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CHERN CLASSES AND PROJECTIVE GEOMETRY

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1. Introduction

Classical projective geometry is rich in relations between the extrinsic invariants (e.g., order, double tangents, number of nodes, triple points, $\cdots$) associated with an algebraic map $f: M \to CP_N$ of a projective algebraic manifold M. It has also long been known that these extrinsic invariants may be sometimes used to define *birational* or *intrinsic* invariants of the manifold.

For example, the Plücker formulas for an algebraic plane curve may be interpreted as a definition of the 1st Chern class of an algebraic manifold of dimension 1, or the postulation formula as the arithmetic genus in terms of the projective characters of an algebraic surface.

The object of this note is:

(a) to obtain descriptions of the Chern classes of an algebraic manifold in terms of the extrinsic invariants associated with an algebraic map $f: M \to CP_N$ which is nonsingular of order 1, i.e., the derivative of f has maximal rank everywhere,

(b) to show how the Chern classes affect well-known geometric invariants, associated with an imbedding satisfying the above condition.

2. The bundle of tangent spaces of a variety

Let $f: M \to CP_N$ be an algebraic map which is nonsingular of order 1, and let $\dim M = n$.

Definition. The tangent projective space to f at $x \in M$ is the unique linear space P, of dimension n, of CP_N, which passes through $f(x)$ such that

$$\operatorname{Im}(Df(x)\cdot(T_xM)) = T_{f(x)}P$$

where T_xM is the tangent space of M at x, and $T_{f(x)}P$ is the tangent space of P at $f(x)$.

Given an algebraic map $f: M \to CP_N$, as above, we shall call f a *cusp-free algebraic variety*. The set of tangent spaces of cusp-free algebraic variety forms a fibre bundle (with fibre CP_n) over M, and, in fact, may be realised as an

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algebraic variety by the construction sketched below. (For details see Pohl [2], [3].)

Let $G_{n,N}$ be the Grassmann manifold of linear n-spaces of CP_N, and $E_{n,N} \xrightarrow{\pi} G_{n,N}$ the tautologous bundle over $G_{n,N}$, with fibre CP_n. Let $\tau\colon E_{n,N} \longrightarrow CP_N$ be the map which takes a point x in an n-linear space P of CP_N to x.

Now there is a map $f'\colon M \longrightarrow G_{n,N}$ called the *dual map*, associated with a cusp-free variety which associates to each $x \in M$, the tangent projective space at x.

Then the pull-back:

$$\begin{array}{ccc} P_f = (\bar{f}')^* E_{n,N} & \xrightarrow{\bar{f}'} & E_{n,N} \\ {\scriptstyle \pi_f}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow[f']{} & G_{n,N} \end{array}$$

yields a CP_n-bundle P_f, over M, which is obviously in (1–1) correspondence with the set of points of the set of tangent projective spaces of M.

P_f is called the *bundle of tangent-spaces* of f, and the map $\tau \circ \bar{f}'\colon P_f \rightarrow CP_N$ realises this manifold as an algebraic variety. P_f is a CP_n-bundle associated to a holomorphic vector bundle $E_f \rightarrow M$ of fibre dimension $n + 1$. Moreover, E_f is topologically, though not analytically, isomorphic to the bundle $f^*\hat{l}_N \oplus (TM \otimes f^*\hat{l}_N)$, where $\hat{l}_N \rightarrow CP_N$ is the tautologous line bundle. Also, $(\tau \circ \bar{f}')^*\hat{l}_N$ is the tautologous line bundle $\hat{l}_f$ over P_f. Let $l_N \rightarrow CP_N$ and $l_f \rightarrow P_f$ be the line bundles conjugate to $\hat{l}_N \rightarrow CP_N$ and $\hat{l}_f \rightarrow P_f$ respectively. Then by the Leray-Hirsch theorem, the map[1]

$$\varphi\colon H^*(M) \otimes H^*(CP_n) \rightarrow H^*(P_f)$$

defined by

$$\varphi(\alpha \otimes x^m) = \pi_f^*(\alpha) \cup \chi_f^m$$

is an isomorphism of $H^*(M)$-modules, where $x \in H^2(CP_n)$ is the Poincaré dual of a hyperplane, and χ_f is the 1st Chern class of l_f: $\chi_f \in H^2(P_f)$.

Moreover, in $H^{2n+2}(P_f)$ we have

$$\chi_f^{n+1} = -\sum_{j=1}^{n+1} c_j(E_f) \cdot \chi_f^{n+1-j} \,,$$

where c_j is the j-th Chern class.

[1] Here, as in what follows, we use rational coefficients for all cohomology groups.

3. Chern classes of varieties

Let $E \to X$ be a complex vector bundle of dimension r, and let us factorise formally the polynomial

$$1 + c_1 t + \cdots + c_r t^r = (1 + \gamma_1 t) \cdots (1 + \gamma_r t^r) ,$$

where $c_n = c_n(E)$ is the n-th Chern class of E. Then the polynomial

$$\chi_q(\gamma) = \sum_{\substack{\alpha_1 + \cdots + \alpha_r = q \\ \alpha_i \geq 0}} \gamma^{\alpha_1} \cdots \gamma_r^{\alpha_r}$$

is clearly symmetric, and can thus be expressed in terms of the Chern classes. We denote this polynomial by $\mathfrak{P}_q(E)$. For example, $\mathfrak{P}_1(E) = c_1$, $\mathfrak{P}_2(E) = c_1^2 - c_2$, $\mathfrak{P}_3(E) = c_1^3 - 2c_1 c_2 + c_3$.

