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A METHOD IN METRIC DIOPHANTINE APPROXIMATION

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1. Introduction

Various well-known theorems can be stated in the following form: A certain *series* of nonnegative functions converges (or diverges) almost everywhere if the series of its *integrals* converges (or diverges). Examples are the Denjoy-Lusin Theorem in trigonometric series, the Borel-Bernstein Theorem on continued fractions, and the Khintchine Theorem on metric Diophantine approximation. In [3] and [4] Cassels studied the last of the above theorems. In this paper we will abstract the method used there by Cassels and first show how it can be used to prove the Denjoy-Lusin Theorem and various generalizations of the Borel-Bernstein and the Khintchine Theorems.

It is fairly clear that the convergence half of the above theorems is nothing more than an application of the monotone convergence theorem. Our method therefore consists in obtaining conditions under which the monotone convergence theorem has a strong converse; i.e., one which implies divergence almost everywhere. One such converse is immediately supplied by the Kolmogorov three series theorem applied to uniformly bounded nonnegative functions. Unfortunately, this converse is not adequate for most applications, since it requires that the functions be independent random variables. However, we show (Theorem 2.5) that the condition of independence for a series $\sum_{n=1}^{\infty} f_n(x)$ can be replaced by the much weaker condition for $m \neq n$,

$$\int_X |f_n(x)f_m(x)|\,\mu(dx) \leq \int_X |f_n(x)|\,\mu(dx) \int_X |f_m(x)|\,\mu(dx) \;;$$

whenever $|f_n(x)| \leq M < \infty$ for all n and almost all x.

The idea of using this sort of a converse of the monotone convergence theorem is not really original with us. Similar ideas have been used not only by Cassels, but also by Kahane [20] and especially by Erdös and Rényi [19]. However, we believe that there is still much to be gained from using it systematically and in connection with various other techniques.

It is of some historical interest to recall that Borel's original proof of the

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Borel-Bernstein theorem was based on an illegitimate application of the Borel-Cantelli lemma, which is a special case of the Kolmogorov theorem. The gap in Borel's reasoning was pointed out by Lebesgue and Bernstein, see [2, p. 208] for references.

Many of our results are illustrated by our generalization of Khintchine's Theorem. Usually, the theorem is stated as follows:

Khintchine's Theorem. *Let $a_1, a_2, \cdots$ be a sequence of positive numbers such that*

$$1/2 \geq a_1 \geq a_2 \cdots . \tag{1.1}$$

Then for almost every $x, 0 \leq x \leq 1$, there are infinitely many (or only finitely many) integers $n > 0$ such that

$$|nx - m| \leq a_n \quad \text{or} \quad |x - m/n| \leq a_n/n \tag{1.2}$$

is satisfied for some integer m according as $\sum_{n=1}^{\infty} a_n = \infty$ (or $\sum_{n=1}^{\infty} a_n < \infty$).

To see that this theorem can be cast in the form discussed above, it suffices to let f_n be the characteristic function of the set of x's satisfying (1.2) for the integer n. This set consists of n intervals, each of length $2a_n/n$, centered at the points m/n (modulo 1) so $\int_0^1 f_n(x)dx = 2a_n$, hence (1.1) is equivalent to

$$1 \geq \int_0^1 f_1(x)dx \geq \int_0^1 f_2(x)dx \geq \cdots , \tag{1.1$'$}$$

and the assertion becomes "$\sum_{n=1}^{\infty} f_n(x)$ diverges (or converges) almost everywhere according as $\sum_{n=1}^{\infty} \int_0^1 f_n(x)dx$ diverges (or converges)".

Our first generalization consists in letting $f_n(x) = g_n(nx)$ where g_n is any periodic function of period one satisfying $g_n(x) = g_n(-x)$, and $g_n(x) \leq g_n(y)$ when $0 \leq y \leq x \leq 1/2$, so in particular, f_n need not be the characteristic function of a set at all. The conditions on the g_n's are rather natural if g_n is to measure the distance from nx to an integer.

A further generalization is obtained by considering nx as a polynomial $P(n, x)$ in n with coefficient x. The original assertion then is that for almost all choices of the coefficient, $\sum_{n=1}^{\infty} g_n(P(n, x))$ diverges (or converges) according as $\sum_{n=1}^{\infty} \int_0^1 g_n(x)dx$ diverges (or converges). A second generalization consists in showing that the above assertions are valid for $P(n, x) = \sum_{i \in S} n^i x_i$, where S is a finite set of nonnegative integers, $S \neq \{0\}$. Here, of course, the coefficient is interpreted as a vector with as many components as there are elements of S. If S has more than one element, then it turns out that (1.1)′ is not needed. The difference between our version of Khintchine's Theorem and the original one is therefore analogous to the difference between Weyl's polynomial version of the Kronecker-Weyl Theorem and the usually stated linear version.

The last generalization of the Khintchine Theorem described, as well as various other results proved here, depends on replacing S^1 by other manifolds and viewing the situation geometrically. We feel that it is appropriate that a paper honoring Chern and Spencer should apply geometrical methods to problems in Diophantine approximation.

Finally, we would like to acknowledge that contacts with various mathematicians, especially Cassels and P. Hartman have enabled us to improve on the original version of this paper.

2. Series on a probability space

Let (X, Ω, μ) be a probability space with X the space, Ω the Borel field of subsets and μ a probability measure on (X, Ω). Let $f_n, n = 1, 2, \cdots$, be a sequence of measurable functions defined on (X, Ω, μ). This section is concerned with the absolute convergence and divergence of the series $\sum_{n=1}^{\infty} f_n(x)$. We will denote the L^p norm by $\|f\|_p$. The following proposition is a version of the monotone convergence theorem.

Proposition 2.1. *If* $\sum_{n=1}^{\infty} \|f_n\|_1 < \infty$, *then* $\sum_{n=1}^{\infty} f_n(x)$ *converges absolutely except on a set measure zero.*

Consequently, our main concern will be with the case where

$$\|f_n\|_2 < \infty \quad \text{for all } n, \text{ but } \quad \sum_{n=1}^{\infty} \|f_n\|_1 = \infty .$$

Following Cassels [3], [4], we will make frequent use of the proposition below. (The Tchebycheff inequality could be used instead, but would lead to a slightly less precise result.)

Proposition 2.2 (*Paley-Zygmund*, *cf.* [3]). *Suppose that* $0 \leq b \leq a \leq 1$ *and* $0 < a\|f\|_2 \leq \|f\|_1$. *Let* $E = \{x \in X : |f(x)| > b\|f\|_2\}$. *Then* $\mu(E) \geq (a-b)^2$.

