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THE NORMAL SINGULARITIES OF A SUBMANIFOLD

I. R. PORTEOUS

0. Introduction

Let $\mathbf{R}^n$ be furnished with its Euclidean bilinear scalar product $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$; $(x, x') \mapsto x \cdot x'$ and associated positive-definite quadratic form $\mathbf{R}^n \to \mathbf{R}$; $x \mapsto x^{(2)} = x \cdot x$, and let M be an m-dimensional smooth (=sufficiently differentiable) submanifold of $\mathbf{R}^n$. Locally M may be represented parametrically as the image of a smooth embedding $g\colon \mathbf{R}^m \rightarrowtail \mathbf{R}^n$. (The tail on the arrow denotes that the domain of g is not necessarily the whole of $\mathbf{R}^m$.) Consider the map

$$\phi\colon M \times \mathbf{R}^n \to \mathbf{R} \times \mathbf{R}^n ; \qquad (w, x) \mapsto ((x - w)^{(2)}, x) ,$$

represented locally by the map

$$f\colon \mathbf{R}^m \times \mathbf{R}^n \rightarrowtail \mathbf{R} \times \mathbf{R}^n ; \qquad (t, x) \mapsto ((x - g(t))^{(2)}, x) .$$

Our purpose is to describe the Whitney-Thom generic singularities $\Sigma^I\phi$ of ϕ ([1] and also [19]), at least for small values of m and n. For the smallest values this turns out to be a re-exposition from a fresh point of view of some well-known facts of elementary differential geometry [21]. The inspiration for studying the map ϕ is a remark of R. Thom in his book [27, Chapter 4], where he justifies the use of the word 'umbilic' to describe certain of the elementary catastrophes. (See § 13 below.)

The main results which seem to be new, at least in detail, are in § 9, on what happens at an umbilic of a generic surface in $\mathbf{R}^3$ and in § 11, on what happens at a parabolic umbilic of a generic surface in $\mathbf{R}^4$. Umbilics have recently been studied by Feldman [7], [8] from a somewhat different point of view. The classical references are Darboux [5], Picard [18] and Gullstrand [11]. In [13], Hartman and Wintner made some corrections to Picard. The author is grateful to many people, especially Professor C. B. Allendoerfer and Professor W. L. Edge, for acquainting him with classical papers he had never read, and for reminding him of things he once knew but had forgotten. The author is also grateful to Professor R. Thom for hinting that there was more to umbilics than Thom actually stated in the first draft of his book.

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Notations used to describe differentials are in the main those of [20]. In particular the differential at a point $x \in X_0 \times X_1$ of a smooth map

$$h \colon X_0 \times X_1 \to Y_0 \times Y_1 ,$$

where X_0, X_1, Y_0 and Y_1 are finite-dimensional real linear spaces, will be denoted by

$$dhx = \begin{pmatrix} d_0h_0x & d_1h_0x \\ d_0h_1x & d_1h_1x \end{pmatrix}$$

with $d_ih_jx \in L(X_i, Y_j)$.

The following short-hand notations for certain forms which occur will be used throughout the paper.

For any $(t, x) \in \operatorname{dom} f$, let

$$P_1(t, x) = (x - g(t))\cdot dgt \in L(\mathbf{R}^m, \mathbf{R}) .$$

Up to a factor -2 this is just $d_0f_0(t, x)$. Then let

$$P_2(t, x) = d_0P_1(t, x) = (x - g(t))\cdot d^2gt - dgt\cdot dgt$$

with the obvious interpretation of $\cdot$ in each case. This is an element of $L_S(\mathbf{R}^m, L(\mathbf{R}^m, \mathbf{R}))$, the linear subspace of $L(\mathbf{R}^m, L(\mathbf{R}^m, \mathbf{R}))$ of symmetric linear maps $\mathbf{R}^m \to L(\mathbf{R}^m, \mathbf{R})$, since

$$P_2(t, x)(v)(u) = P_2(t, x)(u)(v)$$

for each $(u, v) \in \mathbf{R}^m \times \mathbf{R}^m$. Next, let

$$\begin{aligned} P_3(t, x) &= d_0P_2(t, x) \\ &= (x - g(t))\cdot d^3gt - \text{other terms not involving } x, \end{aligned}$$

which is an element of $L_S(\mathbf{R}^m, L_S(\mathbf{R}^m, L(\mathbf{R}^m, \mathbf{R}))$. Finally, define

$$P_{n+1}(t, x) = d_0P_n(t, x)$$

recursively, for $n > 2$.

1. The normal bundle of M

The differential $df(t, x)$ at $(t, x) \in \operatorname{dom} f$ is expressible as the matrix of linear maps:

$$\begin{pmatrix} -2P_1(t, x) & 2(x - g(t))\cdot \\ 0 & 1_{\mathbf{R}^n} \end{pmatrix} .$$

This has kernel rank (or nullity) $m - 1$ except where $P_1(t, x) = 0$, where the kernel rank is m. Now, for any $t \in \mathbf{R}^m$, the affine $(n - m)$-plane

$$\{x \in \mathbf{R}^n: P_1(t, x) = 0\} ,$$

made linear by choosing the point $g(t)$ to be its origin, is the normal plane $NM_{g(t)}$ at $g(t)$ to M in $\mathbf{R}^n$. It follows at once that $\Sigma^m \phi$, the subset of dom ϕ for which the kernel rank of the differential of ϕ is m, is the *normal bundle* NM of M in $\mathbf{R}^n$, presented as a smooth n-dimensional submanifold of $M \times \mathbf{R}^n$. The bundle projection map $NM \to M$ will be denoted by π.

2. The normal focal set

Consider $\phi | NM$ or, rather, its local representation:

$$f | \Sigma^m f: \Sigma^m f \to \mathbf{R} \times \mathbf{R}^n .$$

The kernel of the differential of this map at $(t, x) \in \Sigma^m f$ coincides with the kernel of the differential there of its second component, the map

$$\Sigma^m f \to \mathbf{R}^n ; \qquad (t, x) \mapsto x .$$

For let $(t, x) \in \Sigma^m f$, and let (t', x') be a tangent vector to $\Sigma^m f$ there in the kernel of the differential of the second component, namely the linear map

$$T(\Sigma^m f)_{(t,x)} \to \mathbf{R}^n ; \qquad (t', x') \mapsto x' .$$

Then $x' = 0$. Moreover $P_1(t, x) = 0$. So

$$\begin{pmatrix} -2P_1(t, x) & 2(x - g(t)) \cdot \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix} = 0;$$

That is, $(t', x') \in \ker d(f | \Sigma^m f)(t, x)$, as asserted.

It follows at once from this remark that the higher-order Whitney-Thom singularities of ϕ are nothing other than the singularities of the map

$$\psi: NM \to \mathbf{R}^n ; \qquad (w, x) \mapsto x .$$

This map is, of its nature, not generic in the Whitney-Thom sense, at least when dim $M > 1$, nor is it generic in the setting of Thom's paper on envelopes [26]. What we have done, by relating it to the map ϕ, is to provide a generic setting for it.

The image in $\mathbf{R}^n$ of the set of singularities of ψ is known as the *focal set* of M in $\mathbf{R}^n$. This is the *target-envelope* of the normal bundle NM [26]. We shall call the set of singularities of ψ itself the *normal focal set* of M. This is a subset of NM, not of $\mathbf{R}^n$, and is what Thom calls the *source-envelope* of NM ([26] again).

