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FUNCTION THEORY OF FINITE ORDER ON ALGEBRAIC VARIETIES. I (A)

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1. Introduction

The purpose of this paper is to discuss the theory of analytic functions of finite order on algebraic varieties. The proofs of our results are only sketched as the complete arguments will appear in a more general setting at a later time.

There are two reasons for making this study. The first is that the classical theory of meromorphic functions of finite order [19] and the extensions of this theory to functions on C^n, [18], [24], form a very pretty subject, and one which furnishes us with the most important examples of entire transcendental functions for use in analysis and number theory. As the natural domains of existence for functions of finite order are the algebraic varieties, it seems worthwhile to develop the theory in this setting. The second, and more important, reason is that on an affine algebraic variety A the functions of finite order give the smallest class of functions which might allow one to realize the topological Grothendieck ring $K_{\text{top}}(A)$. Or, to put matters another way, since A is a Stein manifold, Grauerts' proof of the Oka principle [6] gives the isomorphism

$$K_{\text{top}}(A) \cong K_{\text{hol}}(A)$$

between the topological and analytic K-theories on A. What the examples and partial results the author has seem to indicate is the refined isomorphism

$$K_{\text{top}}(A) \cong K_{\text{f.o.}}(A) \tag{1.1}$$

between the topological and finite order K-theories on A. We are able to establish the isomorphism (1.1) in certain special cases, and in general are able to reduce the problem to semi-local questions in several complex variables. If (1.1) were established, then we could be able to measure the obstructions to making an analytic cycle $\Gamma \in H^{2q}(A, Q)$ progressively more algebraic. Some examples suggest (roughly) that the μ^{th} obstruction should be the projection of Γ in $\sum_{|r-s|>\mu} H^{r,s}(A, C)$ (here we are using Deligne's mixed Hodge structure [5] on

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$H^*(A, C)$). Since the analytic cycles generate all of $H^{\text{even}}(A, Q)$, such a mechanism would have obvious importance.

§§ 2, 3, and 4 of this paper contain definitions, some relatively easy propositions, examples, and heuristic comments about function theory of finite order. The main theorems which we are able to prove deal with divisors and are presented in § 5. In § 6 we discuss K-theory with growth conditions and attempt to isolate the essential questions whose solution seems necessary in order that the function theory of finite order should work in general.

2. Basic definitions

(a) *Localization at infinity on algebraic varieties*

Let A be a smooth, quasi-projective algebraic variety over C. A *smooth completion* of A is given by a smooth, projective variety $\bar{A}$ containing A as the Zariski open set obtained by removing a divisor D which has locally normal crossings. Thus, given $x \in \bar{A} - A$ there is a polycylindrical neighborhood $P = \{z \in C^n : |z_j| \leq 1\}$ of x in $\bar{A}$ such that $D \cap P$ is given by an equation $z_1 \cdots z_k = 0$.

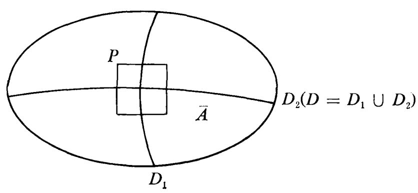

Fig. 1

We set $P^* = P \cap A$ and refer to P^* as a *punctured polycylinder*. If $\Delta = \{z \in C: |z| \leq 1\}$ and $\Delta^* = \Delta - \{0\}$, then obviously

$$P^* \cong (\Delta^*)^k \times \Delta^{n-k} . \tag{2.1}$$

We shall sometimes write $P^*(k)$ for P^* when we want to specify how many locally irreducible branches of D pass through the center of P. Smooth completions exist by the fundamental theorem of Hironaka [13].

The smooth completions of A are not unique. For example, $P_1 \times P_1$ and P_2 are both smooth completions of C^2. However, given two such smooth completions $\bar{A}$ and $\bar{A}'$, there are a third $\bar{A}''$ and a diagram of holomorphic mappings

$$\begin{array}{ccc} & \bar{A}'' & \\ {\scriptstyle \pi} \swarrow & & \searrow {\scriptstyle \pi'} \\ \bar{A} & & \bar{A}' \end{array} \tag{2.2}$$

such that π and π' are both the identity on A. Localizing in punctured cylinders at infinity, the mapping π in (2.2) will be given by

$$z_j = (z_1'')^{\lambda_{j,1}} \cdots (z_{k''}'')^{\lambda_{j,k''}} \qquad (j = 1, \cdots, k) , \tag{2.3}$$

where the $\lambda_{j,\mu}$ are integers.

If A and A' are smooth quasi-projective varieties, and $f: A \to A'$ is a rational holomorphic mapping, then we can find smooth completions $\bar{A}$ and $\bar{A}'$ such that f extends to a holomorphic mapping $\bar{f}: \bar{A} \to \bar{A}'$.

Among the algebraic varieties we shall be especially interested in those which are *affine*. Such a smooth affine variety A may always be realized as an algebraic subvariety of C^N given by polynomial equations $P_\alpha(z_1, \cdots, z_N) = 0$. By projecting onto a generic C^n, we may also realize A as a finite algebraic covering $\pi: A \to C^n$.

(b) *Function theory in punctured polycylinders*

Let $P^*(k)$ be the punctured polycylinder given by

$$\{(z, w) \in C^k \times C^{n-k}: |z_j| \geq 1, |w_\alpha| \leq 1\} .$$

Then $P^*(k)$ is of the form (2.1). Let $\eta(z, w)$ be a holomorphic function in[1] $P^*(k)$. In order to measure the order of growth of $\eta(z, w)$ as $|z| \to \infty$, we define the *maximum modulus*

$$M(\eta, r) = \max_{\substack{|z_j| \leq r \\ |w_\alpha| \leq 1}} \log |\eta(z, w)| . \tag{2.4}$$

Other useful indicators of the order of growth are (i) the *mean value*

$$m(\eta, r) = \left(\frac{1}{2\pi}\right)^n \int \log + | \eta(re^{i\theta}, e^{i\phi})| \, d\theta d\phi , \tag{2.5}$$

where $(re^{i\theta}, e^{i\phi}) = (re^{i\theta_1}, \cdots, re^{i\theta_k}, e^{i\phi_1}, \cdots, e^{i\phi_{n-k}})$ and $d\theta = d\theta_1 \cdots d\theta_k$, $d\phi = d\phi_1 \cdots d\phi_{n-k}$; and (iii) the *order function* (*spherical image*)

$$T(\eta, r) = \int_1^r \left\{ \frac{\sqrt{-1}}{2\pi} \int_{\substack{|z_j| \leq t \\ |w_\alpha| \leq 1}} \frac{d\eta \wedge d\bar{\eta}}{(1 + |\eta|^2)^2} \wedge \omega^{n-1} \right\} \frac{dt}{t} , \tag{2.6}$$

where

$$\omega = \frac{\sqrt{-1}}{2} \left\{ \sum_{j=1}^k \frac{dz_j \wedge d\bar{z}_j}{|z_j|^2} + \sum_{\alpha=1}^{n-k} dw_\alpha \wedge d\bar{w}_\alpha \right\} \tag{2.7}$$

is the Kähler form[2] on $P^*(k)$.

[1] By this we mean that η is defined and holomorphic in the slightly larger open polycylinder $\{(z, w): |z_j| > 1 - \varepsilon, |w_\alpha| < 1 + \varepsilon\}$ for some $\varepsilon > 0$.

[2] Observe that ω is the restriction to $P^*(k)$ of the Euclidean Kähler form on the polycylinder $P \subset C^n$.

Similarly, if $Z \subset P^*(k)$ is a purely q-dimensional analytic subvariety, we let

$$N(Z, r) = \int_1^r \left\{ \int_{Z[t]} \omega^q \right\} \frac{dt}{t} , \tag{2.8}$$

where $Z[t] = Z \cap \{(z, w): |z_j| \leq t\}$.

(2.9) Proposition[3]. (i) *The holomorphic function* η *is holomorphic in the whole polycylinder* $P \Longleftrightarrow M(f, r) = 0(1) \Longleftrightarrow m(f, r) = 0(1)$. (ii) *The closure of* Z *is an analytic subvariety in* $P \Longleftrightarrow N(Z, r) = 0(\log r)$.

