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# FORMAL LINEARIZATION OF VECTOR FIELDS AND RELATED COHOMOLOGY. I

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## 1. Introduction

In [5] it was shown that a semi-simple Lie algebra of vector fields could be linearized formally (i.e., in terms of formal power series) in the neighborhood of an invariant point. Guillemin and Steinberg showed [1] that, in the real analytic case, the linearization could in fact be carried out by a real analytic change of coordinates.

Now, there are three directions in which it would be desirable to generalize the argument. In the first, one would want to consider linearization of non-semisimple Lie algebra of vector fields, around an invariant point. In the second, one would want to discuss linearizations within a subgroup of the group of diffeomorphisms of the manifold. In the third, one would want to consider linearizations in the neighborhood of invariant *submanifolds* [6].

Unfortunately, the ingenious arguments of Guillemin and Steinberg are not well-adapted to these more general, but interesting, problems. This motivates us to investigate in more detail the original cohomology arguments given in [5], and see what light they can shed on these more general problems. In fact, we will show that the deeper reason for the appearance of the Lie algebra cohomology in this problem is the connection between Lie algebra cohomology theory and the theory of "deformations" of Lie algebra homomorphisms.

**Note added in January 1973.** The two parts of this paper were written over five years ago. Its publication has been held up by the first referee, who apparently felt that his comments (presented at the end of Part II) precluded its publication. A later referee has finally suggested that it be printed together with the comments by the first referee.

My original motivation for this work was certain ideas in elementary particle physics, particularly the problem of "kinematic singularities" in the scattering amplitude of a many-particle process. In order to not complicate the exposition, which was diffuse enough as it was, I did not refer to this point. However, it should be noted that the case where the group is either an infinite dimensional one (such as the "gauge" symmetry groups) or does not act transitively on the invariant submanifold is relevant for these potential physical applications.

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The referee's comments are not routinely applicable to these cases, and I feel that it is still worthwhile to present a general geometric formalism. In the meantime, the important work by Gelfand and Fuks (*Cohomology of the Lie algebra of tangential vector fields of a smooth manifold*, Functional Anal. Appl. **3** (1969) 194–210) has appeared and been developed by numerous geometers. Their ideas make it much more feasible now to contemplate the explicit calculuation of the cohomology groups defined in this paper which are obstructions to linearization. In certain cases, I believe that these cohomology groups would be interesting physical invariants.

## 2. Linearization theorems for filtered Lie algebras

Let $L$ be a Lie algebra. A *filtration* on $L$ is defined by a sequence $L^1, L^2, \cdots$ of subalgebras of $L$ such that:

$$L = L^1 \supset L^2 \supset \cdots , \tag{2.1}$$

$$[L^r, L^s] \subset L^{r+s-1} \qquad \text{for } r, s \geq 1 . \tag{2.2}$$

See [2] for a description of the general properties of filtered Lie algebras.

The following problem will be discussed in this section: Let $K$ be a given subalgebra of $L$. Can one find an $X \in L$ such that

$$\operatorname{Exp}(\operatorname{Ad} X)(K) \cap L^2 = (0)? \tag{2.3}$$

In fact, we will be considering a more restrictive problem here; we will attempt to exhibit $X$ *formally* (i.e., without discussion of the convergence) as a limit

$$\cdots \operatorname{Exp}(\operatorname{Ad} X_2) \operatorname{Exp}(\operatorname{Ad} X_1)$$

where $(X_r)$ is a sequence of elements of $L$, with each $X_r$ in $L^r$.

Now $L^2$ is an ideal in $L$. Suppose that the homomorphism $L \to V^1 = L/L^2$ splits, i.e., there is a subalgebra $H$ of $L$ such that

$$L = L^2 + H , \qquad H \cap L^2 = (0) .$$

(We will not consider the more general case in this paper.) Let $\phi_1$ be the projection map $L \to H$. For $r > 1$, define $V^r = L^r/L^{r-1}$, and let $\pi_r$ be the projection map: $L^r \to V^r$. Notice that $\phi_1$ is a homomorphism of $L$ into $H$. Notice also that $[K, L^r] \subset L^r$ for each $r \geq 1$, hence $\operatorname{Ad} K$ passes to the quotient to define a representation, denoted by $\phi_r$, of $K$ by linear transformations in $V^r$.

Let us begin the process of "linearizing" $K$. For $Y \in K$ define:

$$\omega_2(Y) = \pi_2(Y - \phi_1(Y)) .$$

Consider $\omega_2 \colon K \to V^2$ as a 1-cochain of $K$ with respect to the representation $\phi_2$

of $K$ in $V^2$. (For the notations of Lie algebra cohomology theory which we use, see [4].)

