , \qquad XG(C, E) = G(\nabla'_X C, E) + G(C, \nabla'_X E) , \tag{8}$$

where $X \in \mathfrak{X}(M)$, $A, B, C, E \in \mathfrak{X}'(TM)$, and A, B are horizontal.

Theorem 6. *If ∇' is Riemannian with respect to an $\mathfrak{F}'$-metric having the property that horizontal and vertical vectors are orthogonal on TM_0, then D is Riemannian with respect to the induced metric in the horizontal bundle over $TM_0 \approx M$, ∇ is metric with respect to the induced metric in the vertical bundle, and $\alpha \equiv 0$.*

Proof. From the definition of an $\mathfrak{F}'$-metric it is clear that by restricting G to horizontal and vertical vector fields on $TM_0 \approx M$ we obtain metrics on the horizontal and vertical bundles over M. Suppose that $X, Y, Z \in \mathfrak{X}(M)$ with extensions A, B, C respectively to horizontal vector fields on a neighborhood of $p \in TM_0$. If ∇' is Riemannian with respect to the metric G, then on TM_0

$$AG(B, C) = G(\nabla'_A B, C) + G(B, \nabla'_A C) .$$

If $p \in TM_0$, then by Lemma 1 and the definition of an $\mathfrak{F}'$-metric we have

$$X_pG(Y, Z) = G(\nabla'_X Y, Z)_p + G(Y, \nabla'_X Z)_p \ .$$

Thus on $TM_0 \approx M$ we have

$$XG(Y, Z) = G(D_X Y + \alpha(X, Y), Z) + G(Y, D_X Z + \alpha(X, Z)) \ .$$

Also we see that if $\xi \in \mathfrak{X}^v(M)$, then $G(Y, \xi) = 0$ so that

$$XG(Y, \xi) = G(D_X Y + \alpha(X, Y), \xi) + G(Y, \nabla_X \xi) = G(\alpha(X, Y), \xi) = 0 \ ,$$

which implies that $\alpha \equiv 0$. Thus

$$XG(Y, Z) = G(D_X Y, Z) + G(Y, D_X Z)$$

on $TM_0 \approx M$. That $\mathrm{Tor}_D \equiv 0$ follows from Theorem 5. Finally, if $\xi, \eta \in \mathfrak{X}^v(M)$, then on $TM_0 \approx M$

$$XG(\xi, \eta) = G(\nabla'_X \xi, \eta) + G(\xi, \nabla'_X \eta) = G(\nabla_X \xi, \eta) + G(\xi, \nabla_X \eta) \ .$$

Suppose that the covariant derivative $D\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is Riemannian with respect to the metric g on M, and that the covariant derivative $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}^v(M) \to \mathfrak{X}^v(M)$ is metric with respect to the fiber metric h in the vertical bundle over M. If $X \in \mathfrak{X}(M)$, and A is a section of ${}^2_0\Pi\colon {}_2M \to M$, then we define the second order covariant derivative

$$\mathscr{D}_X A = D_X A^h + \nabla_X A^v$$

and the corresponding $\mathfrak{F}'$-derivative, using (3),

$$(\nabla'_B A)_p = ({}^1_0\Pi_* \oplus \tilde{D})_p^{-1}(D_{{}^1_0\Pi_* B^h_p}\, {}^1_0\Pi_* A + \nabla_{{}^1_0\Pi_* B^h_p} \tilde{D}(A)) \ , \tag{9}$$

If $A, B \in \mathfrak{X}'(TM)$, and we take

$$\langle A, B\rangle_p = g({}^1_0\Pi_* A_p, {}^1_0\Pi_* B_p) + h(\tilde{D}(A_p), \tilde{D}(B_p)) \ ,$$

then $\langle\ ,\ \rangle$ is an $\mathfrak{F}'$-metric.

Theorem 7. *∇' is Riemannian with respect to $\langle\ ,\ \rangle$.*

Proof. If $A, B \in \mathfrak{X}'(TM)$ and $X \in \mathfrak{X}(M)$, then we have on TM_0

$$\begin{aligned} X\langle A, B\rangle &= g(D_X\, {}^1_0\Pi_*\, A, {}^1_0\Pi_*\, B) + g({}^1_0\Pi_*\, A, D_X\, {}^1_0\Pi_*\, B) \\ &\quad + h(\nabla_X \tilde{D}(A), \tilde{D}(B)) + h(\tilde{D}(A), \nabla_X \tilde{D}(B)) \ , \end{aligned}$$

since D is Riemannian with respect to g, ∇ is metric with respect to h, and ${}^1_0\Pi_*(A \mid TM_0)$, $\tilde{D}(A \mid TM_0)$ are vector fields. Since