We now introduce the notion of a *λ-ring* [21]. This is given by a collection $\Lambda = \{\lambda(r)\}$ of real-valued functions of $r \in \boldsymbol{R}^+$ such that: (i) each $\lambda(r)$ is continuous and increasing in r; (ii) if λ_1, λ_2 belong to Λ, then so do $\lambda_1 + \lambda_2$, $\lambda_1 \circ \lambda_2$, and $c\lambda_1$ for $c > 0$; and (iii) if $\lambda \in \Lambda$, then $A\lambda(Br) = 0(\lambda'(r))$ for some $\lambda' \in \Lambda$. The following are the λ-rings of which we shall make essential use:

Example 1. $\Lambda = \boldsymbol{R}^+$.

Example 2. $\Lambda = \{1, r\}$, the ring generated by the constant 1 and the function r. Thus Λ consists of all positive polynomials $a_l r^l + \cdots + a_1 r + a_0 (a_\mu \geq 0)$.

Example 3. $\Lambda =$ all increasing, continuous, nonnegative functions of r.

Let Λ be a λ-ring.

(2.10) Definition. The ring $\mathcal{O}_\Lambda(P^*)$ of holomorphic functions in P^*, which have finite Λ-order, is given by those $\eta \in \mathcal{O}(P^*)$ which satisfy $M(\eta, r) = 0(\lambda(r))$ for some $\lambda \in \Lambda$.

Referring to Examples 1, 2,3 above, we obtain respectively: (i) the holomorphic functions in P, (ii) the functions of *finite order* in P^*, and (iii) all holomorphic functions in P^*. These examples will be denoted by $\mathcal{O}(P)$, $\mathcal{O}_{\text{f.o.}}(P^*)$, and $\mathcal{O}(P^*)$.

Let Λ be a λ-ring, and $\mathcal{M}(P^*)$ the field of meromorphic functions in P^*. We assume furthermore that $P^* = P^*(1)$ is a punctured polycylinder with only *one* branch having been deleted from the closed polycylinder P. (We shall do the general case later.)

(2.11) Definition. The field $\mathcal{M}_\Lambda(P^*)$ of meromorphic functions in P^* which have finite Λ-order is given by those meromorphic functions ϕ on P^* which admit a factorization $\phi = \eta/\psi$, where η, $\psi \in \mathcal{O}_\Lambda(P^*)$.

Remark. The reason that we have restricted P^* to be a $P^*(1)$ is that, for a general $P^*(k)$, it is *not* the case that $\mathcal{M}(P^*(k))$ is the quotient field of $\mathcal{O}(P^*(k))$. This restriction can be removed and will be discussed in § 4(*f*) below.

Referring to the three examples of λ-rings given above, we see that $\mathcal{M}_\Lambda(P^*)$ is respectively: (i) the meromorphic functions on P; (ii) the field of meromorphic functions of finite order; and (iii) the field of all meromorphic functions on P^*. These cases will be denoted by $\mathcal{M}(P)$, $\mathcal{M}_{\text{f.o.}}(P^*)$, and $\mathcal{M}(P^*)$.

[3] It may also be shown that η is meromorphic in $P \Longleftrightarrow M(\eta, r) = 0(\log r) \Longleftrightarrow m(\eta, r) = 0(\log r) \Longleftrightarrow T(f, r) = 0(\log r)$. The reason for integrating with respect to dt/t in $T(\eta, r)$ and $N(Z, r)$ comes from *Jensen's theorem* [19, p. 164].

(c) *Holomorphic mappings of finite order between algebraic varieties*

Let A be an algebraic variety as in § 2(a) above. We denote by $\mathscr{R}(A)$ and $\mathscr{M}(A)$ respectively the fields of rational and meromorphic functions on A. For each λ-ring Λ we shall define a sub-field $\mathscr{M}_\Lambda(A)$ of $\mathscr{M}(A)$. To do this we consider a smooth completion $\bar{A}$ of A. Then a neighborhood of $\bar{A} - A$ in A may be covered by finitely many punctured polycylinders $\{P^*\}$, and we say that $\phi \in \mathscr{M}(A)$ is in $\mathscr{M}_\Lambda(A)$ if each restriction $\phi \mid P^*$ is in $\mathscr{M}_\Lambda(P^*)$ for all sufficiently small P^*.

Referring to Examples 1, 2, 3 in § 2(b) above, we obtain respectively (i) the field $\mathscr{R}(A)$ of rational functions on A, (ii) the field $\mathscr{M}_{\text{f.o.}}(A)$ of meromorphic functions of finite order on A, and (iii) the field $\mathscr{M}(A)$ of all meromorphic functions on A.

(2.12) Proposition. (i) *$\mathscr{M}_\Lambda(A)$ is intrinsically defined by the algebraic structure on A*; (ii) *if $f: A \to A'$ is a rational holomorphic mapping, then $f^*(\mathscr{M}_\Lambda(A'))$ is contained*[4] *in $\mathscr{M}_\Lambda(A)$.*

Remark. In the "classical" case $A = \boldsymbol{C}^n$, it is customary to define *the* order $\rho(\eta)$ of an entire *holomorphic* function by

$$\rho(\eta) = \overline{\lim_{r \to \infty}} \left[\frac{\log M(\eta, r)}{\log r} \right] .$$

This definition then leads, as above, to *the* order $\rho(\eta)$ of an entire meromorphic[5] function η. In general, we may define $\rho(\eta)$ relative to a *fixed* smooth completion $\bar{A}$ of A. However, this is not an intrinsic notion, nor does it behave functorially, whereas the notion of η being of finite order does both.

(2.13) Definition. Let A and V be algebraic varieties, and $f: A \to V$ a holomorphic mapping. Then *f has order Λ* if $f^*[\mathscr{R}(V)] \subset \mathscr{M}_\Lambda(A)$.

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[4] It may also be proved that $\mathscr{M}_\Lambda(A)$ is *algebraically closed in $\mathscr{M}(A)$*, in the sense that if we have a polynomial relation

$$\phi^k + \eta_1 \phi^{k+1} + \cdots + \eta_{k-1}\phi + \eta_k = 0 ,$$

where $\eta_j \in \mathscr{M}_\Lambda(A)$ and $\phi \in \mathscr{M}(A)$, then $\phi \in \mathscr{M}_\Lambda(A)$ (cf. [21]).

[5] Actually, this is a little misleading. If ϕ is a *meromorphic* function on $\boldsymbol{C}^n$, then one defines the *Nevanlinna order function* ([19], [29])

$$T_0(\phi, r) = \int_0^r \left\{ \int_{||z|| \le t} dd^c (\log 1 + |\phi|^2) \wedge dd^c \log ||z||^{n-1} \right\} \frac{dt}{t} .$$

In case ϕ is holomorphic, it turns out that

$$\overline{\lim_{r \to \infty}} \left[\frac{\log M(\phi, r)}{\log r} \right] = \overline{\lim_{r \to \infty}} \left[\frac{\log T_0(\phi, r)}{\log r} \right] ,$$

and this defines the order $\rho(\phi)$ of any meromorphic function ϕ. It is then a consequence of [18] or [24] that every meromorphic function ϕ on $\boldsymbol{C}^n$ may be written globally as a quotient $\phi = \psi / \eta$ of holomorphic functions where $\rho(\psi) = \rho(\phi) = \rho(\eta)$.

Remarks. Taking $V = \boldsymbol{C}$ we recover the ring $\mathcal{O}_\Lambda(A)$ of holomorphic functions of order Λ on A. Taking $V = \boldsymbol{P}_1$, we recover the field $\mathcal{M}_\Lambda(A)$. In case $\Lambda = \{1, r\}$ is given by Example 2 above, then we simply say that f has *finite order* if f has order Λ for this particular Λ.