**Lemma 2.1.** $d\omega_2 = 0$, i.e., *$\omega_2$ is a 1-cocycle.*

*Proof.* For $Y_1, Y_2 X \in K$

$$\begin{aligned} d\omega_2(Y_1, Y_2) &= \phi_2(Y_1)(\omega_2(Y_2)) - \phi_2(Y_2)(\omega_2(Y_1)) - \omega_2([Y_1, Y_2]) \\ &= \phi_2(Y_1)(\pi_2(Y_2 - \phi_1(Y_2))) - \phi_2(Y_2)(\pi_2(Y_1 - \phi_1(Y_1))) \\ &\quad - \pi_2([Y_1, Y_2] - \phi_1([Y_1, Y_2])) \\ &= \pi_2([Y_1, Y_2 - \phi_1(Y_2)] - [Y_2, Y_1 - \phi_1(Y_1)] \\ &\qquad - [Y_1, Y_2] + [\phi_1(Y_1), \phi_1(Y_2)]) \\ &= \pi_2([Y_1, Y_2] - [Y_1, \phi_1(Y_2)] + [Y_2, \phi_1(Y_1)] + [\phi_1(Y_1), \phi_1(Y_2)]) \\ &= \pi_2([Y_1 - \phi_1(Y_1), Y_2 - \phi_2(Y_2)]) = 0 , \end{aligned}$$

since both $Y_1 - \phi_1(Y_1)$ and $Y_2 - \phi_2(Y_2)$ are in $L^2$, and $[L^2, L^2) \subset L^3$.

The cohomology class in $H^1(\phi_2)$ determined by $\omega_2$ is the *first obstruction* to linearizing $K$. Suppose it is zero, i.e., there is an element $X_2 \in L^2$ such that

$$d\,\pi_2(X_2) = \omega_2 , \quad \text{or} \quad \omega_2(Y) = [\phi_2(Y), \pi_2(X_2)] = \pi_2([Y, X_2]) \qquad \text{for } Y \in K .$$

Then

$$\begin{aligned} \operatorname{Exp}(\operatorname{Ad} X_2)(Y) &= Y + [X_2, Y] + \frac{1}{2!}[X_2, [X_2, Y]] \\ &\quad + \cdots + \omega_2 (\operatorname{Exp}(\operatorname{Ad} X_2)(Y)) \\ &= \pi_2 (\operatorname{Exp}(\operatorname{Ad} X_2)(Y) - \phi_1 (\operatorname{Exp}(\operatorname{Ad} X_2)(Y)) \\ &= \pi_2\left(Y + [X_2, Y] + \frac{1}{2!}[X_2, [X_2, Y]] + \cdots - \phi_1(Y)\right) \\ &= \pi_2(Y - \phi_1(Y)) + \pi_2([X_2, Y]) = 0 , \end{aligned}$$

i.e.,

$$\operatorname{Exp}(\operatorname{Ad} X_2)(Y) - \phi_1(Y) \in L^3 \qquad \text{for all } Y \in K . \tag{2.4}$$

Now replace $K$ by $K^2 = \operatorname{Exp}(\operatorname{Ad} X_2)(K)$. If (2.4) is satisfied, then we have

$$Y - \phi_1(Y) \in L^3 \qquad \text{for } Y \in K^2 . \tag{2.5}$$

Define $\omega_3(Y) = \pi_3(Y - \phi_1(Y))$.

A similar reasoning shows that $\omega_3$, when interpreted as a 1-cochain defined by the representation $\phi_3$, is a 1-cocyle. Its cohomology class is the second obstruction to linearizing $K$. If it is zero, there is an element $X_3 \in L^3$ so that

$$\omega_3(Y) = \pi_3([Y, X_3]) \qquad \text{for } Y \in K^2 .$$

Define $\boldsymbol{K}^3 = \operatorname{Exp}(\operatorname{Ad} X_3)(\boldsymbol{K}^2)$.

Notice that, since $X_3 \in \boldsymbol{L}^3$,

$$0 = \omega_2(Y) = \omega_2(\operatorname{Exp}(\operatorname{Ad} X_3)(Y)) \qquad \text{for } Y \in \boldsymbol{K}^2 .$$

A similar calculation then shows that

$$\omega_3(K^3) = 0 .$$

We can now continue the process, obtaining a sequence $\boldsymbol{K} = \boldsymbol{K}^1, \boldsymbol{K}^2, \boldsymbol{K}^3 \cdots$ of of subalgebras of $L$.