$$D_X\, {}^1_0\Pi_* A = {}^1_0\Pi_* ({}^1_0\Pi_* \oplus \tilde{D})^{-1}(D_X\, {}^1_0\Pi_* A + \nabla_X \tilde{D}(A)) \quad \text{on} \quad TM_0$$

and a similar expression holds for ∇, we see that

$$X\langle A, B\rangle = \langle \nabla'_X A, B\rangle + \langle A, \nabla'_X B\rangle$$

on TM_0. If $A, B \in \mathfrak{X}'(TM)$ are horizontal and have the restrictions X and Y respectively to $TM_0 \approx M$, then for $p \in TM_0$

$$\mathrm{Tor}\,(A, B)_p = \mathrm{Tor}\,(X, Y)_p = (D_X Y - D_Y X - [X, Y])_p \ .$$

Thus $\mathrm{Tor} \equiv 0$ on TM_0 since D is Riemannian.

If A, B, $C \in \mathfrak{X}'(TM)$, we define

$$R(A, B)C = \nabla'_A \nabla'_B C - \nabla'_B \nabla'_A C - \nabla'_{[A,B]} C \ . \tag{10}$$

Theorem 8.

$$R(A, B)C = -R(B, A)C \ ,$$

and R is $\mathfrak{F}'(TM)$ multilinear.

Theorem 9. *If A, B, $C \in \mathfrak{X}'(TM)$, and A, B are horizontal, then for $p \in TM_0$*

$$R(A_p, B_p)C_p = (R(A, B)C)_p \ .$$

Proof. In terms of a coordinate chart at ${}^1_0\Pi(p)$ we have

$$\begin{aligned} A \,|\, TM_0 &= a^{0i} X^h_i \,|\, TM_0 \ , \qquad B \,|\, TM_0 = b^{0i} X^h_i \,|\, TM_0 \ , \\ C \,|\, TM_0 &= C^{0i} X^h_i \,|\, TM_0 + C^{1i} X^v_i \,|\, TM_0 \ , \end{aligned}$$

where a^{0i}, b^{0i}, C^{0i}, $C^{1i} \in \mathfrak{F}(M)$. Extend these to the vector fields

$$\bar{A} = a^{0i\prime} X^h_i \ , \quad \bar{B} = b^{0i\prime} X^h_i \ , \quad \bar{C} = C^{0i\prime} X^h_i + C^{1i\prime} X^v_i \ ,$$

where the accent denotes vertical lift. From Lemmas 1 and 2 and the definition of R we see that for $p \in TM_0$, R_p depends only upon the values of A, B, and C on TM_0. Consequently, we have on TM_0

$$\begin{aligned} R(A, B)C = R(\bar{A}, \bar{B})\bar{C} &= R(a^{0i\prime} X^h_i, b^{0j\prime} X^h_j)(C^{0k\prime} X^h_k + C^{1k\prime} X^v_k) \\ &= a^{0i\prime} b^{0j\prime} C^{0k\prime} R(X^h_i, X^h_j) X^h_k + a^{0i\prime} b^{0j\prime} c^{1k\prime} R(X^h_i, X^h_j) X^v_k \ . \end{aligned}$$

Thus we see that if A, B or C vanishes at a point $p \in TM_0$, then $(R(A, B)C)_p = 0$, and hence we may take

$$R(A_p, B_p)C_p = (R(A, B)C)_p \ .$$

Remark. This implies that R induces a tensor on $TM_0 \approx M$, since the restriction of $\mathfrak{F}'(TM)$ to $TM_0 \approx M$ may be identified with $\mathfrak{F}(M)$.

Using the fact that ∇' may be restricted to $TM_0 \approx M$ we have for $X, Y, Z \in \mathfrak{X}(M)$

$$\begin{aligned} \nabla'_X\nabla'_Y Z &= \nabla'_X(D_Y Z + \alpha(Y, Z)) = D_X D_Y Z + \nabla_X\alpha(Y, Z) + \alpha(X, D_Y Z) , \\ \nabla'_Y\nabla'_X Z &= D_Y D_X Z + \nabla_Y\alpha(X, Z) + \alpha(Y, D_X Z) , \\ \nabla'_{[X,Y]}Z &= D_{[X,Y]}Z + \alpha([X, Y], Z) . \end{aligned} \tag{11}$$