(2.14) Proposition. (i) *If $f_1: A \to V_1$ and $f_2: A \to V_2$ are of order Λ, then so is the product mapping $f_1 \times f_2: A \to V_1 \times V_2$.* (ii) *If $f: A \to V$ is of order Λ, and $B \subset A$ is an an algebraic subvariety, then the restriction $f: B \to V$ is of order Λ.* (iii) *If $f: A \to A'$ is rational and $g: A' \to V$ is of order Λ, then $f \circ g$ is*[6] *of order Λ.*

Remark. In appendix 2 to [9] we have defined the order function $T(f, r)$ for a holomorphic mapping $f: A \to V$ into a compact Kähler manifold V. In fact, from the definition given there it is clear that essentially

$$T(f, r) = N(\Gamma_f, r) ,$$

where $\Gamma_f \subset A \times V$ is the graph of f, and $N(\Gamma_f, r)$ is obtained by adding up the local $N(\Gamma_f \cap P^*, r)$ as defined by (2.8). Using Proposition (2.9) we see that f is rational $\Longleftrightarrow T(f, r) = 0(\log r)$. However, it is unknown whether $T(f, r)$ has the nice functoriality properties as given by Proposition (2.14). Moreover, it is not known whether the condition

$$T(f, r) = 0(\lambda(r)) \qquad (\lambda \in \Lambda)$$

is the same as $f^*[\mathscr{R}(V)] \subset \mathcal{M}_\Lambda(A)$ in case V is a projective algebraic variety, although the author suspects that this must be the case. In conclusion, so far as the author is aware we do not know how to define what it means for a mapping $f: A \to V$ into a compact Kähler manifold to have order Λ in such a way that Proposition (2.14) holds true and which recovers our old definition when V is algebraic.

3. Algebraic, analytic, and topological K-theory

(a) *The classical comparison theorems*

Let A_{alg} be a smooth quasi-projective variety, and A_{hol} the underlying

[6] Given smooth algebraic varieties A, V we shall denote the tangent bundles by $\boldsymbol{T}(A)$, $\boldsymbol{T}(V)$ respectively. These tangent bundles are also algebraic varieties, and a holomorphic mapping

$$f: A \to V$$

induces the bundle mapping

$$f_*: \boldsymbol{T}(A) \to \boldsymbol{T}(V) .$$

It should be that: f of order $\Lambda \Rightarrow f_*$ is also of order Λ, although we have not tried to prove this. The classical case is that $\rho(d\eta/dz) \leq \rho(\eta)$ for an entire holomorphic function $\eta(z)$ $(z \in \boldsymbol{C})$.

complex-analytic space.[7] Then there are maps

$$(3.1) \qquad K_{\text{alg}}(A) \to K_{\text{hol}}(A) \, ,$$

$$(3.2) \qquad H^q(A_{\text{alg}}, \mathscr{E}_{\text{alg}}) \to H^q(A_{\text{hol}}, \mathscr{E}_{\text{hol}}) \, .$$

In (3.1), $K_{\text{alg}}(A)$ is the algebraic Grothendieck group generated by the algebraic vector bundles on the quasi-projective A_{alg} with the usual equivalence relation [3], and $K_{\text{hol}}(A)$ is the corresponding object in the analytic category. In (3.2), $\mathscr{E}_{\text{alg}}$ is a coherent algebraic sheaf and $H^q(A_{\text{alg}}, \mathscr{E}_{\text{alg}})$ is the cohomology in the Zariski topology, while $\mathscr{E}_{\text{hol}}$ is the corresponding coherent analytic sheaf and cohomology is in the usual topology (cf. [22] and [11]).

The classical comparison theorems (G.A.G.A. [23]) state that both (3.1) and (3.2) are isomorphisms in case A is *complete* (=compact in the usual topology). These results generalize such statements as "a meromorphic function on a complete algebraic variety/$\boldsymbol{C}$ is rational", and "an analytic subvariety of a complete algebraic variety is itself an algebraic subvariety (Chow)".

Such statements are obviously false when A not complete. However, the *linear* G.A.G.A. theorems relating to (3.2) seem to remain close to being valid. To be more explicit, it is frequently, but not always, the case that we have finiteness theorems

$$(3.3) \qquad \begin{array}{ll} \dim H^q(A_{\text{alg}}, \mathscr{E}_{\text{alg}}) < \infty & (q \neq q_0) \, , \\ \dim H^q(A_{\text{hol}}, \mathscr{E}_{\text{hol}}) < \infty & (q \neq q_0) \end{array}$$

such that (3.2) turns out to be an isomorphism for $q \neq q_0$, while the restriction

$$(3.4) \qquad H^{q_0}(A_{\text{alg}}, \mathscr{E}_{\text{alg}}) \to H^{q_0}(A_{\text{hol}}, \mathscr{E}_{\text{hol}})$$

has dense image (*Runge theorem*). For example, if A is affine, then both of these results are[8] true with $q_0 = 0$.

However, the *non-linear* mapping (3.1) is considerably more subtle and is generally very far from being either injective or surjective. One of the main purposes of this work is to try and begin the development of a stepwise procedure (an obstruction theory if you like) to try and analyze the mapping (3.1) in case A is an affine variety.

(b) *Oka's principle* (*following Grauert*)

Let A be a smooth *affine* algebraic variety, which we may assume is given

[7] In this section we wish to keep track of whether we consider A as an algebraic variety, a complex manifold, or a topological space (actually, a finite CW complex).

[8] Results of this sort are discussed by Hartshorne in [12]. Another example is where $A = \bar{A} - S$ with $\bar{A}$ being smooth and projective, and where S is a smooth d-dimensional sub-variety of $\bar{A}$ such that the normal bundle $N \to S$ has signature $(q, d-q)$. Then (3.3) and (3.4) are true with $q_0 = q$. On the other hand, there is an example, where A_{hol} is a Stein manifold but A_{alg} is *not* an affine variety.

by polynomial equations in C^N. Thus A is a *Stein manifold* [11], and we may apply the theorems[9] of Grauert [6]. To state these, we let $GL(m, \mathcal{O})$ be the sheaf of holomorphic mappings of A into $GL(m, C)$, and denote the corresponding continuous sheaf by $GL(m, \mathcal{C})$. Then it is well-known that we have isomorphisms

$$(3.5) \qquad \begin{aligned} H^1(A, GL(m, \mathcal{O})) &\cong \mathrm{Vect}^m_{\mathrm{hol}}(A) , \\ H^1(A, GL(m, \mathcal{C})) &\cong \mathrm{Vect}^m_{\mathrm{top}}(A) , \end{aligned}$$

where $\mathrm{Vect}^m_{\mathrm{hol}}(A)$ is the set of isomorphism classes of holomorphic vector bundles with fibre C^m over A, and similarly for $\mathrm{Vect}^m_{\mathrm{top}}(A)$. One basic result of Grauert's is that the mapping

$$(3.6) \qquad H^1(A, GL(m, \mathcal{O})) \to H^1(A, GL(m, \mathcal{C}))$$

is an isomorphism of sets. For example, in the "abelian" case $m = 1$, this results from the cohomology sequences in the diagram

$$(3.7) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & Z & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}^* & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{C}^* & \longrightarrow & 1 \end{array}$$

together with the vanishing theorem

$$H^q(A, \mathcal{O}) = 0 = H^q(A, \mathcal{C}) \qquad (q > 0) ,$$

and is expressed by the isomorphisms

$$H^1(A, \mathcal{O}^*) \cong H^1(A, \mathcal{C}^*) \cong H^2(A, Z) .$$

We will discuss two variants of this result. For the first, we denote by $K_{\mathrm{hol}}(A)$ and $K_{\mathrm{top}}(A)$ the Grothendieck rings[10] associated respectively to the holomorphic and continuous vector bundles over A.

(3.8) Proposition. *On an affine variety A, the mapping*

$$K_{\mathrm{hol}}(A) \to K_{\mathrm{top}}(A)$$

is a ring isomorphism.

To give the second, we let M be an arbitrary complex manifold, and use the notations

$$[A, M]_{\mathrm{hol}} , \qquad [A, M]_{\mathrm{top}}$$

[9] The theorems discussed in this section depend only on A being a Stein manifold.
[10] An excellent informal presentation of $K_{\mathrm{top}}(A)$ which contains all the information we shall need is the notes of Bott [4].

for the holomorphic homotopy classes of holomorphic maps from A to M, and continuous homotopy classes of continuous maps from A to M respectively.[11]

(3.9) Proposition. *For large N, the mapping*

$$[A,\ \mathrm{Grass}\,(m,N)]_{\mathrm{hol}} \to [A,\ \mathrm{Grass}\,(m,N)]_{\mathrm{top}}$$

is an isomorphism of sets.