3. Second-order normal singularities

The submanifold $\Sigma^m f$ of $\mathbf{R}^m \times \mathbf{R}^n$ is given as the set of zeros of the map

$$\mathbf{R}^m \times \mathbf{R}^n \rightarrowtail L(\mathbf{R}^m, \mathbf{R}) \; ; \qquad (t, x) \mapsto P_1(t, x) \; .$$

So the tangent vectors to $\Sigma^m f$ at (t, x) are those vectors (t', x') such that

$$(P_2(t, x) \quad \cdot dgt) \begin{pmatrix} t' \\ x' \end{pmatrix} = 0 \; .$$

Such a vector will map to zero under the map

$$\Sigma^m f \rightarrow \mathbf{R}^n \; ; \qquad (t, x) \mapsto x$$

if, and only if, $x' = 0$, when $P_2(t, x)(t') = 0$. Therefore $\Sigma^{m,m'} f$, the set of points of $\Sigma^m f$ where the differential of the map to $\mathbf{R}^n$ has kernel rank m', may also be defined as the set of points $(t, x) \in \Sigma^m f$ where

$$\operatorname{kr} P_2(t, x) = m' \; .$$

Recall that

$$P_2(t, x) = (x - g(t)) \cdot d^2 gt - dgt \cdot dgt \; .$$

The bilinear map

$$\mathbf{R}^m \times \mathbf{R}^m \rightarrow \mathbf{R} \; ; \qquad (u, u') \mapsto dgt(u) \cdot dgt(u')$$

is the local representation of the *first fundamental form* at $w = g(t) \in M$, namely the symmetric bilinear map

$$TM_w \times TM_w \rightarrow \mathbf{R} ; \; (s, s') \mapsto s \cdot s' \; .$$

The trilinear map

$$\begin{gathered} (NM)_{g(t)} \times \mathbf{R}^m \times \mathbf{R}^m \rightarrow \mathbf{R} \; , \\ ((x, t), u, u') \mapsto (x - g(t)) \cdot d^2 gt(u)(u') \end{gathered}$$

is the local representative of the *second fundamental form* at w, properly a trilinear map

$$NM_w \times TM_w \times TM_w \rightarrow \mathbf{R} \; .$$

(The second fundamental form is sometimes represented geometrically by the induced quadratic map $TM_w \rightarrow (NM_w)^*$ with image the dual normal space at w, or by the quadratic map $TM_w \rightarrow NM_w$ obtained from the previous one by composition with the isomorphism $(NM_w)^* \to NM_w$ induced by the Euclidean scalar product on NM_w. This latter map may be described simply, in terms of the local representation, as the map

$$\mathbf{R}^m \to \mathbf{R}^n ; \ u \mapsto d^2gt(u)(u)$$

followed by the linear projection of $\mathbf{R}^n$ on to NM_w with kernel TM_w.)

4. Third-order normal singularities

The set $\Sigma^{m,m'}f$ is defined by the equations

$$P_1(t, x) = 0 \quad \text{and} \quad \operatorname{kr} P_2(t, x) = m' \ .$$

Suppose $(0, 0) \in \Sigma^{m,m'}f$. We can choose a new basis for $\mathbf{R}^m$ so that $\operatorname{kr} P_2(0, 0) = \mathbf{R}^{m'} \times \{0\}$, where we identify $\mathbf{R}^m$ with $\mathbf{R}^{m'} \times \mathbf{R}^{m-m'}$. Then near $(0, 0)$ the condition $\operatorname{kr} P_2(t, x) = m'$ is equivalent to the existence of a linear map $s \in L(\mathbf{R}^{m'}, \mathbf{R}^{m-m'})$ such that $P_2(t, x)(1, s) = 0$ in $L_S(\mathbf{R}^{m'}, L(\mathbf{R}^m, \mathbf{R})$, the obvious subspace of $L_S(\mathbf{R}^m, L(\mathbf{R}^m, \mathbf{R}))$ of dimension $m'(m - m') + \frac{1}{2}m'(m' + 1)$. Clearly the graph of s is the kernel of $P_2(t, x)$. Now suppose that at every point of $\Sigma^{m,m'}f$ near $(0, 0)$ the map

$$\begin{array}{ccc} \mathbf{R}^m \times \mathbf{R}^n \times L(\mathbf{R}^{m'}, \mathbf{R}^{m-m'}) & \rightarrowtail & L(\mathbf{R}^m, \mathbf{R}) \times L_S(\mathbf{R}^{m'}, L(\mathbf{R}^m, \mathbf{R})) \\ (t, x, s) & \mapsto & (P_1(t, x), P_2(t, x)(1, s)) \end{array}$$

has surjective differential (the generic case). Then, near $(0, 0)$, $\Sigma^{m,m'}f$ is, by the inverse function theorem, a smooth submanifold of $\Sigma^m f$ of codimension $\frac{1}{2}m'(m' + 1)$.

Consider now $\Sigma^{m,m',m''}f$, the set of points of $\Sigma^{m,m'}f$ where the restriction to $\Sigma^{m,m'}f$ of the projection to $\mathbf{R}^n$ has kernel rank m''. By considering the differential of the above map one can show that $(t', x') \in \mathbf{R}^m \times \mathbf{R}^n$ is a tangent vector to $\Sigma^{m,m'}f$ projecting to zero if, and only if, $x' = 0$, $P_2(t, x)(t') = 0$ and there exists $s' \in L(\mathbf{R}^{m'}, \mathbf{R}^{m-m'})$ such that

$$P_3(t, x)(1, s)(t') + P_2(t, x)(0, s') = 0 \ ,$$

or, equivalently, if

$$x' = 0 \ , \quad P_2(t, x)(t') = 0 \ , \quad P_3(t, x)(1, s)(1, s)(t') = 0 \ .$$

(In proving the reverse implication of the equivalence one uses the fact that by the symmetry of $P_2(t, x)$ the sequence

$$0 \longrightarrow \mathbf{R}^{m'} \xrightarrow{(1, s)} \mathbf{R}^m \xrightarrow{P_2(t, x)} L(\mathbf{R}^m, \mathbf{R}) \xrightarrow{(1, s)^*} L(\mathbf{R}^{m'}, \mathbf{R}) \longrightarrow 0$$

is exact.) So, near $(0,0)$, $\Sigma^{m,m',m''}f$ is definable as the subset of $\Sigma^{m,m'}f$ where $\operatorname{kr} P_3(t,x)(1,s)(1,s)(1,s) = m''$. Note that $(P_3(t,x)(1,s)(1,s)(1,s)$ is in $L_S(\mathbf{R}^{m'}, L_S(\mathbf{R}^{m'}, L(\mathbf{R}^{m'}, \mathbf{R})))$.)

The argument simplifies when $m' = m$, when the last condition reduces to $\operatorname{kr} P_3(t,x) = m''$. It is also simpler when $m' = 1$, as we see in the examples which follow.

Fourth and higher order singularities are in principle also accessible but become progressively more complicated to handle explicitly. We consider one example of a fourth order singularity in § 11, and return to the general case in § 14.

5. Curves in $\mathbf{R}^2$

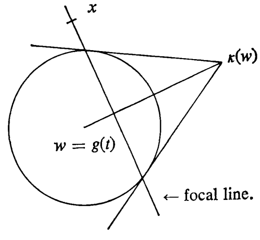
Let $m = 1$, $n = 2$. Then M is a smooth curve in $\mathbf{R}^2$, and $\Sigma^1\phi = NM$, a smooth submanifold of $M \times \mathbf{R}^2$ of dimension 2. The normal focal set $\Sigma^{1,1}\phi$ is a smooth curve in NM, its image in $\mathbf{R}^2$ being the focal set, or *evolute*, of M in $\mathbf{R}^2$, the set of its centres of curvature. Generically $\Sigma^{1,1,1}\phi$ is a discrete set of points of $\Sigma^{1,1}\phi$ whose images in $\mathbf{R}^2$ by ϕ are the cusps on the evolute and whose images in M by π are the *vertices* of M. A non-generic case is the circle, say the unit circle S^1 in $\mathbf{R}^2$. Then NM is the image of the map

$$S^1 \times \mathbf{R} \to S^1 \times R^2 ; \qquad (w, r) \mapsto (w, rw) .$$

The normal focal set is the circle $S^1 \times \{0\}$, and its image, the evolute, is just the centre of the circle, the origin $\{0\}$. In this case, $\Sigma^{1,1,1}f$ coincides with $\Sigma^{1,1}f$, $\Sigma^{1,1,0}$ being null, and every point of the curve is a vertex.