Thus we have proved:

**Theorem 2.1.** *If $H^1(\phi_r) = 0$ for $r = 2, 3, \cdots$, there is a sequence $\boldsymbol{K}^1 = \boldsymbol{K}, \boldsymbol{K}^2, \cdots$ of subalgebras of $L$. Each $\boldsymbol{K}^r$ is conjugate to $\boldsymbol{K}^{r-1}$ within the subgroup* Exp Ad $\boldsymbol{L}^r$ *of the group of inner automorphisms of* $\boldsymbol{L}$. *Also, for $Y \in \boldsymbol{K}^r$,*

$$Y - \phi_1(Y) \in \boldsymbol{L}^{r+1} .$$

Notice that we will have succeeded in "linearizing" $K$, i.e., showing that it is conjugate to a subalgebra of $H$, if

$$\boldsymbol{L}^r = 0 \qquad \text{for } r \text{ sufficiently large.} \tag{2.6}$$

Another hypothesis which will guarantee this linearization is that the "infinite product" $\cdots \operatorname{Exp}(\operatorname{Ad} X_3) \operatorname{Exp}(\operatorname{Ad} X_2)$ converges to an element of the group on inner automorphisms of $L$. However, there is another more general and interesting condition which may be satisfied. Suppose that the "limit" (as explained in [3, Chapter 11]) of the sequence of subalgebras $\boldsymbol{K}^1, \boldsymbol{K}^2, \cdots$ is a subalgebra $\boldsymbol{K}^\infty$. Then

$$Y - \phi_1(Y) = \lim_{r \to \infty} Y_r - \phi_1(Y_r) = 0 ,$$

i.e., the limit algebra $K^\infty$ is a subalgebra of $H$, and hence is "linearized". Now, as explained in [4], there is a close relation between this idea of limit of subalgebras, the Inonu-Wigner "contraction" idea, and the idea of "deformation" of subalgebra, as studied by Kodaira-Spencer, Gerstenhaber, and Nijenhuis-Richardson. Thus we may conjecture that (if the cohomology obstructions vanish) even if the subalgebra itself is not linearizable, one of its contractions is.

## 3. Construction of filtered Lie algebras

Let $\boldsymbol{G}$ be a Lie algebra, and $\boldsymbol{L}$ a subalgebra. We define subspaces $\boldsymbol{L}^r$, $r = 1, 2, \cdots$, of $\boldsymbol{L}$ with

$$\boldsymbol{L} = \boldsymbol{L}^1 \supset \boldsymbol{L}^2 \supset \boldsymbol{L}^3 \cdots \tag{3.1}$$

as follows:

$L^2$ consists of the elements $X \in L$ such that $[X, G] \subset L$.
$L^3$ consists of the elements $X \in L$ such that $[G, [G, X]] \subset L$.
In general, define $L^r$ by induction as the set of elements $X \in L^{r-1}$ such that

$$[G, X] \subset L^{r-1} .$$

**Lemma 3.1.** $[L^r, L^s] \subset L^{r+s-1}$, *i.e., the sequence* (3.1) *forms a filtered Lie algebra.*

*Proof.* Proceed by induction on the total degree $r + s$. Suppose $Y \in G$. Then

$$\begin{aligned}[Y, [L^r, L^s]] &\subset [Y, L^r], L^s] + [L^r, [Y, L^s]] \\ &\subset [L^{r-1}, L^s] + [L^r, L^{s-1}] \\ &\subset L^{r+s-2} , \qquad \text{by induction hypothesis.}\end{aligned}$$

This shows that $[G, [L^r, L^s]] \subset L^{r+s-2}$, which shows that $[L^r, L^s] \subset L^{r+s-1}$. Let $G$ and $L$ be connected Lie groups whose Lie algebras are $G$ and $L$.

**Lemma 3.2.** *Suppose that $L$ has no nonzero ideals which are also ideals in $G$. (Geometrically, this means that $G$ acts almost effectively on $G/L$, i.e., the set of elements $g \in G$ which acts as the identity on $G/L$ as discrete.) If $L^{r-1} \neq 0$, then $L^{r-1} \neq L^r$.*

*Proof.* If $L^{r-1} = L^r$, then $[G, L^{r-1}] \subset L^{r-1}$, i.e., $L^{r-1}$ is an ideal of $G$.

Now let $M$ be the coset space $G/L$. The action of $G$ on $M$ defines, as usual in Lie group theory, an infinitesimal action of $G$, i.e., a homomorphism of $G$ into the Lie algebra (under Jacobi bracket) $V(M)$ of vector fields on $M$. Each element $X \in G$ then determines a vector field, i.e., an element of $V(M)$, which we also denote by $X$. Let $p_0$ be the identity coset. Then

$$X(p_0) = 0 \qquad \text{for } X \in L .$$

Let $V^r, r = 1, 2, \cdots$, be the set of elements $X \in V(M)$ whose coefficients all vanish to at least the $r$-th order at $p_0$.