Here Grass (m,N) is the complex Grassmannian of $(N-m)$-planes through the origin in C^N(cf. [14]). This proposition results from (3.6) and the set isomorphisms

$$\mathrm{Vect}^m_{\mathrm{hol}}(A) \cong [A,\ \mathrm{Grass}\,(m,N)]_{\mathrm{hol}}\ , \qquad \mathrm{Vect}^m_{\mathrm{top}}(A) \cong [A,\ \mathrm{Grass}\,(m,N)]_{\mathrm{top}}\ .$$

A final consequence of the Oka principle, as proved by Grauert, is that the natural mapping

$$[A, GL(m, C)]_{\mathrm{hol}} \to [A, GL(m, C)]_{\mathrm{top}} \tag{3.10}$$

is an isomorphism of groups. Thus, e.g., in the abelian case $m = 1$ we have

$$[A, C^*]_{\mathrm{hol}} \cong [A, C^*]_{\mathrm{top}} \cong H^1(A, Z)\ , \tag{3.11}$$

where the map

$$[A, C^*]_{\mathrm{hol}} \to H^1(A, Z)$$

is given by

$$[f] \to \frac{1}{2\pi i}(d \log f)$$

for a holomorphic mapping $f\colon A \to C^*$. This generalizes to give a homomorphism

$$[A, GL(m, C)]_{\mathrm{hol}} \to H^{2m-1}(A, Z)$$

given by sending a mapping $f = (f_1, \cdots, f_m)\colon A \to GL(m, C) \subset \underbrace{C^m \times \cdots \times C^m}_{m}$ into the cohomology class of the differential form

$$\mu_f = d^c \log \|f_1\| \wedge (dd^c \log \|f_1\|)^{m-1}\ ,$$

which is (essentially) the *Martinelli kernel* constructed from the first column of the matrix f. Because the mapping

$$\pi_{2m-1}(GL(m, C)) \cong Z$$

[11] Two holomorphic maps $f_j\colon A \to M (j = 1,2)$ are *holomorphically homotopic* if there are a connected analytic space T and a holomorphic map $F\colon T \times A \to M$ such that $F(t_1, \cdot) = f_1$ and $F(t_2, \cdot) = f_2$ for $t_1, t_2 \in T$.

given by sending $[f] \in [S^{2m-1}, GL(m, C)]_{\text{top}}$ into

$$\frac{1}{(m-1)!} \int_{S^{2m-1}} \mu_f$$

is an isomorphism (Bott), one might look for some analogue of (3.11) using the complex structure on A_{hol}.

(c) *Analytic K-theory and algebraic de Rham cohomology*

As above we consider a smooth affine variety A, on which we denote respectively by $\boldsymbol{\Omega}^q(A_{\text{hol}})$ and $\boldsymbol{\Omega}^q(A_{\text{alg}})$ the global holomorphic q-forms and global rational, regular q-forms. Thus a holomorphic q-form $\phi \in \boldsymbol{\Omega}^q(A_{\text{hol}})$ may have an *essential singularity* along $\bar{A} - A$, but a $\phi \in \boldsymbol{\Omega}^q(A_{\text{alg}})$ has only a *finite pole* on this divisor. The (de Rham) cohomology of the respective complexes

$$\begin{array}{ccccccc} \cdots \longrightarrow & \boldsymbol{\Omega}^q(A_{\text{hol}}) & \xrightarrow{d} & \boldsymbol{\Omega}^{q+1}(A_{\text{hol}}) & \longrightarrow \cdots, \\ & \uparrow & & \uparrow & \\ \cdots \longrightarrow & \boldsymbol{\Omega}^q(A_{\text{alg}}) & \xrightarrow{d} & \boldsymbol{\Omega}^{q+1}(A_{\text{alg}}) & \longrightarrow \cdots \end{array}$$

will be denoted by

$$H^*_{\text{DR}}(A_{\text{hol}}) , \qquad H^*_{\text{DR}}(A_{\text{alg}}) .$$

Since A is a Stein manifold, we have

$$H^*_{\text{DR}}(A_{\text{hol}}) \cong H^*(A, C) , \tag{3.12}$$

and the Grothendieck comparison theorem [10] gives

$$H^*_{\text{DR}}(A_{\text{alg}}) \cong H^*_{\text{DR}}(A_{\text{hol}}) . \tag{3.13}$$

We now wish to define directly a map

$$d\log\colon \text{Vect}^m (A_{\text{hol}}) \to H^{\text{even}}_{\text{DR}}(A_{\text{hol}}) . \tag{3.14}$$

To do this, we denote by Ω^q the analytic sheaf on A of holomorphic q-forms and let Ω^q_c be the sub-sheaf of closed forms. The holomorphic Poincaré lemma gives

$$0 \longrightarrow \Omega^q_c \longrightarrow \Omega^q \xrightarrow{d} \Omega^{q+1}_c \longrightarrow 0 ,$$

and the exact cohomology of this sequence together with $H^p(A, \Omega^q) = 0$ for $p > 0$ gives the isomorphism

(3.15) $$H^q(A, \Omega_c^q) \cong H_{DR}^{2q}(A_{hol}) .$$

To define the map (3.14), we will use (3.15) and define directly a map

(3.16) $$d\log\colon \mathrm{Vect}_{hol}^m(A) \to \bigoplus_{q=0}^{m} H^q(A, \Omega_c^q) .$$

Let $\{U_\alpha\}$ be a polycylindrical covering of A and $\xi \in \mathrm{Vect}_{hol}^m(A)$ a holomorphic vector bundle given by holomorphic transition functions[12] $f_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(m, \boldsymbol{C})$. Then the matrix-valued 1-forms $d\log f_{\alpha\beta} = df_{\alpha\beta}f_{\alpha\beta}^{-1}$ define a 1-cocycle (cf. Atiyah [1])

$$d\log \xi \in H^1(A, \Omega^1(\mathrm{End}\,\xi)) .$$

Let $P_q(A_1, \cdots, A_q)(A_j \in \mathrm{gl}\,(m, \boldsymbol{C}))$ be the invariant, symmetric multi-linear form which corresponds to the invariant polynomial $P_q(A)$ given by

$$\det(\lambda I_m + A) = \sum_{q=0}^{m} P_q(A)\lambda^{m-q} .$$

Then we may form the cup-product

$$P_q(d\log \xi) \in H^q(A, \Omega_c^q) ,$$

and subsequently define

$$d\log(\xi) = \sum_{q=0}^{m} P_q(d\log \xi) .$$

Under the isomorphism (3.12), we simply obtain the *total Chern class* of the complex vector bundle ξ (cf. Atiyah [1]).

Now we recall the standard isomorphism

(3.17) $$\mathrm{ch}\colon K_{top}(A) \otimes_{\boldsymbol{Z}} \boldsymbol{C} \to H^{even}(A, \boldsymbol{C})$$

given by taking the Chern character of continuous vector bundles over the CW complex A. We may combine (3.8), (3.14), and (3.17) into a big commutative diagram

(3.18) $$\begin{array}{ccc} K_{top}(A) \otimes_{\boldsymbol{Z}} \boldsymbol{C} & \xrightarrow{\mathrm{ch}} & H^{even}(A, \boldsymbol{C}) \\ \downarrow & & \uparrow \\ K_{hol}(A) \otimes_{\boldsymbol{Z}} \boldsymbol{C} & \xrightarrow{d\log} & H_{DR}^{even}(A_{hol}) \end{array}$$

in which all arrows are *isomorphisms*.

[12] By definition, a *polycylindrical covering* $\{U_\alpha\}$ of A is a locally finite covering of A by open sets U_α each of which is biholomorphic to a polycylinder or punctured polycylinder.