Proof. By the Schwarz inequality,

$$\left(\int_E |f|\,\mu(dx)\right)^2 \leq \mu(E)\|f\|_2^2 .$$

Also,

$$\int_E |f|\,\mu(dx) = \|f\|_1 - \int_{X-E} |f|\,\mu(dx) \geq (a-b)\|f\|_2 .$$

The assertion follows trivially from these inequalities.

Theorem 2.3. *Assume*

$$\|f_n\|_2 < \infty \quad \textit{for all } n \textit{ and } \quad \sum_{n=1}^{\infty} \|f_n\|_1 = \infty \ . \tag{2.1}$$

Let $S_N(x) = \sum_{n=1}^{N} |f_n(x)|$ *and define* C *by*

$$C = \limsup \|S_N\|_1/\|S_N\|_2 \quad as \quad N \to \infty \ . \tag{2.2}$$

Then $\sum_{n=1}^{\infty} |f_n(x)|$ *diverges on a set of measure at least* C^2.

Remark. The theorem is vacuous if $C = 0$. The Schwarz inequality implies that $C \leq 1$.

Proof. It is sufficient to show that for any $T > 0$ and $0 < \varepsilon < 2C$, $S_N(x) \geq T$ on a set of measure $(C - \varepsilon)^2$ if N is large enough. By (2.1) and the Schwarz inequality,

$$2T/\varepsilon \leq \|S_N\|_1 \leq \|S_N\|_2 \quad \text{for large} \quad N \ ,$$

and by (2.2) there are arbitrarily large N's such that $(C - \varepsilon/2)\|S_N\|_2 \leq \|S_N\|_1$. Taking $b = \varepsilon/2, a = C - \varepsilon/2$, and $f = S_N$ in Proposition 2.2 gives the desired result.

Theorem 2.4. *Assume* (2.1),

$$\limsup \|f_n f_m\|_1 \|f_n\|_1^{-1} \|f_m\|_1^{-1} = C^{-2} \quad as \quad (n, |n - m|) \to (\infty, \infty) \ , \tag{2.3}$$

and for every fixed $K > 0$,

$$\sum_{n=1}^{N} \sum_{m \in S} \|f_n f_m\|_1 = o(\|S_N\|_1^2) \ , \quad as \quad N \to \infty \ , \tag{2.4}$$

where here and below $S = S(n, N, K)$ *is the set where*

$$\text{Max}\,(0, n - K) \leq m \leq \text{Min}\,(N, n + K) \ .$$

Then $\sum_{n=1}^{\infty} |f_n(x)|$ *diverges on a set of measure at least* C^2.

Proof. It will be shown that $C \leq \limsup \|S_N\|_1/\|S_N\|_2$ as $N \to \infty$, and the conclusion will then follow from Theorem 2.3. For any positive ε let K be such that for $n > K$ and $|n - m| > K$,

$$\|f_n f_m\|_1 \leq (1 + \varepsilon)C^{-2} \|f_n\|_1 \|f_m\|_1 \ .$$

If $N > K$, then

$$\|S_N\|_2^2 = \sum_{n=1}^{N}\left(\sum_{m\in S}\|f_nf_m\|_1 + \sum_{K<|m-n|\le N-n}\|f_nf_m\|_1\right)$$
$$\le o((\|S_N\|_1)^2) + C^{-2}(1+\varepsilon)(\|S_N\|_1)^2\,,$$

hence $C \le \limsup \|S_N\|_1/\|S_N\|_2$ as $N \to \infty$.

Theorem 2.5. *Let $f_1, f_2, \cdots$ satisfy* (2.1), (2.3), *and $|f_n(x)| \le M < \infty$ for some constant M. Then $\sum_{n=1}^{\infty}|f_n(x)|$ diverges on a set of measure C^2.*

Proof. By Theorem 2.4, it is only necessary to verify (2.4). But the left side of (2.4) is no greater than $2KM\sum_{n=1}^{N}\|f_n\|_1$, which is $o((\|S_N\|_1)^2)$ by (2.1).

3. Some applications

(a) **The Denjoy-Lusin Theorem** (*cf.* [9, p. 83] *or* [17, p. 131]). *Suppose that an arbitrary trigonometric series of functions of period one is written in the form $S(x) = \sum_{n=0}^{\infty}\rho_n\cos 2\pi n(x-\alpha_n)$ for $\rho_n \ge 0$, $0 \le \alpha_n < 2\pi$. Then $\sum_{n=0}^{\infty}\rho_n|\cos 2\pi n(x-\alpha_n)|$ diverges* (*or converges*) *almost everywhere if $\sum_{n=0}^{\infty}\rho_n$ diverges* (*or converges*).

Proof. Noting that $\|\rho_n\cos 2\pi n(x-\alpha_n)\|_1 = (2/\pi)\rho_n$ the convergence case is an immediate application of Proposition 2.1. Hence we may assume that $\sum_{n=0}^{\infty}\rho_n = \infty$. There is no loss of generality in assuming that $\rho_n \le M$.

Let $f_n(x) = \rho_n|\cos 2\pi n(x-\alpha_n)|^2$, or by the half angle formula $f_n(x) = \frac{1}{2}\rho_n\{\cos 4\pi n(x-\alpha_n)+1\}$. Then for $n \ne m$, by orthogonality, $\|f_nf_m\|_1 = \|f_n\|_1\|f_m\|_1 = \frac{1}{4}\rho_n\rho_m$, so Theorem 2.5 implies that $\sum_{n=0}^{\infty}f_n(x) = \infty$ almost everywhere. Clearly, $f_n(x) \le \rho_n|\cos 2\pi n(x-\alpha_n)|$, so $\sum_{n=0}^{\infty}\rho_n|\cos 2\pi n(x-\alpha_n)|$ diverges almost everywhere.