6. Curves in $\mathbf{R}^3$

Let $m = 1$, $n = 3$. Then M is a smooth curve in $\mathbf{R}^3$, and $\Sigma^1\phi = NM$ is a smooth submanifold of $M \times \mathbf{R}^3$ of dimension 3. For fixed t the equations $P_1(t,x) = 0$ and $P_2(t,x) = 0$ define a line in the normal plane $NM_{g(t)}$. This line, the *focal line* at $w = g(t)$ of M, is the polar with respect to the unit circle, with centre w, of the end $w + \kappa(w)$ of the curvature vector $\kappa(w) = d^2gt(u)(u)$ at w, u being a unit tangent vector to M at w.



This follows at once from the equation:

$$(x - g(t)) \cdot d^2gt(u)(u) = dgt\, u \cdot dgt\, u = 1 \ .$$

The union of the set of focal lines of M is the normal focal set of M, a submanifold of NM of dimension 2. In the generic case its image by ϕ in $\mathbf{R}^3$ is the *focal developable* of M, with the image of $\Sigma^{1,1,1}\phi$ as *edge* of *regression* or *cuspidal edge*. The set $\Sigma^{1,1,1}\phi$ is a smooth curve in NM, whose intersection with the focal line at w is the *centre* of *spherical curvature* of M at w. Generically also $\Sigma^{1,1,1,1}\phi$ is a discrete set of points of $\Sigma^{1,1,1}\phi$, whose images in $\mathbf{R}^3$ by ϕ are cusps on the edge of regression of the focal developable and whose images on M by π are the *vertices* of M.

7. Surfaces in $\mathbf{R}^3$

Let $m = 2$, $n = 3$. Then $\Sigma^2\phi = NM$. The normal focal set consists of the centres of principal curvature of M, at most two on each fibre of the normal bundle, namely those points $x \in NM_{g(t)}$ where the kernel rank of the map

$$P_2(t, x)\colon \mathbf{R}^2 \to L(\mathbf{R}^2, \mathbf{R}) \ ; \qquad u \mapsto (x - g(t)) \cdot d^2gt(u) - dgt(u) \cdot dgt$$

is >0, that is, $=1$ or 2. Notice the first and second fundamental forms here. The image by ϕ of the normal focal set is the focal set or *centro-surface* of M. Generically the subset $\Sigma^{2,1}\phi$ is the non-singular part of the normal focal set, a smooth submanifold of NM of dimension 2, while $\Sigma^{2,2}\phi$ is its set of singularities, the centres of curvature of the *umbilics* of M, the points of M with 'coincident' centres of curvature.

At a point $w = g(t)$ of M other than an umbilic the *principal directions* are the images by dgt of the kernels of $P_2(t, x)$ at the two centres of principal curvature x', x'' say. These directions are mutually orthogonal, for suppose $P_2(t, x')(u') = 0$ and $P_2(t, x'')(u'') = 0$, that is,

$$(x' - g(t)) \cdot d^2gt(u') = dgt(u') \cdot dgt \ ,$$
$$(x'' - g(t)) \cdot d^2gt(u'') = dgt(u'') \cdot dgt \ .$$

Then

$$\begin{aligned}(x' - g(t)) \cdot d^2gt(u')(u'') &= dgt(u') \cdot dgt(u'') \\ &= (x'' - g(t)) \cdot d^2gt(u')(u'') \ ,\end{aligned}$$

$x'' - g(t)$ being equal to $\lambda(x' - g(t))$ with $\lambda \neq 0, 1$. So $dgt(u') \cdot dgt(u'') = 0$, as asserted.

(For simplicity, we have ignored throughout the above discussion the possibility that one of the centres of curvature may lie 'at infinity'. See paragraph § 12 below for a few remarks on this case.)

Recall that a *line of curvature* on M is a curve C on M such that at each point $w = g(t)$ of C the tangent-direction to C is a principal direction.

Further singularity sets on NM in the generic case are $\Sigma^{2,1,1}\phi$, a smooth curve on $\Sigma^{2,1}\phi$, and $\Sigma^{2,1,1,1}\phi$, a discrete set of points on $\Sigma^{2,1,1}\phi$. The image of $\Sigma^{2,1,1}\phi$ on M by π is then a smooth immersed curve on M. We call the components of $\Sigma^{2,1,1}\phi$ *ribs* of M and their images by π *base ribs* (or *ridges*) of M. A given base rib is usually transverse to the members of one of the two families of lines of curvature on M, those points where it fails to be transverse, that is, where it touches a member of the appropriate family of lines of curvature, being the projection on M by π of points of $\Sigma^{2,1,1,1}\phi$.

We study ribs in more detail in the next section. Meanwhile we remark that an ellipsoid with distinct semi-axes [4] has six ribs, the six base ribs being the major-mean and mean-minor principal conic sections and the components of the complement in the major-minor principal section of the set of umbilics, which all lie on this section and are four in number. Each base rib is, in this case, a plane curve of curvature, and, in this case, the set $\Sigma^{2,1,1,1}\phi$ is null.

Non-generic examples include the ellipsoid of revolution and the sphere.

In the case of an ellipsoid of revolution there are only two umbilics, at the poles of the ellipsoid. The equator is a base rib. The corresponding rib is the only one on its sheet of the normal focal set. The whole of the other sheet belongs to $\Sigma^{2,1,1}\phi$ and has as image in $\mathbf{R}^3$ a curve, namely the axis of the ellipsoid, and not a surface.

In the case of a sphere every point is an umbilic and the whole of the normal focal set lies on $\Sigma^{2,2}\phi$, with image in $\mathbf{R}^3$ a point, namely the centre of the sphere.

8. Ribs

In this and the next section we work, for simplicity, entirely with the local representative f of the map ϕ, rather than with ϕ directly.

We continue to discuss the case of a surface in $\mathbf{R}^3$. At each point (t, x) of $\Sigma^{2,1}f$ there exists a non-zero vector $u \in \mathbf{R}^2$ such that $P_2(t, x)(u) = 0$. The line-bundle over $\Sigma^{2,1}f$, associating to each (t, x) the kernel of $P_2(t, x)$ is the set of zeros of the map

$$\begin{aligned} \mathbf{R}^2 \times \mathbf{R}^3 \times \mathbf{R}^2 &\rightarrowtail L(\mathbf{R}^2, \mathbf{R}) \times L(\mathbf{R}^2, \mathbf{R}) \, , \\ (t, x, u) \quad &\mapsto (P_1(t, x), P_2(t, x)(u)) \, . \end{aligned}$$

The differential at (t, x, u) of this map is the matrix of linear maps

$$\begin{pmatrix} P_2(t, x) & \cdot dgt & 0 \\ P_3(t, x)(u) & \cdot d^2gt(u) & P_2(t, x) \end{pmatrix} ,$$

and, generically, this is a surjective linear map. The tangent vectors to the line bundle are those vectors (t', x', u') such that

$$P_2(t,x)(t') + x'\cdot dgt = 0\ ,$$
$$P_3(t,x)(u)(t') + x'\cdot d^2gt(u) + P_2(t,x)(u') = 0\ ,$$

and the tangent vectors to $\Sigma^{2,1}f$ are those vectors (t', x') for which there exists a non-zero vector $u \in \ker(P_2(t,x))$ and a vector u' such that these equations are satisfied. In particular, when $t' = 0$ and $x' = 0$, u' is any real multiple of u.