The validity of the diagram (3.18) only depends on A being a Stein manifold. To put in the algebraic structure, we combine (3.13) together with bottom row in (3.18) to obtain a commutative diagram

$$\text{(3.19)} \qquad \begin{array}{ccc} K_{\text{hol}}(A) \otimes_{\boldsymbol{Z}} \boldsymbol{C} & \xrightarrow{d\log} & H_{\text{DR}}^{\text{even}}(A_{\text{hol}}) \\ \uparrow & & \uparrow \\ (?) & \dashrightarrow & H_{\text{DR}}^{\text{even}}(A_{\text{alg}}) \end{array}$$

in which all the solid arrows are isomorphisms. It certainly makes sense to ask what we should put in for (?) to make (3.19) into a commutative diagram of isomorphisms. It is easily verified that, if $\xi \in \text{Vect}_{\text{alg}}^m(A)$ is an algebraic vector bundle, then $d\log \xi \in H_{\text{DR}}^{\text{even}}(A_{\text{alg}})$ has[13] only a finite order pole along $\bar{A} - A$. However, and this is one of our main points, there will be very many holomorphic, but non-algebraic, vector bundles ξ such that $d\log \xi$ has only a finite order pole along $\bar{A} - A$.

To obtain some idea of what (?) should be, we recall from classical function theory the following elementary fact [19]:

(3.20) Let $f\colon \boldsymbol{C} \to \boldsymbol{C}^*$ be an entire holomorphic function which omits the value zero. Then $d\log f$ is a *rational* differential form on $\boldsymbol{C} \Longleftrightarrow f$ is of finite exponential order.

Although we can only verify it in some very special cases, our main working hypothesis is that (?) should be the ring $K_{\text{f.o.}}(A) \otimes_{\boldsymbol{Z}} \boldsymbol{C}$ constructed from holomorphic vector bundles of finite order. At any event, we can prove that (?) must contain $K_{\text{f.o.}}(A) \otimes_{\boldsymbol{Z}} \boldsymbol{C}$, so that finite exponential order is the *least* amount of growth which we must have in order to have the refined Oka's principle with growth conditions. To some extent then our main question is the following

(3.21) Problem. What natural conditions should be placed on a holomorphic vector bundle $\xi \in \text{Vect}_{\text{hol}}^m(A)$ in order that the Chern character $d\log \xi \in H_{\text{DR}}^{\text{even}}(A_{\text{hol}})$ should, in a natural way, have only a finite order pole along $\bar{A} - A$?

Remark. Instead of defining Chern classes in cohomology, we may use *analytic Schubert cycles* to define Chern classes in homology, in which case

[13] This follows from the fact that an algebraic vector bundle ξ on an affine variety A has an algebraic connection θ. The curvature Θ is a global, rational holomorphic 2-form with values in $\text{End}(\xi)$, and the equation

$$\sum_{q=0}^{m} P_q(d\log \xi)\lambda^{m-q} = \det(\lambda I_m + \Theta)$$

shows that the $P_q(d\log \xi)$ have only finite poles along $\bar{A} - A$.

(3.17) becomes the isomorphism[14]

(3.22) $$\mathrm{ch}\colon K_{\mathrm{hol}}(A) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \to \boldsymbol{H}_{\mathrm{even}}(A, \boldsymbol{Q}) \ .$$

Thus, over $\boldsymbol{Q}$, all even-dimensional homology is represented by analytic cycles, and furthermore homology is (essentially) the same as analytic equivalence (cf. § 4(e) below). Given an *analytic* cycle $Z \subset A$, the dual cohomology class $\mathrm{cl}\,(Z) \in H^{\mathrm{even}}(A, \boldsymbol{Q})$ is represented by a *rational*, holomorphic differential form (cf. (3.20)). In this context, the problem (3.27) is:

(3.23) What (minimal) growth conditions on Z are consistent with this statement concerning $\mathrm{cl}\,(Z)$?

4. Some examples, comments, and questions

In this section we shall give several examples to illustrate how function theory of finite order arises quite naturally from rather simple and basic problems in algebraic geometry.

(a) *Divisors on algebraic curves*

Let A be an affine algebraic curve, and $\delta = n_1x_1 + \cdots + n_rx_r$ $(n_j \in \boldsymbol{Z})$ an *algebraic* divisor on A. Then δ is always the divisor of a meromorphic function ϕ on A, but in general we cannot take ϕ to be a *rational* function.[15] A very natural question is how much growth must we allow on ϕ in order to have $(\phi) = \delta$? To answer this question, we first recall from § 2(c) that we may define *the* order $\rho(\phi)$ of a meromorphic function $\phi \in \mathscr{M}(A)$; this is because there is a unique smooth completion of A. If ϕ is rational, then $\rho(\phi) = 0$ but not conversely. (The entire function $\sum_{n=0}^{\infty} q^{n^2}z^n$, $|q| < 1$, has order zero. Also, the entire function $\cos\sqrt{z}$ has order $1/2$, which shows that $\rho(\phi)$ need not be an integer; it may, in fact, be any nonnegative real number.)

[14] Throughout this paper $\boldsymbol{H}_*(A)$ denotes homology with infinite chains (non-compact support). The *Poincaré-Lefschetz duality* is then the isomorphism

$$H^{2n-q}(A) \cong \boldsymbol{H}_q(A)$$

(any coefficients are O.K.).

[15] This latter statement may be seen as follows: Let $\bar{A}$ be the *unique* smooth completion of A and write $A = \bar{A} - \{z_1, \cdots, z_N\}$. Denote by $J(\bar{A})$ the *Jacobian variety* of $\bar{A}$, and let $\psi\colon \bar{A} \to J(\bar{A})$ be the usual mapping of $\bar{A}$ into $J(\bar{A})$(choose our base point to be z_1). Then, by *Abel's theorem*, δ is the divisor of a rational function $\Longleftrightarrow$

$$\psi(\delta) = l_1\psi(z_1) + \cdots + l_N\psi(z_N)$$

for suitable integers l_α. It follows that, if the genus $g(A) \geq 1$, then almost all δ are *not* divisors of rational functions on A.

(4.1) Proposition. (i) *We may always find* $\phi \in \mathcal{M}(A)$ *such that* $(\phi) = \delta$ *and the order* $\rho(\phi) \leq 2g$ *where* g *is the genus of* A. *This estimate is the best possible.* (ii) *If* ϕ *is any function of finite order such that* $(\phi) = \delta$, *then* $\rho(\phi)$ *is an integer and* $\rho(\phi) = 0 \Longleftrightarrow \phi$ *is rational.*[16]

(4.2) Remark. This proposition may be thought of as suggesting two general principles: (i) If, on an algebraic variety A, a Cousin-type problem is given in the algebraic category and has a solution in the analytic category, then it has a solution using function theory of finite order. (ii) Having solved the Cousin-type problem in the finite-order category, the obstruction to solving it in the algebraic category is measured by the Hodge structure on $H^*(A, \boldsymbol{C})$.

We shall be extensively discussing these two principles and variations on them. The proofs of some special cases will be sketched in § 5.

(b) *The Oka principle with growth conditions*

Let A be an n-dimensional smooth affine algebraic variety, and $V \subset A$ an algebraic divisor. If the homology class

$$[V] \in \boldsymbol{H}_{2n-2}(V, \boldsymbol{Z})$$

of V is zero, then V is the divisor of a meromorphic function ϕ on A. As before, we cannot in general take ϕ to be a rational function, and we have the

(4.3) Proposition. (i) *We may find a meromorphic function* ϕ *of finite order such that* $(\phi) = V$. (ii) *We may always take* ϕ *to be a rational function* $\Longleftrightarrow H^1(\bar{A}, \boldsymbol{C}) = 0$, *where* $\bar{A}$ *is any smooth completion*[17] *of* A.

This proposition raises the following question: Suppose that Λ is a λ-ring (§ 2(b)) and that $V \subset A$ is an analytic divisor which is of *finite* Λ-*order*; i.e., we have

$$N(V, r) = 0(\lambda(r)) \qquad (\lambda \in \Lambda)$$

(cf. (2.8)). Suppose furthermore that the Cousin problem for V has a topological solution, by which we mean that $V = (\phi)$ is the divisor of a continuous function on A. (This is the same as saying that $[V] = 0$ in $\boldsymbol{H}_{2n-2}(V, \boldsymbol{Z})$.) Then can we write $V = (\phi)$ where $\phi \in \mathcal{M}_\Lambda(A)$ is a meromorphic function having finite Λ-order? In other words, what we are asking is a special case of whether the *Oka principle with* Λ-*growth conditions* is valid for A? Referring to Examples 1–3 in § 2(b) of λ-rings, we see that this special case of the Oka principle is false in the first case (the *algebraic category*), and by Grauert is true in the third case (the *analytic category*). We shall see later (§ 5) that this Oka

[16] The results given in this section follow from Theorems I–III in § 5.