(b) **Functions on T^d.** Let $T^d = R^d/Z^d$ denote the d-dimensional torus, and $F_1, F_2, \cdots$ be a sequence of uniformly bounded measurable functions defined on T^d and satisfying $\sum_{n=1}^{\infty}\|F_n\|_1 = \infty$, where $\|\ \ \|_1$ is taken relative to the measure on T^d induced by Lebesgue measure on R^d. Let $A_1, A_2, \cdots, B_1, B_2, \cdots$ be sequences of elements of $GL(d, Z)$ (or equivalently of endomorphisms of T^d). Consider the functions f_n on T^{2d} defined by $f_n(x, y) = F_n(A_nx + B_ny)$. One easily verifies (cf. appendix) that $\|f_n\|_1 = \|F_n\|_1$ hence $\sum_{n=1}^{\infty}\|f_n\|_1 = \infty$. Now suppose that

$$J_{mn} = \det\begin{pmatrix}A_m, & B_m\\ A_n, & B_n\end{pmatrix} \ne 0 \tag{3.1}$$

is satisfied foı $m \neq n$. Then $\|f_n f_m\|_1 = \|f_n\|_1 \|f_m\|_1$ for $m \neq n$. In fact the change of variables $(u, v) = (A_m x + B_m y, A_m x + B_m y)$ is nondegenerate so

$$\|f_n f_m\|_1 = \int_{T^{2d}} |F_n(A_n x + B_n y) F_m(A_m x + B_m y)| dxdy$$
$$= \int_{T^{2d}} |F_n(u) F_m(v)| dudv = \|f_n\|_1 \|f_m\|_1 .$$

Now Theorem 2.5 shows that

$$(3.2) \qquad \sum_{n=1}^{\infty} |f_n(x, y)| = \infty \qquad \text{almost everywhere.}$$

We have therefore proved:

Theorem 3.1. *Let $f_n(x, y) = F_n(A_n x + B_n y)$, where $F_1, F_2, \cdots$ is a sequence of uniformly bounded function on the torus $T^d = R^d/Z^d$, and $A_1, A_2, \cdots, B_1, B_2, \cdots$ are sequences of integer $d \times d$-matrices satisfying* (3.1). *If $\sum_{n=1}^{\infty} \|F_n\|_1 = \sum_{n=1}^{\infty} \|f_n\|_1 = \infty$, then $\sum |f_n(x, y)| = \infty$ almost everywhere.*

Remark. Special cases of this theorem are Theorem 2 of [3, p. 124] and Theorem I of [4].

(c) **Approximation by polynomials.** The theorem stated in this section has an obvious generalization to the case of "simultaneous approximation", which we omit in order to keep the notation simple. Let $g(n, p)$ be an integer valued function defined for $n = 1, 2, \cdots$ and $p = 1, 2, \cdots, d$ where $d \geq 2$. It will be required that for every m, n, $m \neq n$, there exist i, j depending on m, n such that

$$(3.3) \qquad g(n, i) g(m, j) \neq g(n, j) g(m, i) .$$

Many simple and natural choices of $g(n, p)$ satisfy (3.3), for instance, $g(n, p) = n^p, p^n$, or $(n + p)!$.

Theorem 3.2. *Let $g(n, p)$ be as described above, and let $F_n : R^1 \to R^1$ ($n = 1, 2, \cdots$) be a sequence of uniformly bounded measurable function satisfying $F_n(x + 1) = F_n(x)$. Suppose that $\sum_{n=1}^{\infty} \int_0^1 |F_n(x)| dx = \infty$. Define $f_n(x) = F_n(P(n, x))$ where $P(n, x) = \sum_{p=1}^{d} g(n, p) x_p$ and $x = (x_1, \cdots, x_d)$ is viewed as a point on T^d, so $f_n : T^d \to R^1$. Then $\sum_{n=1}^{\infty} |f_n(x)| = \infty$ almost everywhere on T^d.*

Proof. By Theorem 2.5 it suffices to verify that $\|f_n f_m\|_1 \leq \|f_n\|_1 \|f_m\|_1$ for every pair $m, n, m \neq n$. It can be assumed without loss of generality that (3.3) holds for $i = 1$ and $j = 2$. Define an endomorphism E_{mn} of T^d by $E_{mn}(x) = (u_1(x), \cdots, u_d(x))$ where $u_1(x) = P(n, x)$, $u_2(x) = P(m, x)$, $u_p(x) = x_p$ for $p > 2$.

The Jacobian of the endomorphism is

$$J_{mn} = g(n, 1)g(m, 2) - g(n, 2)g(m, 1) \neq 0 \ ,$$

by (3.3). Therefore

$$\int_{T^d} |f_n(x)f_m(x)|\,dx = \int_{T^d} |F_n(u_1)F_m(u_2)|\,du \ , \tag{3.4}$$

as is shown in the appendix. However, the right side of (3.4) is just

$$\int_0^1 |F_n(u_1)|\,du_1 \int_0^1 |F_m(u_2)|\,du_2 = \int_{T^d} |f_n(x)|\,dx \int_{T^d} |f_m(x)|\,dx \ ,$$

hence $\|f_n f_m\|_1 = \|f_n\|_1 \|f_m\|_1$ and $\sum_{n=1}^{\infty} |f_n(x)| = \infty$ almost everywhere.

(d) **The Borel-Bernstein Theorem.** Let x be an irrational number, $0 < x < 1$, and let $[0, a_1, a_2, \cdots]$ (cf. [8, Chapter X]) be its expression as a continued fraction. Suppose that $\alpha_1, \alpha_2, \cdots$ is a sequence such that α_n^{-1} is a positive integer. The Borel-Bernstein Theorem asserts that for almost all x in $(0, 1)$, $\alpha_n a_n \geq 1$ for infinitely many n if $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\alpha_n a_n \geq 1$ for only finitely many n if $\sum_{n=1}^{\infty} \alpha_n < \infty$. This theorem will be derived from Theorem 2.5.

Let h_n denote the characteristic function of the interval $[0, \alpha_n]$ and let $T(x)$ = fractional part of $1/x$ for $0 < x < 1$. For irrational x, define $f_n(x) = h_n(T^{n-1}(x))$ so that $f_n(x) = 1$ if $a_n \alpha_n \geq 1$ and $f_n(x) = 0$ otherwise, as one easily verifies. T leaves the measure $\mu(dx) = (\log 2)^{-1}(1 + x)^{-1}dx$ invariant, so $\|f_n\|_1 = \log(1 + \alpha_n)$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$ if and only if $\sum_{n=1}^{\infty} \|f_n\|_1 = \infty$. Then the Borel-Bernstein Theorem follows from Theorem 2.5 provided

$$\|f_n f_m\|_1 \leq \|f_n\|_1 \|f_m\|_1 (1 + Cq^{|n-m|}) \ , \tag{3.5}$$

where $C > 0$ and $0 < q < 1$.

The relation (3.5) can be deduced fairly easily from a known result (the Lévy-Kuzmin Theorem). This assertion will not be verified here, since a detailed discussion of a more general situation is contained in a forthcoming paper [21].