A tangent vector (t', x') to $\Sigma^{2,1}f$ maps to 0 under the map $\Sigma^{2,1}f \to \mathbf{R}^3$ if, and only if, $x' = 0$. So $(t, x) \in \Sigma^{2,1,1}f$ if, and only if, there exist a non-zero vector t' and a vector u' such that

$$P_2(t,x)(t') = 0 \quad \text{and} \quad P_3(t,x)(u)(t') + P_2(t,x)(u') = 0\ .$$

Clearly, in such a case t' must be a non-zero real multiple of u. Suppose we take $t' = u$. Then there must be a vector u' such that

$$P_3(t,x)(u)^2 + P_2(t,x)(u') = 0\ .$$

A necessary condition for this is that $P_3(t,x)(u)^3 = 0$ since $P_2(t,x)(u')(u) = 0$, $P_2(t,x)(u)$ being zero. It is easy to see that this condition is also sufficient. (One uses the fact that, by the symmetry of $P_2(t,x)$, the sequence

$$0 \longrightarrow [u] \xrightarrow{\text{incl.}} \mathbf{R}^2 \xrightarrow{P_2(t,x)} L(\mathbf{R}^2, \mathbf{R}) \xrightarrow{ev_u} \mathbf{R} \longrightarrow 0$$

is exact. Cf. the arguments in § 4 and at the end of § 14.) So $(t, x) \in \Sigma^{2,1,1}f$ if, and only if, there exists a non-zero vector $u \in \mathbf{R}^2$ such that

$$P_1(t,x) = 0\ , \qquad P_2(t,x)(u) = 0 \quad \text{and} \quad P_3(t,x)(u)^3 = 0\ .$$

The line-bundle over $\Sigma^{2,1,1}f$ associating to each (t, x), the kernel of $P_2(t,x)$, is therefore the set of zeros of the map

$$\begin{array}{ll}\mathbf{R}^2 \times \mathbf{R}^3 \times \mathbf{R}^2 \rightarrowtail L(\mathbf{R}^2, \mathbf{R}) \times L(\mathbf{R}^2, \mathbf{R}) \times \mathbf{R} \\ \qquad (t, x, u) \qquad \mapsto (P_1(t,x), P_2(t,x)(u), P_3(t,x)(u)^3)\ .\end{array}$$

The differential at (t, x, u) of this map is the matrix of linear maps

$$\begin{pmatrix} P_2(t,x) & \cdot dgt & 0 \\ P_3(t,x)(u) & \cdot d^2gt(u) & P_2(t,x) \\ P_4(t,x)(u)^3 & \cdot d^3gt(u)^3 & 3P_3(t,x)(u)^2 \end{pmatrix},$$

and, generically, this may be expected to be a surjective linear map. In that case the tangent vectors to $\Sigma^{2,1,1}f$ are those vectors (t', x') for which there exists a vector u' such that

$$P_2(t,x)(t') + x'\cdot dgt = 0\ ,$$
$$P_3(t,x)(u)(t') + x'\cdot d^2gt(u) + P_2(t,x)(u') = 0\ ,$$
$$P_4(t,x)(u)^3(t') + x'\cdot d^3gt(u)^3 + 3P_3(t,x)(u)^2(u') = 0\ ,$$

where u is a non-zero vector $\in \ker(P_2(t,x))$.

In general, t' is not a multiple of u, since in general $x'\cdot dgt \neq 0$. Indeed when $x'\cdot dgt = 0$, $x' = 0$; for otherwise, since $P_2(t,x)(u) = 0$,

$$x'\cdot d^2gt(u) = \mu dgt(u)\cdot dgt$$

with $\mu \neq 0$; therefore, taking $t' = u$,

$$P_3(t,x)(u)^3 + \mu dgt(u)\cdot dgt(u) = 0\ ,$$

which is nonsense, since $P_3(t,x)(u)^3 = 0$ and $\mu(dgtu)^{(2)} \neq 0$. That is, t' is a multiple of u if, and only if, $(t,x)\in \Sigma^{2,1,1,1}f$, substantiating the assertion about base ribs made in the last section.

For $(t,x)\in \Sigma^{2,1,1,1}f$, we have the existence of a non-zero element u of the kernel of $P_2(t,x)$ and a vector u' such that

$$P_3(t,x)(u)^2 + P_2(t,x)(u') = 0\ ,$$
$$P_4(t,x)(u)^4 + 3P_3(t,x)(u)^2(u') = 0\ .$$

For each (t,x), let

$$-P_2(t,x)^{-1}(P_3(t,x)(u)^2)$$

denote some vector u' satisfying the first of these two equations. Such a u' certainly exists and any two differ by a real multiple of u. Substituting this expression in the second equation, and observing that any ambiguity in u' is harmless, since $P_3(t,x)(u)^2(u) = 0$, we find, for u, the quartic equation

$$P_4(t,x)(u)^4 - 3(P_3(t,x)(u)^2)(P_2(t,x))^{-1}(P_3(t,x)(u)^2) = 0\ .$$

9. Configurations at an umbilic

Let C be a smooth curve on the normal focal set of the surface M passing through a point of $\Sigma^{2,2}f$ and otherwise lying in $\Sigma^{2,1,1}f$. We speak of C loosely as a *rib* of M *through* the umbilical centre. At each point (t,x) of C lying in $\Sigma^{2,1,1}f$ we have a principal direction (in general *not* the tangent direction), with representative u, say, in $\mathbf{R}^2$, such that

$$P_2(t,x)(u) = 0 \quad\text{and}\quad P_3(t,x)(u)^3 = 0\ .$$

Now let (t,x) be the umbilical centre itself, and u a representative of the limit there of the principal direction along C. Then

$$P_2(t,x) = 0 \quad \text{and} \quad P_3(t,x)(u)^3 = 0 \ .$$

Let us recall [29] some elementary facts about a binary cubic, that is, a homogeneous cubic form

$$\mathbf{R}^2 \to \mathbf{R} \ ; \qquad u \mapsto P_3(u)^3$$

with associated 'symmetric trilinear' form

$$P_3 \colon \mathbf{R}^2 \to L_S(\mathbf{R}^2, L(\mathbf{R}^2, \mathbf{R})) \ .$$

The quadratic form

$$\mathbf{R}^2 \to \mathbf{R} \ ; \qquad u \mapsto \det(P_3(u))$$

is called the *Hessian* of the cubic.

The cubic form determines 'three' directions in $\mathbf{R}^2$, or 'three' points of the real projective line $\mathbf{R}P^1$. In general either one or three of these is real. Non-generically there may be two 'coincident' real directions, the other direction also being real, or all three may coincide. Let us name the three points of the projective line A, B and C. The Hessian form likewise determines 'two' directions in $\mathbf{R}^2$ or 'two' points of $\mathbf{R}P^1$. These are real and distinct, the *hyperbolic* case, when only one of A, B and C is real, and they are complex, the *elliptic* case, when all three of A, B, C are real and distinct. The Hessian points coincide when and only when at least two roots of the cubic are coincident, the *parabolic* case.

Let α and $\beta \in \mathbf{R}^2$ represent the two Hessian directions of the cubic. Then $P_3(\alpha)(\beta) = 0$ and, conversely, if $P_3(\alpha)(\beta) = 0$ for some α, β, then α and β represent the Hessian directions. In particular, if $P_3(u)^2 = 0$ with $u \neq 0$, then $u = \alpha$ and β; and we are in the parabolic case.

Suppose u is a non-zero vector such that $P_3(u)^3 = 0$. Then the quadratic equation $P_3(u)(v)^2 = 0$ determines two directions $[v]$, one the line represented by u and another, represented by $\bar{u}$, say. The latter direction is said to be *conjugate* to the first direction.