Theorem 1. *Let $f\colon M \to CP_N$ be a cusp-free algebraic variety, $N \geq 2n$, Δ_j be the set of points of M such that the tangent projective n-space at $x \in \Delta_j$ meets a generically situated $CP_{N-j} \subset CP_N$ (Δ_j is a subvariety of M), and δ_j be the Poincaré dual of Δ_j. Then for $n \leq j \leq 2n$,*

$$\delta_j = \mathfrak{P}_{j-n}(E_f) .$$

Proof. $\Delta_j = \pi_j(\tau \cdot \bar{f})^{-1}(P_{N-j}))$, P_{N-j} being generic. Now the genericity assumption implies that $\tau \cdot \bar{f}'\colon P_f \to CP_N$ is transversal to P_{N-j}. Hence the fundamental class of $(\tau \cdot f')^{-1}(P_{N-j})$ is given by

$$\mathfrak{D}_{P_f}(\tau \cdot f')^*(x^j) ,$$

where $\mathfrak{D}_{P_f}\colon H^*(P_f) \to H_*(P_f)$ is the Poincaré duality isomorphism. Hence

$$\delta_j = (\mathfrak{D}_M)^{-1} \cdot (\pi_f)_* \cdot (\mathfrak{D}_{P_f}) \cdot (\tau \cdot \bar{f}')^*(x^j) = (\pi_f)_!(\tau \cdot \bar{f}')^*(x^j) = (\pi_f)_!(\chi_f^j) ,$$

where $(\pi_f)_!\colon H^*(P_f M) \to H^*(M)$ is the "*umkehrungshomomorphismus*" or more simply "integration over the fibre". Hence δ_j is the coefficient of χ_f^n in χ_f^j written as a polynomial in $c_j(E_f)$ and χ_f^m, where the degree of the χ_f-terms are $\leq n$.

The result follows from an easy calculation. q.e.d.

The Chern classes of E_f may be computed by well-known formulas (see Hirzebruch [1, p. 64] for example) in terms of the Chern classes of M, and $\xi = f^*(x)$ where ξ is the class dual to a generic hyperplane section of f. We may then use these to compute the δ_j's successively from $j = n + 1, \cdots, 2n$. These expressions for δ_j can now be inverted to obtain expressions for $c_j(M)$, the Chern classes of M, in terms of ξ and δ_j. The actual computations are complicated; so we give some examples:

Theorem 2A. *Let $f\colon M \to CP_N, N \geq 4$, be a cusp-free algebraic surface* (i.e., $\dim_C M = 2$). *Then*

$$c_1(M) = 3\xi - \delta_3 , \qquad c_2(M) = 3\xi^2 - 2\xi\cdot\delta_3 + \delta_3^2 - \delta_4 ,$$

where ξ, δ_i *are as above.*

Theorem 2B. *Let* $f: M \to CP_N$ $(N \geq 6)$ *be a cusp-free variety* $\dim M = 3$. *Then*

$$\begin{aligned} c_1(M) &= 4\xi - \delta_4 , \\ c_2(M) &= 6\xi^2 - 3\xi\cdot\delta_2 + (\delta_4^2 - \delta_5) , \\ c_3(M) &= 4\xi^3 - 3\xi^2\cdot\delta_4 + 2\xi(\delta_4^2 - \delta_5) - 2\delta_4\cdot\delta_5 + \delta_4^3 + \delta_6 . \end{aligned}$$

Remarks. Our method is of course not the first attempt at a geometric formulation of the Chern classes. Indeed Chern classes were first introduced, in the Chow ring of a nonsingular variety by Todd and Eger using the so-called "canonical systems".

Our result is certainly more "geometric"-the classes δ_j are far more intuitive then Todd's canonical classes, which are pullbacks of Schubert cycles under the dual map. Also our method yields results in arbitrary codimensions, provided the singularities of the variety are generic, i.e., those which would arise when a nonsingular variety of dimension n in CP_{2n+k} is projected to a CP_m $(n \leq m)$. In such a case the classes δ_j may be reinterpreted in terms of other geometric invariants. For example, suppose a surface S in CP_3 arises from a generic projection of $F: M \to CP_5$ onto CP_3, F being a cusp-free variety. Then δ_4 clearly does not make sense for S. However, $\delta_4(F)$ is the Poincaré dual of the set of "pinch-points" in the double-curve of S. (See, Semple and Roth [4, p. 202]).

4. Chern classes and extrinsic invariants

In this section we show how the Chern classes of an algebraic manifold M affect the extrinsic invariants of a cusp-free variety $f: M \to CP_N$. We discuss only algebraic surfaces, where the computations are still reasonable and yet there are a great wealth of results.