**Lemma 3.3.** $L^r \subset V^r$, *for all $r$.*

*Proof.* Let $(x_1, \cdots, x_n) = x$ be a coordinate system for $M$ valid in a neighborhood of $p_0$ with $x(p_0) = 0$. Proceed by induction on $r$. Since $L^r \subset L^{r-1}$, we know that $L^r \subset V^{r-1}$.

Let $X \in L^r$. About $p_0$, it can be written in the form

$$X = A_1\partial/\partial x_1 + \cdots + A_n\partial/\partial x_n .$$

The coefficients $A_1, \cdots, A_n$ vanish to $(r - 1)$-st order at $x = 0$. Since $G$ acts transitively on $M$, the coordinate vector fields $\partial/\partial x_1, \cdots, \partial/\partial x_n$ can be, in a neighborhood of $p_0$, written in terms of vector fields of $G$, i.e.,

$$\partial/\partial x_1 = f_1X_1 + \cdots + f_mX_m , \qquad \text{with } X, \cdots, X_m \in G .$$

Now $[X_1, X], \cdots, [X_m, X] \in L^{r-1}$, since $X \in L^r$. Hence also

$$[\partial/\partial x_1, X] \in I^{t-1} .$$

But this equals also

$$\frac{\partial A_1}{\partial x_1}\frac{\partial}{\partial x_1} + \cdots + \frac{\partial A_n}{\partial x_1}\frac{\partial}{\partial x_1} ,$$

which must vanish to order $r - 1$ at $x = 0$. A similar statement is true for $\partial/\partial x_2, \cdots, \partial/\partial x_n$. This implies that the coefficients $A_1, \cdots, A_n$ vanish at least to order $r$ at $x = 0$, i.e., $X \in V^r$. q.e.d.

Conversely, if $G$ is a Lie algebra of vector fields on a manifold $M$, and $L$ is the isotropy subalgebra of $G$ at a point $p_0 \in M$, then $L^r = V^r \cap L$ defines a filtration of $L$, to which we can apply the conjugacy arguments of § 2, and deduce, from the abstract theorem of § 2, the results that under certain conditions Lie algebras of vector fields $K$ can be linearized by a change of coordinates (perhaps, if $G$ is infinite dimensional, requiring a formal power series definition, whose convergence is still unknown) about a common zero point for the elements of $K$. This returns us to the treatment given in [5].

As an example, suppose that $K$ is one-dimensional, generated by a single element $X$. The "cohomology groups" take a very simple form, of course: Suppose $\phi$ is a representation of $K$ on a vector space $V$. Let $\omega\colon K \to V$ be a 1-cochain. It is automatically a 1-cocycle, since $K$ is a one-dimensional. It cobounds if and only if there is a vector $v \in V$ such that

$$\omega(X) = \phi(X)(v) ,$$

i.e., the first cohomology group is zero if and only if $\phi(X)$ maps $V$ onto $V$, so that if $V$ is finite dimensional, $\phi(X)$ must be one-one.

For example, consider the case where $G$ is the Lie algebra $V(M)$ itself, and $L$ is the subalgebra of those vector fields which vanish at $p_0$. Suppose $X \in L$ is of the form

$$X = A_1\partial/\partial x_1 + \cdots + A_n\partial/\partial x_n$$

with $A_i(0) = 0$ for $i = 1, \cdots, n$.

Suppose that Taylor expansion of $A_i$ about $x = 0$ is of the form:

$$A_i(x) = \sum_{i,j} \lambda_{ij}x_i\partial/\partial x_j + \cdots .$$

It is readily verified that Ad $Z$ acting in $V^r/V^{r-1}$ is one-one if the matrix $(\lambda_{ij})$ is diagonalizable, and if its eigenvalues are nonzero. The problem of linearization of $X$ by a change of variable is, of course, a classical problem first considered by Poincaré, and brought to definitive form by S. Sternberg (see [8], and the references quoted there).

## 4. Filtrations defined by submanifolds

First, we will present an algebraic construction, then explain how it applies to a problem (but not the most general) of "linearizing" a Lie algebra of vector fields near an invariant submanifold.