Returning to our problem, we say that an umbilical centre (t,x) is *hyperbolic* or *elliptic* according as the cubic form

$$u \mapsto P_3(t,x)(u)^3$$

is hyperbolic or elliptic. For a rib C through the umbilical centre the limiting direction of the principal direction along C is one of the one or three determined by the cubic. Such a direction will be called a *principal direction* at the umbilic.

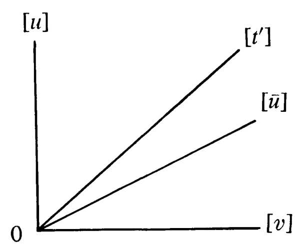
Next, by going again to the limit, a tangent vector (t', x') to C at the umbilical centre must satisfy the equations

$$x' \cdot dgt = 0 \quad \text{and} \quad P_3(t, x)(u)(t') + x' \cdot d^2gt(u) = 0 \; .$$

By a previous argument $x' \cdot d^2gt(u)$ is then a real multiple of $dgt(u) \cdot dgt$. So, if we choose v orthogonal to u, $dgt(u) \cdot dgt(v) = 0$, and

$$P_3(t, x)(u)(t')(v) = 0 \; ;$$

that is, the direction $[t']$ is the harmonic conjugate of the direction $[v]$ with respect to the directions $[u]$ and $[\bar{u}]$.



The vector t' is tangential at the umbilic t (or, rather, $g(t)$) to the base rib $\pi(C)$. It can happen that two principal directions at the umbilic are mutually orthogonal. Suppose, for example, that

$$P_3(t, x)(u)^3 = 0 \quad \text{and} \quad P_3(t, x)(v)^3 = 0$$

with v orthogonal to u, that is,

$$dgt(u) \cdot dgt(v) = 0 \; .$$

Then, since the equation

$$P_3(t, x)(u)(t')(v) = 0$$

can be rewritten as

$$P_3(t, x)(v)(t')(u) = 0 \; ,$$

we see that two base ribs, with different principal directions $[u]$ and $[v]$, can pass through the umbilic in the same direction $[t']$. It is easily seen, however, that the corresponding ribs in the normal focal set are not tangential at the umbilical centre.

There is a nice situation when $\bar{u}$ happens to be orthogonal to u, for then $[t'] = [\bar{u}] = [v]$ and is orthogonal to $[u]$. Notice also that t' is not a multiple of u unless $[\bar{u}] = [u]$, that is, unless the umbilical centre is parabolic.

Next, suppose that C is a smooth curve passing through the umbilical centre, such that the tangent vectors at the centre are of the form $(t', 0)$, and let u be a representative of the limiting principal direction at the umbilic. Then $P_3(t,x)(u)(t') = 0$, so that u and t' represent the Hessian directions at the umbilical centre.

Finally, let C be a line of curvature on M passing through an umbilic $g(t)$ with centre x. In this case all along the curve, and therefore also in the limit, tangent vectors are multiples of u. So, at the umbilical centre,

$$x' \cdot dgt = 0 \quad \text{and} \quad P_3(t,x)(u)^2 + x' \cdot d^2gt(u) = 0 \ ,$$

implying that $P_3(t,x)(u)^2(v) = 0$, where v is orthogonal to u.

Now, if $u \mapsto P_3(u)^3$ is a binary cubic on $\mathbf{R}^2$ and $u \mapsto Q_2(u)^2$ is a binary quadratic on $\mathbf{R}^2$ then, classically [6], the cubic form

$$u \longrightarrow \det \begin{pmatrix} P_3(u)^2 \\ Q_2(u) \end{pmatrix}$$

is known as the *Jacobian* or *first trasvectant* of the cubic and quadratic. For example, the first transvectant of a cubic and its Hessian quadratic form is the cubic form (the cubicovariant) determining the three conjugate directions.

The equation

$$P_3(t,x)(u)^2(v) = 0$$

with $dgt(u) \cdot dgt(v) = 0$ implies that $[u]$ is one of the three directions determined by the first transvectant of the original cubic and the first fundamental form. The reality conditions for this new cubic are *not* in accord with those for the cubic $u \mapsto P(t,x)(u)^3$. Things are, however, again nice if $\bar{u}$ is orthogonal to u, for then $P_3(\bar{u})^2(u) = 0$ with $\bar{u}$ orthogonal to u. In this case the tangent directions of a base-rib and a line of curvature through the umbilic coincide. We noticed this happening in the case of an ellipsoid with distinct semi-axes.

Another case of interest is where two of the directions $[u]$, $[v]$ for lines of curvature through an umbilic are orthogonal. For them we have

$$P_3(t,x)(u)^2(v) = 0 \quad \text{and} \quad P_3(t,x)(v)^2(u) = 0 \ ,$$

implying that

$$P_3(t,x)(u)(v) = 0 \ ,$$

since u, v form a basis for $\mathbf{R}^2$. So $[u]$ and $[v]$ are the Hessian directions. Conversely, if the Hessian directions are orthogonal then two of the directions for lines of curvature are orthogonal. It is a nice problem to sketch the lines of curvature near the umbilic in such a case. This is an example where there are three directions for lines of curvature at the umbilic, but only one rib through the umbilical centre.

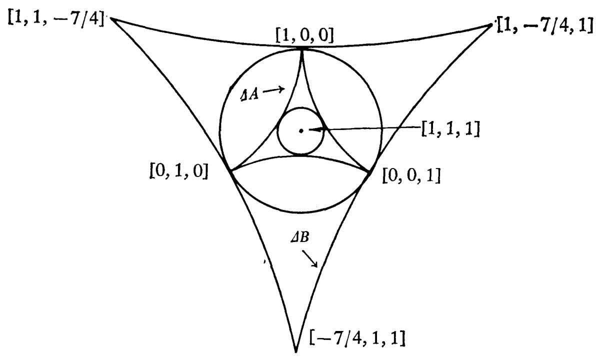
The problem of describing the lines of curvature near an umbilic was studied by Hamilton, Frost [9] Cayley [2], [3] and others, and is discussed fully, in the generic case, by Darboux [5]. When there is only one curvature direction at the umbilic the configuration is what one might expect, with a single line of curvature in the given direction and the lines of curvature resembling confocal coaxial parabolas, with axis the line of curvature and with focus the umbilic. However, when there are three possible directions at the umbilic there are two cases according as the three directions can or cannot be contained within a sector of angle $\pi/2$. When they can be so contained the picture is surprising— the 'middle' direction is tangential to a whole family of lines of curvature through the umbilic. When they cannot, there is just one line of curvature in each of the three directions. See [9], [5, p. 455] and [11, pp. 91–92], for pictures. Gullstrand, whose interest in umbilics arose out of work on eyesight aberration [10], goes father than Darboux. He is aware of the importance of the ribs near an umbilical centre [11, pp. 87–93], though he contrives to give the impression that base ribs approach umbilics in principal curvature directions. He seems also to be the first to put due emphasis on the discriminant of the cubic form giving the principal directions at an umbilic.

The *index* of an umbilic is defined by the configuration of lines of curvature there. It plays an important role in work on the Caratheodory conjecture (see, for example [12] and [28]), and has also been studied recently by Feldman [8] amongst others. When there is only one curvature direction at the umbilic, the index of the umbilic in $1/2$. When there are three curvature directions, the index is either $1/2$ or $-1/2$ according as the three directions can or cannot be contained within a sector of angle $\pi/2$. In the case where two of the curvature directions are mutually orthogonal, the index may then be $1/2$, 0 or $-1/2$.