4. A generalization of Khintchine's theorem

Khintchine's Theorem [10], [11] on metric Diophantine approximation has been generalized in various directions [4], [5], [6], [12], [13], [14]. The generalization below (Theorem 4.1) contains some of the others as special cases.

Pick a basis $\{e_1, \cdots, e_d\}$ of R^d and denote a typical element by $x = (x^1, \cdots, x^d) = \sum_{i=1}^{d} x^i e_i$. A Lebesgue integrable function $f\colon R^d \to R^1$ is called a *W-function* if for every x in R^d and $i = 1, \cdots, d$,

$$f(x) = f(x - 2x^i e_i) \ . \tag{4.1}$$

(That is, f is symmetric in each coordinate hyperplane) and

$$f(x + te_i) \leq f(x - te_i) \quad \text{if} \quad tx^i \geq 0 \ . \tag{4.2}$$

When (4.1) holds, (4.2) simply means that $f(x)$ does not decrease as the hyperplane $x^i = 0$ is approached along a line orthogonal to it.

The properties (4.1) and (4.2) are natural ones if $f(x)$ is to represent a measure of how close x is to the set of coordinate hyperplanes. Later such functions will be used to measure the closeness of a d-tuple of numbers to the integer lattice in R^d.

First we shall develop some properties of W-functions.

Lemma 4.1 (*Wintner* [16, p. 30 *and* p, 32]). *If f and g are W-functions, so is their convolution $h(x) = \int_{R^d} f(y)g(x - y)dy$.*

Proof. Since f and g satisfy (4.1),

$$\begin{aligned} h(x) &= \int_{R^d} f(y - 2y^i e_i)g(x - y)dy = \int_{R^d} f(y)g(x + 2y^i e_i - y)dy \\ &= \int_{R^d} f(y)g(x - 2x^i e_i - y)dy = h(x - 2x^i e_i) \ , \end{aligned}$$

so h satisfies (4.1).

To show that h satisfies (4.2), it suffices to discuss the case where $x^i \geq 0$ and $t \geq 0$. Consider

$$\begin{aligned} \Delta_i h &= h(x - te_i) - h(x + te_i) \\ &= \int_{R^d} f(y)\{g(x - te_i - y) - g(x + te_i - y)\}dy \ . \end{aligned}$$

Now write the integral as the sum of integrals over the set where $y^i \geq x^i$ and the set where $y^i \leq x^i$. In these two sets make the substitutions $y = u + x$ and $y = u + x - 2u^i e_i$ respectively. The result after using the fact that g satisfies (4.1) is

$$\Delta_i h = \int_{u_i \geq 0} \{f(u + x) - f(u + x - 2u^i e_i)\}\{g(-u - te_i) - g(-u + te_i)\}du \ .$$

Since $u^i \geq 0, u^i + x^i \geq 0$, and $t > 0$, (4.2) for f and g implies that each term is nonpositive and $\Delta_i h \geq 0$.

Theorem 4.2. *Let $Q = \{x \in R^d : |x^i| \leq 1/2, i = 1, \cdots, d\}$ and suppose that $f_1, f_2, \cdots$ is a sequence of W-functions satisfying*:

$$0 \leq f_n(x) \leq D < \infty \qquad \text{for } x \text{ in } R^d \,, \tag{4.3}$$

$$f_n(x) = 0 \qquad \text{for } x \text{ outside } Q \,, \tag{4.4}$$

$$\|f_1\|_1 \geq \|f_2\|_1 \geq \cdots \,. \tag{4.5}$$

Let $F_1, F_2, \cdots$ be the sequence of functions such that $F_n(x) = f_n(x)$ for x in Q and $F_n(x + z) = F_n(x)$ for z in Z^d. If $\sum_{n=1}^{\infty} \|f_n\|_1 < \infty$ (or $= \infty$), then the series $\sum_{n=1}^{\infty} F_n(nx)$ converges (or diverges) almost everywhere.

Notation. F_n will be viewed as a function on $T^d = R^d/Z^d$ so that

$$\|f_n\|_1 = \int_{R^d} |f_n(x)|dx = \int_{T^d} |F_n(x)|dx = \|F_n\|_1 \,.$$

Proof. As usual, the convergence case is just Proposition 2.1, so it can be assumed that

$$\sum_{n=1}^{\infty} \|f_n\|_1 = \infty \,. \tag{4.6}$$

Our proof uses many of the ideas of Cassel's proofs [3], [4]. In particular,

Lemam 4.3. *Assume* (4.5). *Then for some $B > 0$ and independent of N, $\sum_{n=1}^{N} \|f_n\|_1 \leq B \sum_{n=1}^{N} (\varphi(n)/n)^d \|f_n\|_1$, where φ is the Euler function, i.e., $\varphi(n) =$ the number of the integers less than and prime to n.*

For a proof see [3, p. 127 and p. 131].

Now let $P(n)$ denote the set of d-tuples $k = (p_1, \cdots, p_d)$ of integers p_i with $0 < p_i < n$ and p_i relatively prime to n. Note that $P(n)$ has $(\varphi(n))^d$ elements. Let $g_n(x) = \sum_{p \in P(n)} f_n(nx - p)$ and note that

$$\|g_n\|_1 = (\varphi(n)/n)^d \|F_n\|_1 \,. \tag{4.7}$$

Now (4.6) and Lemma 4.3 imply that

$$\sum_{n=1}^{\infty} \|g_n\|_1 = \infty \,. \tag{4.8}$$

Next it will be shown that

$$\|g_n g_m\|_1 \leq \|F_n\|_1 \|F_m\|_1 \quad \text{for} \quad n \neq m \,. \tag{4.9}$$

In fact, if $R(n, m)$ denotes the set of pairs $(p; q) = (p_1, \cdots, p_d; q_1, \cdots, q_d)$ where $mp_i \neq nq_i, 0 < p_i < n, 0 < q_i < m$, then $R(m, n) \supseteq P(n) \times P(m)$ and

$$\|g_n g_m\|_1 = \sum_{P(n)} \sum_{P(m)} \int_{R^d} f_n(nx - p) f_m(mx - q) dx$$
$$\leq \sum_{R(m,n)} \int_{R^d} f_n(nx - p) f_m(mx - q) dx = \sum_{R(m,n)} h_{mn}(p/n - q/m) ,$$

where h_{mn} is convolution of $f_n(nx)$ with $f_m(mx)$. If r is the greatest common divisor of m and $n, p/n - q/m = rk/mn$ can occur for a given $k \in Z^d$ at most r^d times for distinct (p, q) in $R(m, n)$. Therefore,

$$\|g_n g_m\|_1 \leq r^d \sum_{\substack{k \in Z^d \\ k \neq 0}} h_{mn}(rk/mn) .$$

Now Lemma 4.1 and the fact that $f_n(nx)$ and $f_m(mx)$ are W-functions imply that

$$\|g_n g_m\|_1 \leq r^d \int_{R^d} h_{mn}(rx/mn) dx = \|F_n\|_1 \|F_m\|_1 ,$$

which proves (4.9).