Let $F$ be an algebra over the real numbers, whose elements we denote by $f, g$, etc. Let $V$ be the Lie algebra of derivations of $F$. Elements of $V$ will be denoted by $X, Y$, and the action of $X \in V$ and $f \in F$ by $X(f) \in F$. Let $F^1$ be a subalgebra of $F$, and $V^1$ the subalgebra of $V$ consisting of the elements $X \in V(F)$ such that

$$X(F^1) \subset F^1 .$$

Define $F^r$ as the subalgebra of polynomials of degree $\geq r$ in the elements of $F^1$. Then

$$F^r \cdot F^s \subset F^{r+s} .$$

Define $V^r = \{X \in V : X(F^1) \subset F^r\}$. Then

$$V^r(F^s) \subset F^{r+s-1} .$$

Now consider $X \in V^r, Y \in V^s, f \in F^t$,

$$[X, Y](f) = X(Y(f)) - Y(X(f)) .$$

Hence we have proved:

**Lemma 4.1.** $[V^r, V^s](F^t) \subset F^{r+s+t-1}$.

**Lemma 4.2.** $[V^r, V^s] \subset V^{r+s-1}$.

Thus $V^1 \supset V^2 \supset \cdots$ forms a filtered Lie algebra to which we can apply the general procedure given in § 2.

The geometric situation which we have in mind can be described as follows: Let $M$ be a manifold, and $F(=F(M))$ the algebra of $C^\infty$ real-valued functions. Then $V(=V(M))$ is the Lie algebra of vector fields on $M$. Suppose $F^1$ is a subalgebra of $F$, and $N$ is a submanifold of $M$ defined as the set of points of $M$ where all the functions of $F^1$ vanish. Then $V^r$ consists of vector fields which are tangent to $N$ to the $r$-th order, but does not contain all such vector fields (unless $N$ reduces to a single point). To see what is involved in this point, suppose that $M = R^2$, the Euclidean plane, with $x, y$ the Euclidean coordinate functions. Suppose that $F^1$ is the subalgebra of $F$ generated by $x$, so that $N$ is the plane $x = 0$. Suppose $f \in F^r$. Then $X(x) = a_r x^r + \cdots$, i.e.,

$$X = (a_r x^r + \cdots)\partial/\partial x + B\partial/\partial y ,$$

where $B$ is any function $C(x, y)$, and the coefficients $a_r, \cdots$ are real numbers. Of course, this is not the most general form of vector field which is tangent to $N$ to the $r$-th order, since it omits those of the type:

$$X = (a_r(y)x^r + \cdots)\partial/\partial x + B\partial/\partial y ,$$

but these remarks do give us one type of linearization theorem. For example, if we write

$$X = x^r a(x, y)\partial/\partial x + B\partial/\partial y ,$$

with $a(0, y) \neq 0$, then we can define

$$X' = X/a = x^r\partial/\partial x + B'\partial/\partial y .$$

The integral curves of $X$ and $X'$ only differ by a change in parameterization, and we can apply the general theory. A similar remark applies to a single vector field which is tangent to a hypersurface in a general manifold $M$.

## 5. Contraction and deformation of Lie algebra homomorphisms

We temporarily leave the problem of linearizing a Lie algebra of vector fields near an invariant submanifold in order to treat a more abstract problem which will be shown later to be relevant.

Suppose $K$ and $L$ are Lie algebras, and $\phi, \phi'$ are homomorphisms: $K \to L$. $\phi$ and $\phi'$ are said to be *related by a deformation* if there is a one-parameter family $\lambda \to \phi_\lambda$ of homomorphisms: $K \to L$ such that

(a) $\phi_0 = \phi$ , $\phi_1 = \phi'$ , (b) $\phi_\lambda$ depends analytically on $\lambda$.

Then we can form the Taylor expansion:

$$\phi_\lambda(x) = \sum_{j=0}^{\infty} \theta_j(x)\lambda^j , \qquad \text{for } X \in K .$$

Let $\alpha$ be the following representation of $K$ by linear transformation on $L$:

$$\alpha(X)(Y) = [\phi(X), Y] , \qquad \text{for } X \in K, Y \in L .$$

Then the $\theta_j$'s are 1-cochains of $K$ with coefficients in $L$. The relation of the corresponding cohomology groups and the "triviality" of the deformation has been investigated in [7] and [4, IV].

Recall that the deformation $\lambda \to \phi_\lambda$ is said to be *trivial* if there is a one-parameter family $\lambda \to A_\lambda$ of automorphisms of $L$ such that

(5.1)
(a) $\phi_\lambda(X) = A_\lambda\phi(X)$ , for $X \in K$ and all $\lambda$,
(b) $A_\lambda$ depends smoothly on $\lambda$.

Now (modelling our terminology on that used by Inonu and Wigner in a similar case, the deformation theory of Lie algebra structures) let us say that $\phi$ is a *contraction* of $\phi'$ if

(a) there is a one-parameter family of homomorphisms $\phi_\lambda\colon \boldsymbol{K} \to \boldsymbol{L}$ such that $\phi_0 = \phi, \phi_1 = \phi'$,

(b) $\lambda \to \phi_\lambda$ depends continuously on $\lambda$ for $0 \leq \lambda \leq 1$,

(c) $\phi_\lambda$ depends analytically on $\lambda$ for $\lambda \neq 0$.