To illustrate the various possibilities in relation to one another consider those cubic forms (a, b, c) on $\mathbf{R}^2$ defined, in polar coordinates, by the formula

$$(a, b, c)(r \cos\theta, r\sin\theta) = r^3(a\cos^3\theta + b\cos^3(\theta + 2\pi/3) + c\cos^3(\theta - 2\pi/3))$$

(These form a three-dimensional subspace of the four-dimensional linear space of all real cubic forms on $\mathbf{R}^2$.) We regard two of these forms as equivalent if one is a non-zero real multiple of the other. The equivalence classes of non-zero forms then form a real projective plane. The Jacobian of a cubic form $A = (a, b, c)$ and the quadratic form $(r\cos\theta, r\sin\theta) \mapsto r^2$ is the cubic form $B = \partial A/\partial\theta$. The discriminants ΔA and ΔB of A and B are each tricuspidal quartics in the projective plane, the cusps of ΔA lying at the vertices of the triangle of reference and corresponding to cubic forms with three coincident roots. The cusps of ΔA lie on ΔB as shown in the following sketch.



Inside ΔA, A and B each have three real distinct roots, two of the roots of A being mutually orthogonal at points of the smaller 'circle'. Inside ΔB, but outside ΔA, A has one real root, while B has three, two of the roots of B being mutually orthogonal at points of the larger 'circle'. Outside ΔB, A and B each have only one real root. When A is the cubic giving the principal directions at an umbilic and the quadratic form is the first fundamental form there, the index of the umbilic is $-1/2$ or $1/2$ according as A lies inside or outside the larger 'circle'. In this case, B is the cubic giving the directions of lines of curvature at the umbilic.

10. Surfaces in $\mathbf{R}^4$

Let $m = 2$, $n = 4$. Again $\Sigma^2\phi = NM$. On any normal plane NM_w the focal set is a conic which has no tangents passing through w [14]. Generically there is a curve, the curve of umbilics [17], in M over each point of which the focal conic is a pair of real intersecting lines. The set $\Sigma^{2,1}\phi$ is the non-singular part of the focal set, while $\Sigma^{2,2}\phi$ consists of the nodes of the focal line-pairs. The reciprocal of the focal conic with respect to the unit circle is an ellipse, the *ellipse of curvature*, which degenerates to a line segment in the case that the focal conic is a line-pair. (Cf. also [30], [22], [23] and [16]. Other earlier references are to be found in [25] and [17].)

Further singularities in the generic case are $\Sigma^{2,1,1}\phi$, a smooth surface in NM, $\Sigma^{2,1,1,1}\phi$, the inverse image by ϕ of the cuspidal edge or edge of regression of the image of $\Sigma^{2,1,1}\phi$ in $\mathbf{R}^4$ and, finally, $\Sigma^{2,1,1,1,1}\phi$, a discrete set of points on $\Sigma^{2,1,1,1}\phi$, mapped by ϕ to cusps on the cuspidal edge of the image of $\Sigma^{2,1,1}\phi$.

11. Parabolic umbilics

The details, for a surface in $\mathbf{R}^4$, are to a large extent simple modifications of the details for a surface in $\mathbf{R}^3$, with $\mathbf{R}^4$ replacing $\mathbf{R}^3$. As in that case

$$P_1(t,x) = 0 \quad \text{on} \quad \Sigma^2 f\,,$$
$$P_1(t,x) = 0 \quad \text{and} \quad P_2(t,x)(u) = 0 \quad \text{on} \quad \Sigma^{2,1} f\,,$$
$$\text{and} \quad P_1(t,x) = 0\,, \quad P_2(t,x)(u) = 0 \quad \text{and} \quad P_3(t,x)(u)^3 = 0 \quad \text{on} \quad \Sigma^{2,1,1} f\,,$$

the vector u in the last two cases being some non-zero element of $\mathbf{R}^2$. Again, as in that case, the additional condition for (t, x) to belong to $\Sigma^{2,1,1,1}f$ is that

$$P_4(t,x)(u)^4 - 3(P_3(t,x)(u)^2)(P_2(t,x))^{-1}(P_3(t,x)(u)^2) = 0\,.$$

The set $\Sigma^{2,1,1,1}f$ is generically a curve in $\mathbf{R}^2 \times \mathbf{R}^4$. Again also, at a point (t, x) of $\Sigma^{2,2}f$,

$$P_1(t,x) = 0 \quad \text{and} \quad P_2(t,x) = 0\,,$$

and there is defined there a cubic form

$$\mathbf{R}^2 \to \mathbf{R}\,;\ u \mapsto P_3(t,x)(u)^3\,,$$

according to which the points of $\Sigma^{2,2}f$, the *umbilical centres* of M, may be classified as elliptic, hyperbolic or parabolic, just as before. The only difference is that parabolic points may be expected to occur generically. The set $\Sigma^{2,2}f$ generically is a curve in $\mathbf{R}^2 \times \mathbf{R}^4$ and the parabolic points form a discrete set of points on the curve.

Consider a smooth curve C on the normal focal set passing through a point of $\Sigma^{2,2}f$ and otherwise lying in $\Sigma^{2,1,1,1}f$, of dimension 1 as we remarked earlier. At all points of C, other than the umbilical centre, $P_2(t, x)$ has rank 1, but at the centre itself it has rank 0, being 0. Taking the expression

$$P_4(u)^4 - 3(P_3(u)^2)P_2^{-1}(P_3(u)^2) = 0$$

to the limit one can easily see that, for the limiting direction $[u]$ at the umbilical centre (t, x),

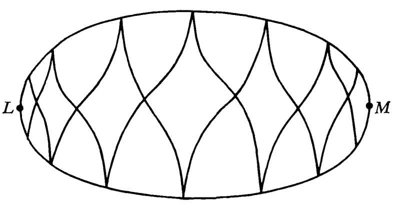
$$P_3(t,x)(u)^2 = 0\,.$$

But this is just the condition for the umbilical centre to be parabolic. As for the tangent direction to C at U, let (t', x') be some tangent vector there. Then

$$x' \cdot dgt = 0\,, \quad x' \cdot d^2gt(u)^2 = 0\,, \quad \text{and} \quad P_3(t,x)(u)(t') + x' \cdot d^2gt(u) = 0\,,$$

and these are in general sufficient to determine (t', x') up to a real factor. (Since the dimension of the normal space is now >1, it is no longer true that $x' \cdot dgt = 0$ implies that x' is a multiple of $x - g(t)$.)

It is natural to suppose at first that the parabolic points of $\Sigma^{2,2}f$ belong to $\Sigma^{2,2,1}f$ but this is not so. The singularity type $\Sigma^{2,2,1}$ has codimension 7 and therefore does not occur generically for surfaces in $\mathbf{R}^4$. It will occur for surfaces in $\mathbf{R}^5$. In that case, for a surface M, the focal set in each normal three-dimensional space is a quadric cone, and points $g(t)$ of M may be classified as elliptic, hyperbolic or parabolic according to the nature of the cubic form $P_3(t, x)$ at the vertex of the cone in the normal space at $g(t)$. The parabolic points form a curve on the surface separating the elliptic and hyperbolic regions. At a hyperbolic point two real directions on the surface are determined by the Hessian of the cubic. One gets situations such as that illustrated by the figure



where the oval consists of parabolic points, with elliptic points outside and hyperbolic points inside, the points L and M being points where the vertex of the normal focal quadric belongs to $\Sigma^{2,2,1}f$. The lines shown inside the oval are some of the Hessian lines. Further details are left to the reader!