[17] The group $H^1(\bar{A}, \boldsymbol{C})$ is independent of the particular smooth completion $\bar{A}$ of A. In general, the vector space $H^{q,0}(\bar{A})$ of everywhere holomorphic q-forms is independent of $\bar{A}$, and $H^1(\bar{A}, \boldsymbol{C}) \cong H^{1,0}(\bar{A}) \oplus \overline{H^{1,0}(\bar{A})}$.

principle is also valid in the second case (*finite order category*). From (4.3) we have the

(4.4) Proposition. *On an affine algebraic variety A, the function theory of finite order is the smallest category which contains the algebraic function theory and for which the Oka principle with Λ-growth conditions might be true.*

This whole business is a little reminscent of taking the algebraic closure of ordinary fields. Thus every polynomial equation with coefficients in $\boldsymbol{Q}$ has roots in $\boldsymbol{C}$, but there are certainly algebraically closed fields containing $\boldsymbol{Q}$ which are much smaller than $\boldsymbol{C}$. To some extent, the author's working hypothesis may be stated as saying that *the Oka principle with Λ-growth conditions is valid provided that there are no Hodge conditions in $H^*_{\mathrm{DR}}(A_{d \log \Lambda})$*. In other words, once we go up to function theory of order Λ for which the Hodge conditions disappear, then this function theory should admit solutions to all Cousin type problems which may be solved topologically. It is also the author's feeling that the function theory of finite order is, so to speak, the *Cousin-closure* (like algebraic closure) of algebraic function theory. The rough parallel with ordinary fields is therefore

$$\boldsymbol{Q} \leftrightarrow \{\text{algebraic category}\} \ ,$$
$$\bar{\boldsymbol{Q}} \leftrightarrow \{\text{finite order category}\} \ ,$$
$$\boldsymbol{C} \leftrightarrow \{\text{analytic category}\} \ .$$

(c) *Homotopy classes of mappings into $\boldsymbol{C}^*$*

Let A be an affine variety, and M a projective variety. Then we have defined what it means for a holomorphic mapping $f\colon A \to M$ to have order Λ for a λ-ring Λ (§ 2). Two such mappings f_1, f_2 will be said to be Λ-homotopic if there are an algebraic variety T and a holomorphic mapping $F\colon T \times A \to M$ such that F has order Λ and specializes to f_1 and f_2 at points t_1 and t_2 in T. We denote by $[A, M]_\Lambda$ the Λ-homotopy classes of mappings of order Λ from A to M. Referring to Examples 1–3 of § 2(b), we will use the notations $[A, M]_{\mathrm{alg}}$, $[A, M]_{\mathrm{f.o.}}$, and $[A, M]_{\mathrm{hol}}$ for the three examples[18] of λ-rings Λ. We recall from (3.11) the isomorphisms

$$[A, \boldsymbol{C}^*]_{\mathrm{hol}} \cong [A, \boldsymbol{C}^*]_{\mathrm{top}} \cong H^1(A, \boldsymbol{Z}) \ .$$

(4.5) Proposition. (i) *The mapping $[A, \boldsymbol{C}^*]_{\mathrm{f.o.}} \to [A, \boldsymbol{C}^*]_{\mathrm{hol}}$ is always an isomorphism.* (ii) *If $A = \bar{A} - D$ where D is sufficiently ample smooth divisor, then the mapping $[A, \boldsymbol{C}^*]_{\mathrm{alg}} \to [A, \boldsymbol{C}^*]_{\mathrm{f.o.}}$ is an isomorphism $\Longleftrightarrow H^1(\bar{A}, \boldsymbol{C}) = 0$.*

Remarks. We will explain what it means for D to be "sufficiently ample". Letting $\boldsymbol{L} \to \bar{A}$ be the line bundle determined by D, we should have that

[18] It is not at all clear that this definition of $[A, M]_{\mathrm{hol}}$ coincides with the one given in § 3(b). However, for the cases $M = GL(m, \boldsymbol{C})$ and $M = \mathrm{Grass}\,(m, N)$, these definitions agree (using Grauert's result again), and these are the only situations which we shall consider.

$H^1(\bar{A}, \Omega^1_{\bar{A}}(kL)) = 0$ for $k \geq 1$. This proposition gives another example where the finite order Oka principle holds, and where the obstruction to dropping down to the algebraic category is measured by the Hodge structure on $H^*(A, \boldsymbol{C})$. The author does not know if Proposition (4.5) remains valid with $GL(m, \boldsymbol{C})$ replacing $\boldsymbol{C}^*$ (cf. (3.10)). It is true if we replace $\boldsymbol{C}^*$ by a solvable algebraic group, as follows from (4.5) together with the results in § 4 below.

(d) *Analytic curves on algebraic surfaces*

Let D be a nonsingular plane curve of genus g, and $A = \boldsymbol{P}_2 - D$ the affine algebraic surface obtained by deleting D from $\boldsymbol{P}_2$. Then

$$H^2(A, \boldsymbol{Z}) \cong \underbrace{\boldsymbol{Z} \oplus \cdots \oplus \boldsymbol{Z}}_{2g} \oplus \boldsymbol{Z}/d\boldsymbol{Z}, \tag{4.6}$$

where d is the *degree* of D (thus $g = \frac{1}{2}(d-1)(d-2)$). The following is easy to verify:

(4.7) Lemma. *In* (4.6), *the only part of* $H^2(A, \boldsymbol{Z})$ *which is represented by an algebraic curve is the torsion piece* $\boldsymbol{Z}/d\boldsymbol{Z}$.

Thus, on the very simple affine surface A, there are $2g$ independent classes in $H^2(A, \boldsymbol{Z})$ which are carried by analytic, but *not* algebraic, curves (cf. the cohomology sequences arising from (3.7) which give the isomorphism

$$H^1(A, \mathcal{O}^*_{\text{hol}}) \cong H^2(A, \boldsymbol{Z}) .)$$

(4.8) Proposition. *The free part of* $H^2(A, \boldsymbol{Z})$ *is represented by curves having finite order.*

We will not give a proof of (4.8) here (cf. § 5 below), but we will outline a proof of an analogous example which illustrates very well why we may realize all of the homology by analytic curves of finite order. Thus, suppose that $\bar{A}$ is an *algebraic elliptic surface* (cf. Kodaira [16]) which is represented as a fibre space of elliptic curves over $\boldsymbol{P}_1$ by a mapping

$$\pi: \bar{A} \to \boldsymbol{P}_1 .$$

Let $E_t = \pi^{-1}(t)$, and assume for simplicity that the singular fibres are irreducible and have as singularity only a single node (they are singular fibres of type I_1 in Kodaira's list). Assume also that E_∞ is nonsingular, and let $A = \bar{A} - E_\infty$ so that we have the representation

$$\pi: A \to \boldsymbol{C} . \tag{4.9}$$

There is an exact sequences of sheaves over $\boldsymbol{P}_1$ (cf. [16, Theorem 11.2]):

$$0 \longrightarrow R^1_{\pi_*}(\boldsymbol{Z}) \longrightarrow \mathcal{J} \xrightarrow{\exp} \mathcal{G} \longrightarrow 0 \tag{4.10}$$

whose terms have the following interpretation: $\mathscr{G}$ is the sheaf of groups[19] over C of holomorphic cross-sections of the fibre space of complex Lie groups associated to (4.9); $\mathscr{J}$ is the corresponding locally free coherent analytic sheaf of Lie algebras; and $R^1_{\pi_*}(\boldsymbol{Z})$ is the Leray direct image sheaf for the constant sheaf $\boldsymbol{Z}$ over $\bar{A}$. If $\mathscr{J}(k)$ are the sections of $\mathscr{J}$ which have a pole of order k at $t = \infty$, then $H^1(\boldsymbol{P}_1, \mathscr{J}(k)) = 0$ for $k \geq k_0$. Using this, the exact cohomology sequence of (4.10), together with the degeneracy of the Leray spectral sequence of (4.9), gives the exact sequence

$$H^0(\mathscr{G}(k)) \to H^2(A, \boldsymbol{Z}) \to 0 \qquad (k \geq k_0) \ . \tag{4.11}$$

The geometric interpretaton of (4.11) is the following: Every cohomology class in $H^2(A, \boldsymbol{Z})$ comes from an analytic curve C traced out by a section σ of $\mathscr{G} \to \boldsymbol{P}_1$ such that: (1) σ is holomorphic over C, and (ii) near $t = \infty$, σ is the exponential of a section of $\mathscr{J}$ which has a pole of order k at $t = \infty$.