Let $S_N(x) = \sum_{n=1}^{\infty} g_n(x)$. Then (4.9), Lemma 4.3, and (4.3) imply that

$$\|S_N^2\|_1 \leq \left(\sum_{n=1}^{N} \|F_n\|_1\right)^2 + \sum_{n=1}^{N} \|g_n^2\|_1 \leq B^2 \|S_N\|_1^2 + D\|S_N\|_1 ,$$

hence for any $\varepsilon > 0$, and large N, (4.6) implies

$$\|S_N^2\|_1 \leq (B^2 + \varepsilon)\|S_N\|_1^2 .$$

Applying Theorem 2.1 gives the result that $\sum_{n=1}^{\infty} g_n(x) = \infty$ for x on a set of positive measure. The definition of g_n shows that $g_n(x) \leq F_n(nx)$ so $\sum_{n=1}^{\infty} F_n(nx) = \infty$ on a set of positive measure.

Now let $1 \geq a_1 > a_2 > a_3 \cdots > 0$ be such that $\lim a_n = 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} a_n^d \|f_n\|_1 = \infty$. Define $f_n^*(x) = f_n(x/a_n)$ and let F_n^* be to f_n^* as F_n is to f_n. Since $\|f_n^*\|_1 = a_n^d \|f_n\|_1$, what has already been proved implies that $\sum_{n=1}^{\infty} F_n^*(nx) = \infty$ on a set S of positive measure.

Now it will be shown that if $x = Ny + q$ for some y in S, and q in Z^d and if N is large enough so that $a_n N \leq 1$, then

$$(4.10) \qquad F_n^*(ny) \leq F_n(nx) .$$

This will complete the proof since (4.10) clearly implies that $\sum_{n=1}^{\infty} F_n(nx) = \infty$ for $x = Ny$ and y in S, and almost all points x are of this form; cf. [3, p. 126].

To verify (4.10), first let p_n be an element of Z^d such that $|ny - p_n| \leq 1/2$. If $|ny - p_n| > a_n/2$, the definition of f_n^* together with (4.4) shows that $F_n^*(ny) = 0$. Clearly (4.10) holds in that case, so suppose that $|ny - p_n| \leq a_n/2$. Then

$$\begin{aligned} F_n^*(ny) &= f_n^*(ny - p_n) = f_n((ny - p_n)/a_n) \\ &= f_n((nx - nq - Np_n)/(Na_n)) \leq f_n(nx - nq - Np_n) \leq F_n(nx) , \end{aligned}$$

where the first inequality follows from (4.1), (4.2) and $Na_n \leq 1$. This proves (4.10) and completes the proof of Theorem 4.2.

The arguments of Cassels [4] show that Theorem 4.2 is also true if one considers the series $\sum_{n=1}^{\infty} F_n(b_n x)$ where $b_1, b_2, \cdots$ is a $\sum$-sequence in the terminology of [4]. In particular, the conclusion holds for the series $\sum_{n=1}^{\infty} F_n(n^p x)$ for a fixed positive integer p. Combining this extension of Theorem 4.2 with Theorem 3.2 gives the following extension of the Khintchine Theorem.

Theorem 4.3. *Let the functions $F_1, F_2, \cdots$ be as in Theorem 4.2, $S \neq \{0\}$ be a set of nonnegative integers with $d \geq 1$ elements, and $P(n, x) = \sum_{i \in S} n^i x_i$ be a polynomial in n whose coefficients are defined by an element of R^d. If $\sum_{n=1}^{\infty} \|f_n\|_1 < \infty$ (or $= \infty$), then $\sum_{n=1}^{\infty} F_n(P(n, x))$ converges (or diverges) for almost every x in R^d.*

5. Nilmanifolds

The theorem of Khintchine can be extended in another direction by replacing the reals and the integers, respectively, by a nilpotent group and a discrete subgroup. In this section we are going to give one example of such a generalization.

Our example concerns the 3-dimensional nilpotent Lie group N whose definition is as follows:

As a differentiable manifold, N is just R^3 itself. The group operation, in terms of the coordinates (x, y, z) in R^3, is given by

$$(5.1) \qquad (x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy') .$$

In particular, we have

$$(5.2) \qquad (x, y, z)^{-1} = (-x, -y, -z + xy) .$$

Let Γ denote the subset of N consisting of those points (x, y, z) in N with x, y, and z all in Z. It is easy to see that Γ is a subgroup of N and that N/Γ is compact. In fact, denoting by Q the unit cube $\{(x, y, z): 0 \leq x, y, z < 1\}$, we have that for every $n \in N$ there are unique elements $g \in Q$ and $\gamma \in \Gamma$ satisfying $n = g\gamma$. Lebesgue measure $d\mu = dxdydz$ is both the left and right Haar measure on N, and Lebesgue measure restricted to Q defines a measure ν on N/Γ, which is invariant under translation by elements of N.

One final piece of notation and we shall be ready to state our theorem. Notice that for each real number $t > 0$, the map $(x, y, z) \to (tx, ty, t^2z)$ is an *automorphism* of N. This automorphism is the analogue in the present context of the operation of scalar multiplication by t in R^d. For that reason we will use $t(x, y, z)$ to denote the point (tx, ty, t^2z). Some related notation we shall use is: for each $n \in Z^+$, we set $\Gamma/n = \{n^{-1}\gamma: \gamma \in \Gamma\}$, and for each real $t > 0$, we set $Q(t) = \{tg: g \in Q\}$. (Note that $Q(t)$ is a parallelipiped of measure t^4.)

Theorem 5.1. *Let $a_1, a_2, \cdots$ be a sequence of positive real numbers each of which is no larger than* 1.

(1) *If $\sum_{n=1}^{\infty} a_n^4 < \infty$, then we have that for almost every $g \in Q$ there are only finitely many $n \in Z^+$ for which*

$$n \cdot g \in Q(a_n)\Gamma \; . \tag{5.3}$$

(2) *If $\sum_{n=1}^{\infty} a_n^4 = \infty$, then we have that for almost all $g \in Q$ there are infinitely many $n \in Z^+$ for which* (5.3) *holds.*

Remark. Note that in contrast to the assumption (4.5) in Theorem 4.2, the sequence $a_1, a_2, \cdots$ is not assumed here to be monotone.