Let $f: M \to CP_n$ be an algebraic surface; $n \geq 5$.

Definition. The projective characters of f are defined as follows:

The *order* μ_0 is the number of points at which a generic CP_{n-2} meets f. If Γ is the curve cut out on M by a generic hyperplane section, the *rank* μ_1 is the number of tangents of Γ which meet a generic CP_{n-2}. The *class* μ_2 is the number of hyperplanes belonging to a generic pencil which are tangent to f. The *ceto* or *type* ν_2 is the number of tangent planes which meet a generic CP_{n-4}.

Theorem 3. *Let* $f: M \to CP_n, n \geq 5$, *be a cusp-free surface,* $\xi \in H^2(M)$ *be the class dual to a hyperplane section, and* c_1, c_2 *be the* 1st *and* 2nd *Chern classes of* M. *Then*

$$\mu_0 = \langle \xi^2, [M] \rangle ,$$
$$\mu_1 = \langle 3\xi^2 - \xi \cdot c_1, [M] \rangle ,$$
$$\mu_2 = \langle 3\xi^2 - 2\xi \cdot c_1 + c_2, [M] \rangle ,$$
$$\nu_2 = \langle 6\xi^2 - 4\xi \cdot c_1 + c_1^2 - c_2, [M] \rangle ,$$

where $[M]$ is the fundamental class, and $\langle \; , \; \rangle$ the Kronecker pairing.

Proof. The formula for μ_0 is trivial, and that of ν_2 obvious, since $\nu_2 = \langle \delta_4, [M] \rangle$. To obtain the formulas for μ_1, μ_2 we use the following result, which is very easy to prove (see Semple and Roth [4, p. 194] for example):

Let P, P' be two 3-codimensional linear spaces generically situated with respect to f, and Δ_3, Δ_3' be the algebraic varieties (defined in §3) which arise from P and P'. Then these are algebraic curves, μ_1 is the number of points of intersection of Δ_3 with a generic hyperplane, and $(\mu_1 + \nu_2)$ is the number of points of intersection of Δ_3 and Δ_3'. Thus

$$\begin{aligned} \mu_1 &= \langle x, f_*(\mathfrak{D}_M \cdot \delta_3) \rangle = \langle \xi, \mathfrak{D}_M \cdot \delta_3 \rangle = \langle \xi, \delta_3 \cap [M] \rangle \\ &= \langle \xi \cup (3\xi - c_1), [M] \rangle = \langle 3\xi^2 - \xi \cdot c_1, [M] \rangle , \\ \mu_2 &= \langle \delta_3 \cup \delta_3, [M] \rangle - \nu_2 \\ &= \langle 9\xi^2 - 6\xi c_1 + c_1^2, [M] \rangle - \langle 6\xi^2 - 4\xi + c_1^2 - c_2, [M] \rangle \\ &= \langle 3\xi^2 - 2\xi \cdot c_1 + c_2, [M] \rangle . \quad \text{q.e.d.} \end{aligned}$$

Using these and the Cayley-Zeuthen relations, we can obtain the geometrical invariants of a generic surface in CP_3. (See [4].) Let $f \colon M \to CP_3$ be a generic surface, i.e., a cusp-free variety with Γ as double curve on which there are t triple-points which are also triple points of Γ, and with ν points on Γ at which the two tangent planes coincide.

Let ε_0 be the order of Γ, i.e., the number of intersections with a generic plane, and ε_1 the class of ε, i.e., the number of tangents with meet a generic line. Then

$$\begin{aligned} \varepsilon_0 &= \{\langle \xi^2, [M] \rangle\}^2 - \langle 2\xi^2 - \xi \cdot c_1, [M] \rangle , \\ \varepsilon_1 &= \{\langle \xi^2, [M] \rangle\}^2 - \langle \xi^2, [M] \rangle \cdot \langle \xi \cdot c_1, [M] \rangle \\ &\quad - \left\langle 21\xi^2 - 13\xi \cdot c_1 + \frac{7}{2} c_1^2 - \frac{5}{2} c_2, [M] \right\rangle , \\ t &= \binom{\langle \xi^2, [M] \rangle}{3} - \frac{1}{2} \langle \xi^2, [M] \rangle \langle \xi \cdot c_1, [M] \rangle \\ &\quad + \left\langle 8\xi^2 - \frac{14}{3} \xi \cdot c_1 + \frac{4}{3} c_1^2 - \frac{2}{3} c_2, [M] \right\rangle , \end{aligned}$$

and the number d of bitangents passing through a fixed point is

$$d = 12\binom{\langle \xi^2, [M] \rangle}{4} - 4\varepsilon_0\binom{\langle \xi^2, [M] \rangle - 2}{2} - 2\varepsilon_1 - 12t + 2\varepsilon_0(\varepsilon_0 - 1) .$$

Remark. It is instructive to note how radically different (and more complicated) these formulas are compared to the case of immersions in Euclidean space. For example, the number of triple points of an immersion of a compact 4-manifold in E^6 is simply the topological index of the manifold-a topological invariant. This certainly is not so in projective geometry.

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