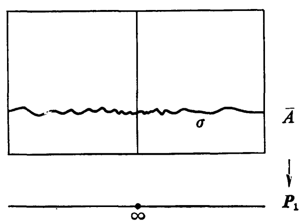


Fig. 2

We may use the Weierstrass functions p and p' to write out explicitly what all of this means, and it turns out exactly that the graph of σ is a curve of finite order on A.

By refining this argument a little, we find that, in general, the best possible estimate is that C should have finite order *zero* if $h^{2,0}(\bar{A}) = 0$, and finite order *one* if $h^{2,0}(\bar{A}) \neq 0$ (cf. § 5 for general results which explain this).

(e) *Algebraic, analytic, and homological equivalence*

Let A be a smooth affine variety. The notion of *algebraic cycles* on A and *algebraic equivalence* between such cycles is well known.[20] The graded group of algebraic equivalence classes of algebraic cycles will be denoted by

$$\mathscr{C}_*(A_{\text{alg}}) = \bigoplus_{q=0}^{n} \mathscr{C}_q(A_{\text{alg}}) \ ,$$

[19] Thus the fibre $\mathscr{G}_t$ is the elliptic curve E_t in case the latter is nonsingular, while $\mathscr{G}_t$ is a multiplicative group $\boldsymbol{C}^*$ in case E_t has a node. There is a general discussion of this situation in [7].

[20] The fact that A is not complete means, among other things, that an *effective* algebraic cycle may be algebraically equivalent to zero. In particular, a 0-cycle is always algebraically (but usually *not* rationally) equivalent to zero. A discussion of homological vs. algebraic equivalence is given in [7].

the grading being by codimension of cycles. Similarly, we may define

$$\mathscr{C}_*(A_{\text{hol}}) = \bigoplus_{q=0}^{n} \mathscr{C}_q(A_{\text{hol}})$$

to be the analytic cycles modulo analytic equivalence.

In general, if Λ is a λ-ring, then we have defined what it means for an analytic sub-variety $V \subset A$ to have order Λ. We may define analytic equivalence of order Λ by considering analytic cycles on the product $T \times A$ of A with an algebraic variety T (the parameter variety). This leads to the graded group

$$\mathscr{C}_*(A_\Lambda) = \bigoplus_{q=0}^{n} \mathscr{C}_q(A_\Lambda)$$

of analytic cycles of order Λ modulo analytic equivalence of order Λ. Referring to Examples (i)–(iii) of λ-rings in § 2(b), we obtain, respectively,

$$\mathscr{C}_*(A_{\text{alg}}) , \quad \mathscr{C}_*(A_{\text{f.o.}}) , \quad \mathscr{C}_*(A_{\text{hol}}) .$$

If $\Lambda \subset \Lambda'$, there is an obvious map

$$\mathscr{C}_*(A_\Lambda) \to \mathscr{C}_*(A_{\Lambda'}) ,$$

which in the above cases leads to the following maps

$$(4.12) \qquad \begin{aligned} \mathscr{C}_*(A_{\text{alg}}) &\xrightarrow{\alpha_*} \mathscr{C}_*(A_{\text{hol}}) , \\ \mathscr{C}_*(A_{\text{alg}}) &\xrightarrow{\beta_*} \mathscr{C}_*(A_{\text{f.o.}}) , \\ \mathscr{C}_*(A_{\text{f.o.}}) &\xrightarrow{\gamma_*} \mathscr{C}_*(A_{\text{hol}}) , \\ \mathscr{C}_*(A_{\text{hol}}) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} &\xrightarrow{\delta_*} H^{\text{even}}(A, \boldsymbol{Q}) . \end{aligned}$$

Regarding these maps, the following is known:

(4.13)
- (i) $\beta_1 \colon \mathscr{C}_1(A_{\text{alg}}) \to \mathscr{C}_1(A_{\text{f.o.}})$ is injective but not surjective (cf. § 4(a));
- (ii) $\gamma_1 \colon \mathscr{C}_1(A_{\text{f.o.}}) \to \mathscr{C}_1(A_{\text{hol}})$ is an isomorphism (cf. § 5);
- (iii) $\delta_*\alpha_* \colon \mathscr{C}_*(A_{\text{alg}}) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \to H^{\text{even}}(A, \boldsymbol{Q})$ is, in general, neither injective nor surjective (cf. [7]; this is essentially the statement that homological and algebraic equivalences differ if codim > 1): and
- (iv) $\delta_* \colon \mathscr{C}_*(A_{\text{hol}}) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \to H^{\text{even}}(A, \boldsymbol{Q})$ *should* be an isomorphism.[21]

[21] This would follow from (3.8) if we know that every analytic sub-variety Z' on A was analytically equivalent to a sub-variety Z such that a bounded number of holomorphic functions locally generate the ideal sheaf $\mathscr{I}_Z$ of Z. In this case we could make a *finite* resolution of $\mathscr{I}_Z$ by locally free sheaves. It seems possible to the author that a category argument might be used to prove this statement about Z'.

Example. Referring to (iii) and (iv), we may find a 3-dimensional affine variety A and two smooth algebraic curves and C_1 and C_2 on A such that C_1 is *analytically*, but not *algebraically*, equivalent to C_2. It is certainly an intriguing question as to just how transcendental we must allow the analytic curves used to deform C_1 into C_2 to be.

At this point we may give one of the main problems which has arisen in function theory of finite order. Namely, both of the graded groups

$$\mathscr{C}_*(A_{\text{alg}}) , \qquad \mathscr{C}_*(A_{\text{hol}})$$

are graded rings, the product being induced by putting cycles in general position and then intersecting them. However, it seems to be unknown whether the intersection $V \cdot V'$ of two sub-varieties V and V', which are assumed to be in general position and each of which has order Λ, again has order Λ.

Problem A. Does the intersection of cycles induce a ring structure on $\mathscr{C}_*(A_\Lambda)$?

Remarks. This question has been discussed in appendix 2 to [9], where it is observed that it will suffice to take either V or V' to be algebraic. Assuming that V' is algebraic, we may find a smooth algebraic embedding $A \subset \boldsymbol{C}^N$ such that V' is contained in the intersection of A with a linear space. Utilizing a generic projection $\pi: A \to \boldsymbol{C}^n$, we may reduce our problem (in the finite order case) to the following

(4.14) Question. Let $V \subset \boldsymbol{C}^n$ be a purely k-dimensional analytic subvariety such that the Euclidean area

$$\int_{V[r]} \phi^k = 0(r^\mu) ,$$

where $V[r] = \{z \in V: \|z\| \leq r\}$ and $\phi = \dfrac{\sqrt{-1}}{2}\left(\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j\right)$ is the standard Kähler form on $\boldsymbol{C}^n$. Suppose that the residual intersection $W = V \cdot \boldsymbol{C}^l$ has pure dimension $k + l - n$. Then do we have

$$\int_{W[r]} \phi^{k+l-n} = 0(r^\nu)$$

for some ν?

Along the lines of the above discussion, the author would like to recall here the problem posed by Serre [22]: Let $\boldsymbol{E}_{\text{alg}} \to \boldsymbol{C}^n$ be an algebraic bundle. Then is $\boldsymbol{E}_{\text{alg}}$ algebraically trivial? Now there will be a *holomorphic* mapping

$$F: \boldsymbol{C}^n \times \boldsymbol{C} \to \text{Grass}\,(m, N)$$

such that: (i) $F(\cdot\,, 0)$ is *rational* and realizes $\boldsymbol{E}_{\text{alg}}$, and (ii) $F(\cdot\,, 1)$ is a constant map. We may then ask how transcendental F must be?