Proof. Let B_n denote the subset $\{g\Gamma: g \in N \text{ and } ng \in Q(a_n)\Gamma\}$ of N/Γ, and recall that ν is the measure on N/Γ defined by Lebesgue measure on N.

Lemma 5.2. *B_n is the disjoint union of the subsets $Q(a_n/n)\gamma\Gamma$, where γ traces $(\Gamma/n) \cap Q$. In particular, $\nu(B_n) = a_n^4$.*

Proof. $(\Gamma/n) \cap Q$ is a complete set of coset representatives for Γ in Γ/n. Furthermore, if γ and λ are in Γ/n, then $Q(a_n/n)\gamma\Gamma$ and $Q(a_n/n)\lambda\Gamma$ are disjoint if and only if $\gamma\Gamma \neq \lambda\Gamma$. The lemma follows easily. q.e.d.

Part (1) of Theorem 5.1 follows immediately from Lemma 5.2 by applying Proposition 2.1 to the characteristic functions of the sets B_n. As for part (2), we shall follow the argument of Gallagher [6].

Given $n \in Z^+$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \Gamma$, we shall use (n, γ) to denote the *g.c.d.* of n, γ_1, γ_2, and γ_3, and also use $\Gamma(n)$ to denote $\{\gamma \in \Gamma: (n, \gamma) = 1\}$.

Lemma 5.3. *Let $\xi(n)$ denote the cardinality of $\Gamma(n) \cap Q(n)$. Then there is a constant $c > 0$ such that for all $n \in Z^+, \xi(n) \geq cn^4$.*

Proof following Gallagher [6]. For this proof *only*, μ will denote the Mobius function on Z^+. If $n \in Z^+$ and $n \neq 1$, then $\sum_{d|n} \mu(d) = 0$. Hence

$$\xi(n) = \sum_{\gamma}{}' \sum_{d|(n,\gamma)} \mu(d) \; , \tag{5.4}$$

where the first summation is over all $\gamma \neq 0$ in $Q(n) \cap \Gamma$. Since a given d which divides n will occur n^4/d^3 times as a summand in (5.4), we have $\xi(n) = n^4 \sum_{d|n} \mu(d)/d^3 = n^4 \prod_{p|n} (1 - p^{-3})$, the product being taken over the primes that divide n. But the product $\prod (1 - p^{-3})$ over all primes converges, hence $\xi(n) \geq cn^4$. q.e.d.

Let $C_n = U\{Q(a_n/n)\lambda\Gamma\colon n\lambda \in \Gamma(n) \cap Q(n)\}$. In order to prove part (2) of the theorem, we need only show that if $\sum_n a_n^4 = \infty$, then almost every point of N/Γ is in infinitely many sets of C_n. Lemma 5.5 is our main tool; its proof requires the following lemma:

Lemma 5.4. *For each pair $\{a, b\}$ of positive real numbers, let $S(a, b) = \sum_{\gamma \neq 0 \in \Gamma} \mu(Q(a) \cap Q(b)\gamma)$. Then there is a constant $c > 0$, not depending on either a or b, such that $S(a, b) \leq ca^4b^4$.*

Proof. Since $S(a, b) = S(b, a)$, we may assume that $b \leq a$. If $b \leq a$ and $a \leq 1/2$, then $S(a, b) = 0$. We assume, henceforth, that $a \leq 1/2$ and $b \leq a$.

For any subset V of N, let $\zeta(V)$ denote the cardinality of $V \cap \Gamma$. We begin the proof by showing that there is a constant $c > 0$ such that for all $g \in N$ and all real $a \geq 1/2$,

$$\zeta(gQ(a)) \leq ca^4 . \tag{5.5}$$

Let $g = (g_1, g_2, g_3)$ be given, and suppose $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is in $gQ(a) \cap \Gamma$. The condition, then, on γ is that each $\gamma_i \in Z$ $(i = 1, 2, 3)$ and $g^{-1}\gamma \in Q(a)$, which means that

$$\begin{gathered} g_1 \leq \gamma_1 \leq a + g_1 , \\ g_2 \leq \gamma_2 \leq a + g_2 , \\ g_3 - \gamma_2 g_1 + g_1 g_2 \leq \gamma_3 \leq a^2 + g_3 - \gamma_2 g_1 + g_1 g_2 . \end{gathered} \tag{5.6}$$

Note that once γ_1 and γ_2 are chosen, there are $0(a^2)$ choices for γ_3 satisfying the third inequality in (5.6). This proves (5.5).

Now

$$\begin{aligned} S(a, b) &\leq \sum_{\gamma \in \Gamma} \mu(Q(a) \cap Q(b)\gamma) \\ &= \int_{Q(b)} \zeta(x^{-1}Q(a))\mu(x) \leq ca^4\mu(Q(b)) = ca^4b^4 , \end{aligned}$$

by virtue of (5.5).

Lemma 5.5. *There is a constant $K > 0$ such that if m and n are in Z^+ and $m \neq n$, then $\nu(C_m \cap C_n) \leq K^2\nu(C_m)\nu(C_n)$.*

Proof. Set $g = g.c.d.(m, n)$. Then

$$\begin{aligned} \nu(C_m \cap C_n) &= \sum \nu(\{Q(a_m/m)(\gamma/m)\Gamma\} \cap \{Q(a_n/n)(\eta/n)\Gamma\}) \\ &= (g/mn)^4 \sum \mu(Q(na_m/g) \cap \{Q(ma_n/g)g^{-1}[(m\eta)(n\gamma^{-1})]\}) , \end{aligned}$$

the sum being taken over $\gamma \in \Gamma(m) \cap Q(m)$ and $\eta \in \Gamma(n) \cap Q(n)$. Now $(m\eta)(n\gamma^{-1}) = g\kappa$ for some $\kappa \in \Gamma$, and a given $\kappa \in \Gamma$ will satisfy $(m\eta)(n\gamma^{-1})g\kappa$ for at most g^4 pairs $\eta \in \Gamma(n) \cap Q(n)$ and $\gamma \in \Gamma(m) \cap Q(m)$. (This can easily be seen by writing everything out in coordinates.) Furthermore, the conditions $(m, \gamma) = (n, \eta) = 1$ guarantee that $(m\eta)(n\gamma^{-1}) \neq 0$. It follows that

$$\begin{aligned}\nu(C_m \cap C_n) &\leq (g^2/mn)^4 \sum_{\kappa \neq 0 \in \Gamma} \mu(Q(na_m/g) \cap \{Q(ma_n/g)\kappa\}) \\ &= (g^2/mn)^4 S(na_m/g, ma_n/g) \\ &\leq ca_m^4 a_n^4 ,\end{aligned}$$

by the previous lemma. From Lemma 5.2 it follows that $\nu(C_n) = \xi(n)\nu(Q(a_n/n)\Gamma) = \xi(n)n^{-4}a_n^4$. On the other hand, $\xi(n)n^{-4}$ is bounded away from 0 by Lemma 5.3. Thus, the inequality (5.5) yields $\nu(C_m \cap C_n) \leq K^2\nu(C_m)\nu(C_n)$ for some $K > 0$. q.e.d.