Let us present a set-up which leads to such notions in a very natural way. Suppose in addition that $F$ is a vector space, and $\boldsymbol{L}$ is a Lie algebra of linear transformation on $F$, with the bracket in $\boldsymbol{L}$ given by commutator of liner transformations. Thus $\phi'$ are representations of $\boldsymbol{K}$ by linear transformations on $F$. Suppose that $\lambda \to B_\lambda$ is a one-parameter family of linear transformations: $F \to F$ such that

(a) $B_\lambda$ depends analytically on $\lambda$ for all $\lambda$,

(b) $B_\lambda^{-1}$ exists only for $\lambda \neq 0$,

(c) $A_\lambda(Y) = B_\lambda Y B_\lambda^{-1}$ for $Y \in \boldsymbol{L}$.

In this case, (5.1) takes the form

$$\phi_\lambda(X) = B_\lambda \phi(X) B_\lambda^{-1} \qquad \text{for } X \in \boldsymbol{K} . \tag{5.2}$$

Thus we have the possibility of the singularity in $B_\lambda^{-1}$ at $\lambda = 0$ generating non-trivial deformations between $\phi$ and $\phi'$.

Before proceeding further with the general algebraic theory, let us turn to the geometric situation which motivates our work, the linearization problem for Lie algebras of vector fields near invariant submanifolds.

## 6. The linearization problem near an invariant submanifold

Suppose $M$ is a manifold, $F = F(M)$ is the ring of $C^\infty$ real-valued functions on $M$, and $V(M)$ is the derivations of $F(M)$, i.e., the Lie algebra of vector fields on $M$. Let $N$ be a submanifold of $M$, and $\boldsymbol{K}$ a subalgebra of $V(M)$ which is tangent to $N$. Since we will only be working locally for the moment, suppose $(x_i)$, $1 \leq i, j, \cdots \leq m = \dim M$, is a coordinate system for $M$ such that

$$x_u = 0 , \qquad n + 1 \leq u, v, \cdots \leq m ; \dim N = n$$

defines $N$. (Adopt the summation convention.) Suppose that $\lambda \to B_\lambda$ is the following one-parameter family of linear transformations:

$$F(M) \to F(M) ,$$

$$B_\lambda(f)(x_1, \cdots, x_m) = f(x_1, \cdots, x_n, \lambda x_{n+1}, \cdots, \lambda x_m) .$$

Let $X \in \boldsymbol{K}$. Suppose

$$X = f_i \partial / \partial x_i , \text{ i.e., } X(x_i) = f_i .$$

Suppose $\phi_\lambda$ is given by (5.2). Then

$$(6.1) \qquad \phi_\lambda(X)(x_i) = B_\lambda X B_\lambda^{-1}(x_i) = \begin{cases} \lambda^{-1}B_\lambda(f_i) & \text{for } i > n , \\ B_\lambda(f_i) & \text{for } 1 \leq i \leq n . \end{cases}$$

Suppose now that $X$ is tangent to $N$, i.e.,

$$f_u(x_1, \cdots, x_n, 0) = 0 .$$

Then $f_u$ admits a Taylor expansion of the form:

$$\begin{aligned} f_u(x_1, \cdots, x_m) &= f_{uv}(x_1, \cdots, x_n)x_v + f_{uvw}(x_1, \cdots, x_n)x_v x_w + \cdots , \\ \phi_\lambda(X)(x_u) &= f_{uv}(x_1, \cdots, x_n)x_v + \lambda f_{uvw}(x_1, \cdots, x_m)x_v x_w + \cdots , \\ \phi_\lambda(X)(x_a) &= f_a(x_1, \cdots, x_n, \lambda x_{n+1}, \cdots, \lambda x_m), 1 \leq a \leq n . \end{aligned}$$

We see that $\phi_\lambda$, considered as a homomorphism: $\boldsymbol{K} \to V(M)$, is perfectly analytic at $\lambda = 0$ despite the fact that the transformation $B_\lambda^{-1}$ used to define it has a pole at $\lambda = 0$. Further,

$$(6.2) \qquad \phi_0(X) = \frac{\partial f_u}{\partial x_v}(x_1, \cdots, x_n, 0)x_v \frac{\partial}{\partial x_u} + f_a(x_1, \cdots, x_n, 0)\frac{\partial}{\partial x_a} , \quad \text{for } X \in \boldsymbol{K} .$$

The subalgebra $\phi_0(\boldsymbol{K})$ is then the linearization of $L$. Linearization" of $\boldsymbol{K}$ itself is equivalent to proving triviality of the deformation in the neighborhood of $\lambda = 0$, a problem which is solved, in the formal sense at least, by the cohomology theory of [8] and [4, IV]. Now we turn to the task of freeing this argument from local coordinate systems, thus enabling one to apply to it situations in differential topology, partial differential equations and continuum mechanics. (In the last discipline, one will be interested in seeing how the argument goes for infinite dimensional manifolds.)