12. Flexional points

As we remarked earlier in § 7 we have ignored the nature of the focal set at infinity. In the case of a surface in $\mathbf{R}^3$ a point where one of the two principal centres of curvature lies at infinity is classically said to be a *parabolic point*, but to avoid confusion with the other use of the word 'parabolic' in this paper we refer to such a point as a *flexional point* of the surface, in accord with the even simpler case of a curve in $\mathbf{R}^2$, in which case a *flex* or *inflexion* is a point where the unique centre of curvature lies at infinity. The inclusion of flexional points in the argument may be effected by a simple 'compactification'

$$\bar{f}\colon \mathbf{R}^m \times \mathbf{R}P^{n+1} \rightarrowtail S^1 \times \mathbf{R}P^{n+1}$$

of the original local map f. Details are left to the reader.

13. Thom's elementary catastrophes

Thom's elementary catastrophes [22, Chapter 4] provide analogues to the material of this paper.

As an example consider the function

$$h\colon \mathbf{R}^2 \to \mathbf{R}\ ; \qquad (x, y) \mapsto x^3 + y^3\ .$$

Then, as Thom has remarked and Kuo [15] and Mather have verified, any function close to h is topologically or differentiably equivalent to one of the forms

$$\mathbf{R}^2 \to \mathbf{R} \; ; \qquad (x, y) \mapsto x^3 + y^3 + axy + bx + cy \; ;$$

which leads us to consider the map

$$f \colon \mathbf{R}^2 \times \mathbf{R}^3 \to \mathbf{R} \times \mathbf{R}^3 \, ,$$
$$((x, y), (a, b, c)) \mapsto (x^3 + y^3 + axy + bx + cy, (a, b, c)) \; .$$

What can one say of the Whitney-Thom singularities of this map? Clearly, $\Sigma^2 f$ may be regarded as a line-bundle over $\mathbf{R}^2$, for the condition for $((x, y), (a, b, c))$ to be a point of $\Sigma^2 f$ is that

$$3x^2 + ay + b = 0 \quad \text{and} \quad 3y^2 + ax + c = 0 \, ,$$

defining, for each $(x, y) \in \mathbf{R}^2$, an affine line in $\mathbf{R}^3$. The reader may care to pursue further the analogy with a surface in $\mathbf{R}^3$, and to show that the origin in $\mathbf{R}^2 \times \mathbf{R}^3$ is in this case a point of $\Sigma^{2,2} f$ of hyperbolic type with a single rib passing through it.

In similar fashion the map

$$f \colon \mathbf{R}^2 \times \mathbf{R}^3 \to \mathbf{R} \times \mathbf{R}^3 \, ,$$
$$((x, y), (a, b, c)) \mapsto (x^3 - 3xy^2 + a(x^2 + y^2) + bx + cy, (a, b, c))$$

exhibits an elliptic umbilical centre at the origin, with three ribs passing through it.

Finally, a parabolic umbilical centre is exhibited at the origin by the map

$$f \colon \mathbf{R}^2 \times \mathbf{R}^4 \to \mathbf{R} \times \mathbf{R}^4 \, ,$$
$$((x, y), (a, b, c, d)) \mapsto (x^2 y + y^4 + ax^2 + by^2 + cx + dy, (a, b, c, d)) \, ,$$

the analogy here being with a surface in $\mathbf{R}^4$.

14. Intrinsic differentials

This paper provides several examples of the *intrinsic higher order differentials* of a smooth map. These differentials, first presented in [14] and in a short talk at the Moscow International Congress in 1966, are also discussed in [1]. The following is a brief sketch of the author's original approach to them.

Let $f \colon X \to Y$ be a smooth map, X and Y being smooth manifolds. The first intrinsic differential $\partial_1 f$ of f is the tangent map $Tf \colon TX \to TY$. Let $\mathscr{G}_a(TX)$ denote the Grassmannian of a-planes in the fibres of TX. Then there is a smooth injection $s^a\colon \Sigma^a f \to \mathscr{G}_a(TX)$ sending each point x of $\Sigma^a f$ to the kernel of the tangent map there $Tf_x\colon TX_x \to TY_{f(x)}$. On $\mathscr{G}_a(TX)$ we have following diagram of bundles

$$\begin{tikzcd} K_a \arrow[r] \arrow[drr, "f^a"'] & TX \arrow[r] \arrow[dr, "Tf"] & M_a \\ & & TY \end{tikzcd}$$

where TX and TY should be replaced, strictly, by their pull-backs to $\mathscr{G}_a(TX)$ by the obvious routes. The bundle K_a is the canonical a-plane bundle over $\mathscr{G}_a(TX)$, associating to each point of $\mathscr{G}_a(TX)$ itself as an a-plane. The bundle M_a is the quotient bundle TX/K_a, with rank $\dim X - a$.

Clearly $s^a(\Sigma^a f)$ is a subset of the set of zeros of f^a, the subset $\Sigma^a f^a$ of $\mathscr{G}_a(TX)$, and this is the set of zeros of the induced section of the bundle $\operatorname{Hom}(K_a, TY)$. The generic case is when this section is transversal to the zero section. In that case, $\operatorname{Hom}(K_a, TY)$, restricted to the set of zeros, is naturally isomorphic to the normal bundle of the set of zeros in $\mathscr{G}_a(TX)$. Moreover the tangent bundle along the fibres of $\mathscr{G}_a(TX)$ is naturally isomorphic to $\operatorname{Hom}(K_a, M_a)$ (Cf. [20, p. 411]). Now, on $s^a(\Sigma^a f)$, the bundle K_a is exactly the kernel of Tf, so the diagram of bundles extends to

$$\begin{tikzcd} K_a \arrow[r] \arrow[drr, "f^a"'] & TX \arrow[r] \arrow[dr, "Tf = \partial_1 f"] & M_a \arrow[d] \\ & & TY \arrow[d] \\ & & Q_a \end{tikzcd}$$

where $Q_a = TY/M_a$. From an easy argument it follows that the normal bundle of $\Sigma^a f$ in X is naturally isomorphic to $\operatorname{Hom}(K_a, Q_a)$, pulled back to $\Sigma^a f$ from $\mathscr{G}_a(TX)$ by s^a. On $\Sigma^a f$ we have the diagram of bundles:

$$\begin{tikzcd} & K_a \arrow[d] \arrow[dr, "\partial_1 f"] & \\ T\Sigma^a f \arrow[r] \arrow[dr] & TX \arrow[r] \arrow[d] \arrow[dr, "Tf"] & \operatorname{Hom}(K_a, Q_a) \\ & M_a \arrow[r] & TY \end{tikzcd}$$

For $\Sigma^{a,b}f$ we take those points where $Tf\,|\,T\Sigma^a f$ has kernel rank b, but, by easy diagram-chasing, this is exactly where the induced map

$$\partial_2 f\colon K_a \to \operatorname{Hom}(K_a, Q_a)$$

has kernel rank b. This map, which can be shown to be symmetric in the obvious sense, is what we call the *second intrinsic differential* of f. It is defined at each point $x \in X$, $(K_a)_x$ being the kernel of $(\partial_1 f)_x$ and $(Q_a)_x$ being the cokernel of $(\partial_1 f)_x$.

In the examples discussed in the paper it is easy to see that at a point (t, x) of $\Sigma^2 f$ in $\mathbf{R}^2 \times \mathbf{R}^3$, $(K_2)_{(t,x)}$ is isomorphic to $\mathbf{R}^2$, naturally, $(Q_2)_{(t,x)}$ is isomorphic to $\mathbf{R}$, naturally, and up to that factor -2, the map

$$P_2(t, x)\colon \mathbf{R}^2 \to L(\mathbf{R}^2, \mathbf{R})$$

is just the second intrinsic differential at $(t, x) \in \Sigma^2 f$.