We are now ready to prove part (2) of Theorem 5.1. Since $\xi(n)n^{-4}$ is bounded away from 0, by Lemma 5.3 we see that $\sum_n \nu(C_n) = \sum_n \xi(n)n^{-1}a_n = \infty$ if $\sum_n a_n = \infty$. It follows then from Lemma 5.5 and Theorem 2.5 that if $\sum_n a_n = \infty$, there is a subset S of N/Γ of positive measure such that for all $g \in S$ there are infinitely many $n \in Z^+$ for which $g \in C_n$. Arguing as in [6] we see that this implies that almost every $g \in N/\Gamma$ lies in infinitely many of the sets C_n. q.e.d.

The proof of Theorem 5.1 works for some non-abelian groups other than N. We shall close this section with the statement of one such general result. Our result concerns connected, simply connected nilpotent Lie groups. It is a theorem that every such Lie group G is, as a differentiable manifold, equal to R^d for some $d \in Z^+$, and furthermore we can choose the coordinates $(x_1, \cdots, x_d)$ so that the group operation in G takes the form $(x_1, \cdots, x_d)(y_1, \cdots, y_d) = (z_1, \cdots, z_d)$, where $z_1 = x_1 + y_1, z_2 = x_2 + y_2$, and for $i \geq 3$,

$$z_i = x_i + y_i + f_i(x_1, \cdots, x_{i-1}; y_1, \cdots, y_{i-1}) , \tag{5.8}$$

where f_i is a polynomial in $x_1, \cdots, x_{i-1}, y_1, \cdots, y_{i-1}$ with real coefficients and $f_i(x_1, \cdots, x_{i-1}; 0, \cdots, 0) = 0 = f_i(0, \cdots, 0; y_1, \cdots, y_{i-1})$.

The group G is said to be *rationally presented* if the polynomials f_i all have *integer coefficients*. When G is rationally presented, the subset $\Gamma = \{(x_1, \cdots, x_d)\colon x_i \in Z \text{ for all } i\}$ of G is a subgroup of G (as in readily verified by induction on d) with compact quotient G/Γ. Let $Q = \{(x_1, \cdots, x_d)\colon 0 \leq x_i < 1 \text{ for all } i\}$. Then every element of G can be written in precisely one way as a product $g\gamma$ with $g \in Q$ and $\gamma \in \Gamma$, when G is rationally presented.

The left and right Haar measure on G is Lebesgue measure.

Theorem 5.6. *Let G be a rationally presented nilpotent Lie group, the group operation being given by* (5.8). *Assume further that there are integers $\alpha_1, \cdots, \alpha_d > 0$ such that for all real numbers $t > 0$, the map $t^\alpha\colon G \to G$ given by*

$$t^\alpha(x_1, \cdots, x_d) = (t^{\alpha_1}x_1, \cdots, t^{\alpha_d}x_d)$$

is an automorphism of G. Finally, assume that $d \geq 2$. *Let* $a_1, a_2, \cdots$ *be a sequence of positive numbers each of which is no larger than* 1, *and let* $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

(1) *If* $\sum_{n=1}^{\infty} a_n^{|\alpha|} < \infty$, *then there are, for almost all* $g \in Q$, *only finitely many integers* $n > 0$ *for which*

$$n^\alpha g \in Q(a_n)\Gamma \ , \tag{5.9}$$

where $Q(a_n) = \{a_n^\alpha x \colon x \in Q\}$.

(2) *If* $\sum_{n=1}^{\infty} a_n^{|\alpha|} = \infty$, *then there are, for almost all* $g \in Q$, *infinitely many integers* $n > 0$ *for which* (5.9) *holds.*

This theorem is proved exactly as Theorem 5.1 is. In the next section we will comment on the existence of the multi-index $\alpha = (\alpha_1, \cdots, \alpha_d)$. Finally, we note that the hypothesis $d \geq 2$ is needed for the analogue of Lemma 5.3. The theorem as stated is false when $d = 1$.

In closing, let us write out what Theorem 5.1 says in the special case $a_n = n^{-1/4}$. Let (x, y, z) denote a typical point in the unit cube in N, and consider the inequalities:

$$\begin{aligned} &0 \leq nx - m < n^{-1/4} \ , \\ &0 \leq ny - m' < n^{-1/4} \ , \\ &0 \leq n^2z + nxm' + m'' < n^{-1/2} \ . \end{aligned} \tag{$*$}$$

Theorem 5.1 says that, for almost all (x, y, z) in Q, there are infinitely many positive integers n for which the inequalities $(*)$ have a solution $(m, m', m'') \in Z^3$. More intricate groups than N yield inequalities like those in $(*)$, but of greater complexity. The exact role of the term nxm' is not yet understood.

6. Dilation automorphisms

Let G denote a nilpotent Lie group whose underlying differentiable manifold is R^d and whose group operation is given by (5.8) with respect to the coordinates in R^d. Let $\alpha = (\alpha_1, \cdots, \alpha_d)$ be a d-tuple of positive integers, and for each real $t > 0$ let $t^\alpha \colon G \to G$ denote the map $(x_1, \cdots, x_d) \to (t^{\alpha_1}x_1, \cdots, t^{\alpha_d}x_d)$. We are going to give an inductive procedure for choosing $\alpha_1, \cdots, \alpha_d$ so that either t^α is an automorphism of G for all $t > 0$ or t^α becomes an automorphism on performing a slight change in the group operation of G.