Let $N$ be a submanifold of $M$, $V(M, N)$ be the Lie algebra of vector fields on $M$ which are tangent to $N$, and $\boldsymbol{K}$ be a subalgebra of $V(M, N)$. Suppose that $\lambda \to \beta_\lambda$ is a one-parameter family of mappings of $M \to M$ such that

(a) $\beta_\lambda$ is a diffeomorphism for $\lambda > 0$, and depends smoothly on $\lambda$ for $\lambda \geq 0$.

(b) $\beta_\lambda(p) = p$ for $p \in N$.

(c) For each $p \in M$, the curve $\lambda \to \beta_\lambda(p)$ proceeds toward $N$ smoothly and transversally as $\lambda \to 0$. Precisely, $\beta_0(p)$ has a neighborhood with a coordinate system $(x_1, \cdots, x_m)$ having the properties described above.

Now we can define $B_\lambda$: $F(M) \to F(M)$ as follows:

$$\begin{aligned} B_\lambda(f) &= \beta_\lambda^*(f) \qquad \text{for } f \in F(M), \text{ i.e.,} \\ B(f)(p) &= f(\beta_\lambda(p)) \qquad \text{for } p \in M . \end{aligned}$$

Define $\phi_\lambda$: $\boldsymbol{K} \to V(M, N)$ as follows:

$$\phi_\lambda(X)(f) = B_\lambda X B_\lambda^{-1}(f) \qquad \text{for } f \in F(M), X \in K .$$

Then the local argument given above can be used to show that $\phi_\lambda(X)$ is well defined and smooth as $\lambda \to 0$. $\phi_0(X)$ can be considered as the "linearization" of the vector field $X$, relative to the homotopy $\lambda \to \beta_\lambda$ used to retract $M$ into $N$.

What does "linearization" of $K$ mean? Obviously, that the "deformation" $\varphi_\lambda$ of homomorphisms: $K \to V(M, N)$ is "trivial" in the neighborhood of $\lambda = 0$. Then the construction of cohomology obstructions to linearization follows from the general theory.

One might also pose a more restricted linearization problem which also has geometric interest. Suppose that $L$ is a given Lie algebra of vector fields on $N$. Denote by $V(M, N, L)$ the set of all vector fields $X$ on $M$ such that

a) $X$ is tangent to $N$, b) $X$ restricted to $N$ belongs to $L$.

Then $V(M, N, L)$ is a subalgebra of $V(M, N)$. Suppose $K \subset V(M, N)$ is given so that

$$K \subset V(M, N, L) .$$

Notice then that

$$\varphi_\lambda(K) \subset V(M, N, L) .$$

Thus we may say that to *linearize $K$ within $L$* is to prove the triviality of the deformation $\varphi_\lambda$ in the neighborhood of $\lambda = 0$, considering $\varphi_\lambda$ is a homomorphism: $K \to V(M, N, L)$.

## 7. Cohomology invariants for homomorphism deformations

Let us now return to the abstract point of view of § 5. Suppose that $K$ and $L$ are Lie algebras, and $\lambda \to \varphi_\lambda$ is a one-parameter family of homomorphisms: $K \to L$ which can be expanded in a formal power series in the parameter $\lambda$:

$$\varphi_\lambda(X) = \varphi_0(X) + \sum_{j=1}^{\infty} \theta_j(X)\lambda^j \qquad \text{for } X \in K . \tag{7.1}$$

Let $\varphi'$ be the following representation of $K$ by operators in $L$:

$$\varphi'(X)(Y) = [\varphi_0(X), Y] \qquad \text{for } Y \in L . \tag{7.2}$$

Then the $\theta_j$ are 1-cochain of $K$ determined by the representation $\varphi'$. The fact that each $\varphi_\lambda$ is a representation translates into the condition:

$$0 = d\theta_j + \sum_{k=1}^{j-1} \alpha(\theta_{j-k}, \theta_k) , \tag{7.3}$$

where $\alpha$ is the "cup product" on 1-cochain induced by the Lie algebras structure on $\boldsymbol{L}$ [4, IV]. In particular, $\theta_1$ is a 1-cocycle.