(f) *The K-ring of a punctured polycylinder*

Perhaps the simplest affine algebraic variety for which $K_{\mathrm{alg}}(A) \neq K_{\mathrm{hol}}(A)$ is the rational variety

$$(4.15) \qquad \begin{aligned} A_n &= \overbrace{C^* \times \cdots \times C^*}^{n}, \text{ or equivalently },\\ A_n &= \{(z_1, \cdots, z_n) \in C^n : z_1 \cdots z_n \neq 0\} . \end{aligned}$$

In this case, $K_{\mathrm{alg}}(A_n) = 0$,[22] but

$$(4.16) \qquad K_{\mathrm{hol}}(A_n) \cong H^{\mathrm{even}}(A_n, Z) \neq 0 \qquad (n \geq 2) .$$

To describe $K_{\mathrm{hol}}(A_n)$, we observe that $H^{\mathrm{even}}(A_n, Z)$ is generated by $H^2(A_n, Z) \cong Z^{\binom{n}{2}}$. For $i < j$, the differential form

$$(4.17) \qquad \omega_{ij} = \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \frac{dz_i}{z_i} \wedge \frac{dz_j}{z_j}$$

gives a class in $H^2(A_n, Z)$, and indeed the ω_{ij} $(i < j)$ give a basis for this group. Let $\Gamma \subset C$ be the lattice of Gaussian integers $\{m + n\sqrt{-1} : m, n \in Z\}$.

(4.18) Definition. The divisor D_{ij} on A_n is defined by the equations[23]

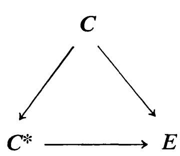
$$\log z_i \equiv \log z_j (2\pi\sqrt{-1} \cdot \Gamma) .$$

(4.19) Proposition. *The dual cohomology class of*

$$[D_{ij}] \in H_{2n-2}(A_n, Z) \ \ \textit{is} \ \ \omega_{ij} \in H^2(A_n, Z) .$$

[22] This follows from the well-known fact (cf. Hodge-Pedoe [14]) that every algebraic sub-variety of P_n is *rationally* equivalent to a linear subspace (counted multiply). Thus every algebraic cycle on A_n may be rationally "pushed to infinity".

[23] *A* more geometric description of D_{ij} ts the following: Let $E = C/\Gamma$ be the elliptic curve with complex multiplication "$\sqrt{-1}$" given by the Gaussian lattice Γ. Then there is a commutative diagram of holomorphic mappings



where $\pi(z) = (\log z)/2\pi i(\Gamma)$ $(z \in C^*)$. Let $Z_{ij} \subset \underbrace{E \times \cdots \times E}_{n}$ be the graph of multiplication "$\sqrt{-1}$" between the i^{th} and j^{th} factors (thus $Z_{ij} = \{(z_1, \cdots, z_i^i, \cdots, \sqrt{-1}z_i^j, \cdots, z_n) \bmod \underbrace{\Gamma \times \cdots \times \Gamma}_{n}\}$). Then $D_{ij} = \pi^{-1}(Z_{ij})$.

Using this proposition, we see that the divisors D_{ij} generate $K_{\mathrm{hol}}(A_n)$, and the relations are exactly those of the $\{\omega_{ij}\}$ in $H^{\mathrm{even}}(A_n, \boldsymbol{Z})$. The order function $N(D_{ij}, r)$, which we recall was defined by (2.8) and which measures the amount of transcendence of the analytic variety D_{ij}, is given by

(4.20) Proposition. (i) *$N(D_{ij}, r) \sim (\log r)^2$, and* (ii) *if D'_{ij} is any analytic divisor on A_n with $[D'_{ij}] = [D_{ij}]$, then*

$$\varliminf_{r\to\infty} \frac{N(D'_{ij}, r)}{(\log r)^2} \geq 1 \ .$$

Remarks. We recall from (2.9) that an analytic sub-variety $V \subset A_n$ is algebraic $\Longleftrightarrow N(V, r) = 0(\log r)$. Thus, (i) says that D_{ij} is transcendental and has finite order zero, and (ii) states that any other divisor in the same homology class as D_{ij} must be "at least as transcendental" as D_{ij}.

In the case of a *punctured polycylinder*

$$P^*(k) = \{(z, w) \in \boldsymbol{C}^k \times \boldsymbol{C}^{n-k} : 0 < |z_j| \leq 1,\ |w_\alpha| \leq 1\} \ ,$$

we may let D^*_{ij} be the divisor

$$\left\{(z, w) \in P^*(k) : \frac{\log z_i}{2\pi\sqrt{-1}} \equiv \frac{\log z_j}{2\pi\sqrt{-1}}(\Gamma)\right\} \ ,$$

and obtain

(4.21) Proposition. *The groups $K_{\mathrm{hol}}(P^*(k)) \cong H_{\mathrm{even}}(P^*(k), \boldsymbol{Z})$ are generated by the divisors D^*_{ij} $(1 \leq i < j \leq k)$.*

Remark. This proposition is somewhat analogous to the lemma of Atiyah-Hodge [2] concerning the local analytic de Rham group $H^*_{\mathrm{DR}}(P^*_{\mathrm{hol}})$. Roughly speaking their lemma states that the natural mappings[24]

$$H^*_{\mathrm{DR}}(P^*_{\mathrm{alg}}) \to H^*_{\mathrm{DR}}(P^*_{\mathrm{hol}}) \to H^*(P^*, \boldsymbol{C}) \tag{4.22}$$

are isomorphisms. Our lemma suggests that the natural mappings

$$K_{\mathrm{f.o.}}(P^*) \to K_{\mathrm{hol}}(P^*) \to K_{\mathrm{top}}(P^*) \tag{4.23}$$

should be isomorphisms, even though we have not yet defined the group $K_{\mathrm{f.o.}}(P^*)$.

Using Proposition (4.18), we may complete our definition of the groups $\mathcal{M}_\Lambda(P^*(k))$ for a general punctured polycylinder (cf. Definition (2.11) for the case $k = 1$). For simplicity we shall only consider the λ-rings given by Examples (i)–(iii) in § 2(b). In the first case, $\phi \in \mathcal{M}(P^*(k))$ will be in $\mathcal{M}_\Lambda(P^*(k)) \Longleftrightarrow$ (i) the divisor (ϕ) extends to analytic sub-variety in the whole polycylinder P;

[24] $H^*_{DR}(P^*_{\mathrm{alg}})$ means cohomology computed from the holomorphic differentials on P^* which have finite poles along the divisor $z_1 \cdots z_k = 0$.

and (ii) because of (i) we will have a factorization $\phi = \eta/\psi$ where η, ψ are holomorphic in $P^*(k)$, and we may proceed as before. To give the finite order case, we let $p(w)$ be the *Weierstrass p-function* for the lattice $2\pi\sqrt{-1}\Gamma$; and we define meromorphic functions p_{ij} on $P^*(k)$ by

$$p_{ij}(z, w) = p(\log z_i - \sqrt{-1} \log z_j) \qquad (i < j) .$$

It follows from Proposition (4.21) that any meromorphic function $\phi \in \mathcal{M}(P^*(k))$ has a factorization

$$\phi^2 = \prod_{i<j} (p_{ij})^{l_{ij}} \left(\frac{\eta}{\psi}\right) \qquad (l_{ij} \in \boldsymbol{Z}) , \tag{4.24}$$

where η, $\psi \in \mathcal{O}(P^*(k))$. For any λ-ring Λ which contains $\{1, r\}$, we say that $\phi \in \mathcal{M}(P^*(k))$ is in $\mathcal{M}_\Lambda(P^*(k)) \Longleftrightarrow \eta$ and $\psi \in \mathcal{O}_\Lambda(P^*(k))$ in the factorization (4.24).

(4.25) Proposition. *For the three λ-rings in* § 2(b), *a meromorphic function ϕ on $P^*(k)$ is in $\mathcal{M}_\Lambda(P^*(k)) \Longleftrightarrow$ the order function* (2.6) *satisfies*[25]

$$T(\phi, r) = 0(\lambda(r)) \qquad (r \in \Lambda) .$$

Added in proof. Question (4.14) has recently been shown to be false by M. Cornalba and B. Shiffman.

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[25] Observe that (2.6) is well-defined even if η is meromorphic. This is because locally $\eta = \xi/\zeta$ is the quotient of holomorphic functions and $\dfrac{d\eta \wedge d\bar{\eta}}{(1 + |\eta|^2)^2} = \partial\bar{\partial} \log (|\xi|^2 + |\zeta|^2)$.