To begin the induction, we choose α_1 and α_2 to be any two integers greater than 0. Suppose now that $\alpha_1, \cdots, \alpha_{i-1}$ are chosen. We will now define α_i. There are two cases:

(1) If the polynomial f_i in (5.8) is 0, then α_i may be chosen arbitrarily from the integers.

(2) If $f_i \neq 0$, then $f_i(t^{\alpha_1}x_1, \cdots, t^{\alpha_{i-1}}x_{i-1}; t^{\alpha_1}y_1, \cdots, t^{\alpha_{i-1}}y_{i-1})$ will be a non-zero polynomial over R in the variables $t, x_1, \cdots, x_{i-1}, y_1, \cdots, y_{i-1}$. We take for α_i the largest power of t occurring in this polynomial.

We have now defined $\alpha = (\alpha_1, \cdots, \alpha_d)$. Our definition of α guarantees that if $f_i \neq 0$, then

$$\begin{aligned} &f_i(t^{\alpha_1}x_1, \cdots, t^{\alpha_{i-1}}x_{i-1}; t^{\alpha_1}y_1, \cdots, t^{\alpha_{i-1}}y_{i-1}) \\ &\quad = g_i(x_1, \cdots, x_{i-1}; y_1, \cdots, y_{i-1})t^{\alpha_i} + h_i(t; x_1, \cdots, x_{i-1}; y_1, \cdots, y_{i-1}) , \end{aligned}$$

where g_i and h_i are polynomials over R, and h_i is either 0 or of degree less than α_i in t. It is easy to verify that t^α is an automorphism of G if, and only if, $h_i = 0$ for all i. In case the h_i's do not vanish we have the following result:

Theorem 6.1. *Define a binary operation $R^d \times R^d \to R^d$ by $(x_1, \cdots, x_d) \circ (y_1, \cdots, y_d) = (z_1, \cdots, z_d)$, where $z_1 = x_1 + y_1, z_2 = x_2 + y_2$, and for $3 \leq i \leq d$,*

$$z_i = x_i + y_i + g_i(x_1, \cdots, x_{i-1}; y_1, \cdots, y_{i-1}) ,$$

where g_i is defined from f_i as above ($g_i = 0$ if $f_i = 0$). Then R^d with the operation $x \circ y$ is a nilpotent Lie group, denoted G_0. The group G_0 will be non-abelian whenever G is non-abelian, and for all real $t > 0$ the map $t^\alpha: G_0 \to G_0$ is an automorphism.

Remark. There exist real nilpotent Lie algebras all of whose automorphisms have only eigenvalues of absolute value 1; see [18], for instance. Thus the present theorem is not without content.

Proof of Theoem 6.1. It is clear that $(0, \cdots, 0)$ is an identity for the operation in G_0, and an easy argument by induction on d shows that for all $g \in G_0$ there is an element $g' \in G_0$ satisfying $g \circ g' = (0, \cdots, 0)$. It remains to check associativity. In order to do so, we will use the fact that the product $x \circ y$ in G_0 can be got from the product xy in G as follows:

$$x \circ y = \lim_{t \to \infty} t^{-\alpha}((t^\alpha x)(t^\alpha x)) .$$

If follows that

$$(6.1) \qquad (x \circ y) \circ z = \lim_{s \to \infty} s^{-\alpha}(s^\alpha(\lim_{t \to \infty} t^{-\alpha}[\{t^\alpha x\}\{t^\alpha y\}](s^\alpha z))) .$$

It is easy to verify that the iterated limit in (6.1) can be replaced by a double limit over (s, t). It follows that we can take the limit in (6.1) along the line $s = t$, and hence

$$(6.2) \qquad (x \circ y) \circ z = \lim_{s \to \infty} s^{-\alpha}((\{s^\alpha x\}\{s^\alpha y\})(s^\alpha z)) .$$

Similarly,

$$(6.3) \qquad x \circ (y \circ z) = \lim_{s \to \infty} s^{-\alpha}((s^\alpha x)(\{s^\alpha y\}\{s^\alpha z\})) .$$

Since multiplication in G is associative, it follows from (6.2) and (6.3) that $(x \circ y) \circ z = x \circ (y \circ z)$.

The remaining assertions of the theorem are obvious.

Appendix

Let G be a compact topological group and denote its left invariant Haar measure by dx. Suppose that $A: G \to G$ is measurable surjective endomorphism, and denote by $A_*(dx)$ the measure induced on G by A, so that for every continuous function f on G,

$$\int_G f(A(x))dx = \int_G f(x)A_*(dx) . \tag{A.1}$$

Then $A_*(dx) = dx$, that is, for every continuous f,

$$\int_G f(A(x))dx = \int_G f(x)dx . \tag{A.2}$$

To prove this well-known fact, let $z = A(y)$ be any element of G. Then (A.1) and the fact that dx is left invariant imply

$$\begin{aligned}\int_G f(zx)A_*(dx) &= \int_G f(zA(x))dx \\ &= \int_G f(A(yx))dx = \int_G f(A(x))dx = \int_G f(x)A_*(dx) .\end{aligned}$$

Therefore the left side does not depend on z, and $A_*(dx)$ is proportional to Haar measure. Taking $f \equiv 1$ shows that $A_*(dx) = dx$, that is, (A.2) holds.

Our interest here is in the case where $G = T^k = R^k/Z^k$ so that A is defined by a $k \times k$ matrix with integer entries and nonzero determinant. (It is clear that such a matrix defines an endomorphism with a closed image. The non-vanishing of the determinant implies that the image is also open so the endomorphism is onto.) In this case (A.2) takes the form

$$\int_{T^k} f(A(z))dz = \int_{T^k} f(z)dz . \tag{A.3}$$

Several cases of (A.3) have been used in § 3. In § 3(b), the proof that $\|f_n\|_1 = \|F_n\|_1$ follows from

$$\begin{aligned}\|f_n\|_1 &= \int_{T^d}\left[\int_{T^d} |F_n(A_n x + B_n y)|\,dx\right]dy = \int_{T^d}\left[\int_{T^d} |F_n(A_n x)|\,dx\right]dy \\ &= \int_{T^d} |F_n(A_n x)|\,dx = \int_{T^d} |F_n(z)|\,dz = \|F_n\|_1 .\end{aligned}$$

In the same section, the proof that $\|f_n f_m\|_1 = \|f_n\|_1 \|f_m\|_1$ is the case of (A.3) where $k = 2d$, $z = (x, y)$, $A(z) = (A_m x + B_m y, A_n x + B_n y)$. Finally, in § 3(c), formula (3.4) is obtained by taking $z = x$, $A(z) = u$ in the notation of that section.

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