Let us now consider deformations of a special form, namely,

$$\varphi_\lambda(X) = A_\lambda\varphi_1(X) \qquad \text{for } X \in \boldsymbol{K}\,, \tag{7.4}$$

where $\lambda \to A_\lambda$ is a one-parameter family of automorphisms of $\boldsymbol{L}$ (for $\lambda \neq 0$), of the form

$$A_\lambda(X) = \exp\left((\log\lambda)\operatorname{Ad} Z\right)(X)\,, \tag{7.5}$$

where $Z$ is a fixed element of $\boldsymbol{L}$. (The geometric motivation for this assumption is explained in § 6.) Think of $\boldsymbol{L}$ as $V(M, N), \boldsymbol{K}$ as a subalgebra of $V(M, N), Z$ as a vector field on a tubular neighborhood which, in local coordinates, has the form

$$X_{n+1}\partial/\partial x_{n+1} + \cdots + X_m\partial/\partial x_m\,.$$

Let us now compare (7.5) with (7.1). For $X \in \boldsymbol{K}$,

$$\varphi_0(X) = \lim_{\lambda\to 0}\exp\left((\log\lambda)\operatorname{Ad} Z\right)(\varphi_1(X))\,,$$

$$\begin{aligned} \theta_1(X) &= \frac{d}{d\lambda}\varphi_\lambda(X)|_{\lambda=0} \\ &= \lim_{\lambda\to 0}\frac{1}{\lambda}[Z, \varphi_\lambda(X)] \lim_{\lambda\to 0}\frac{1}{\lambda}[Z, \varphi_0(X)] + \sum_{j=1}^{\infty}[Z, \theta_j(X)]\lambda^{j-1}\,. \end{aligned} \tag{7.6}$$

This gives the relations:

$$[Z, \varphi_0(\boldsymbol{K})] = 0\,, \tag{7.7}$$

$$\theta_1(X) = [Z, \theta_1(X)] \qquad \text{for all } X \in \boldsymbol{K}\,. \tag{7.8}$$

Set:

$$V^j = \{Y \in \boldsymbol{L}: [Z, Y] = (j-1)Y\} \qquad \text{for } j = 1, 2\cdots\,. \tag{7.9}$$

Then (7.7), (7.8), (7.10), (7.9) imply respectively:

$$\varphi_0(\boldsymbol{K}) \subset V^1\,, \tag{7.10}$$

$$\theta_1(\boldsymbol{K}) \subset V^2\,, \tag{7.11}$$

$$[\varphi_0(\boldsymbol{K}), V^j] \subset V^j\,, \qquad \varphi^k(\boldsymbol{K})(V^j) \subset V^j\,, \tag{7.12}$$

$$[V^j, V^k] \subset V^{j+k-1}\,. \tag{7.13}$$

Let $\varphi^j$ be the representation of $K$ obtained by restricting $\varphi'(k)$ to $V^{j+1}$, and $\omega_1$ be the map: $K \to V^2$ defined as follows:

$$\omega_1(X) = \theta_1(X) \qquad \text{for } X \in K . \tag{7.14}$$

Then $\omega_1$ is a 1-cocycle of $K$ with coefficients in the representation $\varphi^1$. Let us suppose that it cobounds, i.e., there is a $Y \in V^2$ such that

$$\theta_1(X) = [\varphi_0(X), Y] \qquad \text{for all } X \in K .$$

Then we can modify $\varphi_\lambda$ as follows:

$$\varphi_\lambda(X) \to \exp(\operatorname{Ad} \lambda Y)(\varphi_\lambda(X)) = \varphi'_\lambda(X) .$$

Thus following the pattern of [5] (and notice that this is merely an algebraic version of the argument of [5]), the new deformation has a Taylor's series of the form

$$\varphi'_\lambda(X) = \varphi_0(X) + \lambda^2\theta'_2(X) + \cdots ,$$

with $\theta'_2(K) \subset V^2$.

The argument can be repeated now—we see that the "obstructions" to the formal "linearization" of $\varphi(K)$, i.e., an equivalence under a formal inner automorphism of $L$, are the cohomology groups $H^1(K, \varphi^j)$, $j = 1, 2, \cdots$. We can state this as follows:

**Theorem 7.1.** *Suppose $H^1(K, \varphi^j) = 0$ for $j = 1, 2, \cdots$, and let L be the group of inner automorphisms of $L$. There is then a sequence $l_1, l_2, \cdots$ of elements of L with $l_j \in \exp(V^{j+1})$ such that formally, $\varphi_0(X) = \lim_{n\to\infty} \operatorname{Ad} l_n \cdots \operatorname{Ad} l_1(\varphi_1(X))$.*

If each $V^j$ is finite dimensional, and $K$ is semisimple, then Theorem 7.1 is an immediately useful result, since we know that the first cohomology groups automatically vanish. If $L = V(M, N)$, each $V^j$ is not finite dimensional, however. We will investigate the geometric meaning of the cohomology groups in the second part of this paper.

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