If D is torsion free, then

$$D_X Y - D_Y X = [X, Y] ,$$

so that

$$\nabla_{[X,Y]}Z = D_{[X,Y]}Z + \alpha(D_X Y, Z) - \alpha(D_Y X, Z) .$$

Using (11) we see that the horizontal component of R is

$$(R(X, Y)Z)^h = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z = \tilde{R}(X, Y)Z , \tag{12}$$

where $\tilde{R}$ is the curvature of the (first order) induced connection. (12) is analogous to the equation of Gauss. The vertical component of R is

$$\begin{aligned} (R(X, Y)Z)^v = \nabla_X\alpha(Y, Z) &+ \alpha(X, D_Y Z) - \alpha([X, Y], Z) \\ &- \nabla_Y\alpha(X, Z) - \alpha(Y, D_X Z) . \end{aligned}$$

Taking

$$\tilde{\nabla}_X\alpha(Y, Z) = \nabla_X\alpha(Y, Z) - \alpha(D_X Y, Z) - \alpha(Y, D_X Z)$$

we have in the case where D is torsion free

$$(R(X, Y)Z)^v = \tilde{\nabla}_X\alpha(Y, Z) - \tilde{\nabla}_Y\alpha(X, Z) , \tag{13}$$

which is formally the same as the equation of Codazzi. Finally, we have for $\xi \in \mathfrak{X}^v(M)$

$$\nabla'_X\nabla'_Y\xi = \nabla_X\nabla_Y\xi , \quad \nabla'_Y\nabla'_X\xi = \nabla_Y\nabla_X\xi , \quad \nabla'_{[X,Y]}\xi = \nabla_{[X,Y]}\xi ,$$

and hence

$$R(X, Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi = \bar{R}(X, Y)\xi . \tag{14}$$

We call $\bar{R}$ the "vertical curvature tensor" of M.

If $\langle\ ,\ \rangle$ is an $\mathfrak{F}'$-metric on TM, and ∇' is Riemannian with respect to $\langle\ ,\ \rangle$, then we define, for $A, B, C, D \in \mathfrak{X}'(TM)$,

$$R(A, B, C, D) = \langle A, R(C, D)B \rangle \ . \tag{15}$$

Theorem 10. *If A, $B \in \mathfrak{X}'(TM)$ and X, $Y \in \mathfrak{X}(M)$, then*

$$R(A, B, X, Y) = -R(A, B, Y, X) \ , \qquad R(A, B, X, Y) = -R(B, A, X, Y) \ .$$

Proof. The first of these follows from the skew-symmetry of $R(X, Y)B$, and the second from the fact that, since $\langle \ , \ \rangle$ is Riemannian,

$$XY\langle A, B\rangle = \langle \nabla'_X\nabla'_Y A, B\rangle + \langle \nabla'_Y A, \nabla'_X B\rangle + \langle \nabla'_X A, \nabla'_Y B\rangle + \langle A, \nabla'_X\nabla'_Y B\rangle \ ,$$
$$[X, Y]\langle A, B\rangle = \langle \nabla'_{[X,Y]}A, B\rangle + \langle A, \nabla'_{[X,Y]}B\rangle$$

on $TM_0 \approx M$. Hence

$$\begin{aligned} XY - YX - [X, Y] &= 0 \\ &= \langle R(X, Y)A, B\rangle + \langle A, R(X, Y)B\rangle \ . \end{aligned}$$

Let $G(A, B) = \langle A, A\rangle\langle B, B\rangle - \langle A, B\rangle^2$, and

$$K(A, B) = R(A, B, A^h, B^h)/G(A, B) \ . \tag{16}$$

Theorem 11. *If $p \in TM_0 \approx M$ and A, $B \in {}^2M_p$, then the scalar $K(A, B)$ depends only upon the hyperplane of ${}^2M_p$ spanned by A and B.*

Proof. We see that

$$K(A, B) = K(B, A) = K(rA, sB) = K(A + tB, B) \ .$$

Thus if $ad - cb \neq 0$, then

$$K(A, B) = K(aA + bB, cA + dB) \ .$$

Corollary. *If $\alpha \equiv 0$ and A, $B \in \mathfrak{X}'(TM)$ are horizontal with $A \mid TM_0 = X$, $B \mid TM_0 = Y$, then*

$$K(A, B) = \tilde{K}(XY) \ ,$$

where $\tilde{K}$ is the curvature of the induced (first order) connection on M.

If $X \in \mathfrak{X}(M)$, let $X^* \in \mathfrak{X}^v(M)$ denote the vertical vector field having the property that $D(X^*) = X$. Then to complement the first order or horizontal curvature $\tilde{K}$ we define the vertical curvature

$$\bar{K}(X, Y) = R(X^*, Y^*, X, Y)/G(X, Y) \ . \tag{17}$$

References

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