In the general case one proceeds further by analysing the bundle map

$$\partial_2 f\colon K_a \to \operatorname{Hom}(K_a, Q_a)$$

over $\Sigma^a f$ in like fashion. The normal bundle of $\Sigma^{a,b}f$ in $\Sigma^a f$ is generically isomorphic to $\operatorname{Hom}_S(K_{a,b}, Q_{a,b})$ where $K_{a,b}$ is the kernel of $\partial_2 f$ over $\Sigma^{a,b}f$ and $Q_{a,b}$ is the cokernel of $\partial_2 f$ over $\Sigma^{a,b}f$, the S denoting that the bundle homomorphisms $K_{a,b} \to Q_{a,b}$ which we consider are all symmetric in the appropriate sense of that word. The third-order singularities of f are then definable in terms of the kernel rank of the third intrinsic differential

$$\partial_3 f\colon K_{a,b} \to \operatorname{Hom}(K_{a,b}, Q_{a,b}) ,$$

and the fourth intrinsic differential in terms of its kernel and symmetric maps of its kernel to its cokernel, and so on.

In the examples discussed in the paper,

$$P_3(t, x)\colon R^2 \to L_S(R^2, L(R^2, R))$$

is essentially the third intrinsic differential of f, for $(t, x) \in \Sigma^{2,2}f$.

For $(t, x) \in \Sigma^{2,1}f$, $(K_{2,1})_{(t,x)} = [u]$. Moreover, since $\partial_2 f\colon R^2 \to (R^2)^* = L(\mathbf{R}^2, \mathbf{R})$ is a self-dual map, $(Q_{2,1})_{t,x}$ is isomorphic to $[u]^* = L([u], \mathbf{R})$, the sequence

$$\{0\} \longrightarrow [u] \xrightarrow{\ i\ } \mathbf{R}^2 \xrightarrow{\ \partial_2 f\ } L(\mathbf{R}^2, \mathbf{R}) \xrightarrow{\ i^*\ } L([u], \mathbf{R}) \longrightarrow \{0\} ,$$

where i is the inclusion, and i^* its dual, being exact. So the restriction of $P_3(t, x)$ to $[u]$,

$$[u] \to L([u], L([u], \mathbf{R})) ,$$

is just the third intrinsic differential of f, for $(t, x) \in \Sigma^{2,1}f$. As we saw in § 8, this is zero exactly when $(t, x) \in \Sigma^{2,1,1}f$.

Higher-order intrinsic differentials become more complex in their expression in terms of local coordinates, as is exemplified by the fourth intrinsic differential $\partial_4(t, x)$ of f at a point (t, x) of $\Sigma^{2,1,1}f$, which (cf. the end of § 8) is the map $[u] \to L([u], L([u], L([u], \mathbf{R})))$ defined by

$$\partial_4 f(t, x)(u)^4 = -2(P_4(t, x)(u)^4 - 3(P_3(t, x)(u)^2)(P_2(t, x))^{-1}(P_3(t, x)(u)^2)) .$$

This is a linear map

$$(K_{2,1,1})_{(t,x)} \to L_S(K_{2,1,1})_{(t,x)}, (Q_{2,1,1})_{(t,x)}) ,$$

as it should be, since $(K_{2,1,1})_{(t,x)} = [u]$ and

$$(Q_{2,1,1})_{(t,x)} = L([u], L([u], \mathbf{R})) .$$

References

[1] J. M. Boardman, *Singularities of differentiable maps*, Inst. Hautes Études Sci. Publ. Math. No. 33 (1967) 21–57.

[2] A. Cayley, *On differential equations and umbilici*, Philos. Mag. **26** (1863) 373–379, 441–452 (Collected Works, Vol. V, paper 330).

[3] ——, *Note on Mr. Frost's paper on the direction of lines of curvature in the neighbourhood of an umbilicus*, Quart. J. Math. **10** (1870) 111–113 (Collected Works Vol. VII, paper 460).

[4] ——, *On the centro-surface of an ellipsoid*, Trans. Cambridge Philos. Soc. **12** (1873) 319–365 (Collected Works Vol. VIII, paper 520).

[5] G. Darboux, *Leçons sur la théorie générales des surfaces*, Vol. 4, Gauthier-Villars, Paris, 1896.

[6] E. B. Elliott, *An introduction to the algebra of quantics*, Clarendon, Oxford, 1895.

[7] E. A. Feldman, *The geometry of immersions*. II, Bull. Amer. Math. Soc. **70** (1964) 600–607.

[8] ——, *On parabolic and umbilic points of immersed hypersurfaces*, Trans. Amer. Math. Soc. **127** (1967) 1–28.

[9] P. Frost, *On the directions of lines of curvature in the neighbourhood of an umbilicus*, Quart. J. Math. **10** (1870) 78–86 and Plate 1 facing p. 96.

[10] A. Gullstrand, *Allgemeine Theorie der monochromatischen Aberrationen und ihre nächsten Ergebnisse für die Ophthalmologie*, Nova Acta Regiae Soc. Sci. Ups. Ser. III (1900).

[11] ——, *Zur Kenntniss der Kreispunkte*, Acta Math. **29** (1905) 59–100.

[12] H. Hamburger, *Beweis einer Caratheodoryschen Vermutung*. I, Ann. of Math. (2) **41** (1940) 63–86; II, III, Acta Math. **73** (1941) 175–228, 229–332.

[13] P. Hartman & A. Wintner, *On the singularities in nets of curves defined by differential equations*, Amer. J. Math. **75** (1953) 277–297.

[14] K. Kommerell, *Riemannschen Flächen im ebenen Raum von vier Dimensionen*, Math. Ann. **60** (1905) 548–596.

[15] T. C. Kuo, *On C^0-sufficiency of jets of potential functions*, Topology **8** (1969) 167–171.

[16] J. A. Little, *On singularities of submanifolds of higher dimensional Euclidean spaces*, Ann. Mat. Pura Appl. (IV) **83** (1969) 261–336.

[17] D. Perepelkine, *Sur la courbure et les espaces normaux d'une V_m dans R_n*, Rec. Math. (Mat. Sbornik) N.S. **42** (1935) 81–100.

[18] E. Picard, *Traité d'analyse*. III, Gauthier-Villars, Paris, 1896, 225–227.
[19] I. R. Porteous, *Simple singularities of maps,* Columbia Universtiy notes, 1962; reprinted with slight revision in Proc. Liverpool Singularities Sympos. I, Lecture Notes in Math. Vol. 192, Springer, Berlin, 1971, 286–307.
[20] ——, *Topological geometry*, Van Nostrand Reinhold, London, 1969.
[21] ——, *Geometric differentiation—a Thomist view of differential geometry,* Proc. Liverpool Singularities Sympos. II, Lecture Notes in Math. Springer, Berlin, 1971.
[22] K. S. Ramazanova, *On the theory of two-dimensional surfaces in* E_4, Volzh. Mat. Sb. Vyp. **3** (1965) 296–311 (Russian).
[23] ——. *The theory of curvature of* X_2 *in* E_4, Izv. Vyssh. Učhebn. Zaved. Mat. **6** (**55**) (1966) 137–143 (Russian).
[24] S. A. Robertson, *Toroidal maps*, Sympos. seminar on singularities of smooth manifolds and maps (Liverpool), 1969–70.
[25] J. A. Schouten & D. J. Struik, *Einführing in die neueren Methoden der differential Geometrie*, Vol. 2, Groningen-Batavia, 1938.
[26] R. Thom, *Sur la théorie des enveloppes*, J. Math. Pures Appl. **41** (1962) 177–192.
[27] ——, *Stabilité structurelle et morphogénèse*, Benjamin, New York, to appear.
[28] C. J. Titus, *A proof of the Caratheodory conjecture concerning umbilic points*, to appear.
[29] J. A. Todd, *Projective and analytical geometry*, Pitman, London, 1947.
[30] Y. C. Wong, *A new curvature theory for surfaces in a Euclidean 4-space,* Comment Math. Helv. **26** (1952) 152–170.

University of Liverpool