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SPECTRUM AND SEMIGROUPS OF ELLIPTIC OPERATORS ON HOMOGENEOUS SPACES

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1. Introduction and notation

For a manifold X we denote the bounded continuous real (resp. complex) valued functions on X by $\mathscr{B}(X)$(resp. $\mathscr{B}(X, C)$). For $f \in \mathscr{B}(X)$(resp. $\mathscr{B}(X, C)$) we set $\|f\| = \sup_{x \in X} |f(x)|$. Let $\mathscr{B}^n(X)$(resp. $\mathscr{B}^n(X, C)$) denote the real (resp. complex) valued functions which are bounded and continuously differentiable of order n.

Let G be a Lie group and H a closed subgroup. Let Δ be a second order elliptic differential operator on G/H, which is invariant under the left action of G. Assume that in any local coordinate neighborhood Δ is of the form:

$$a_{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + b_k(x)\frac{\partial}{\partial x^k} ,$$

where $(a_{ij}(x))$ is a positive definite symmetric matrix and the $b_k(x)$'s are real. (We use Einstein summation convention.) Unless otherwise stated we will assume G, H and Δ as above.

In § 2, we show that Δ generates a continuous semigroup of probablility measures on $\beta(G/H)$, the Stone-Čech compactification of G/H. This extends a result of Hunt [3]. We also obtain restrictions on the spectrum of Δ. If, moreover, G/H admits a G-invariant Riemannian metric and Δ is the Laplacian of this metric, then the above results are strengthened.

Throughout this paper, the crucial technique is given by Lemma 2.1 which has been proven by Omori [4] under somewhat different circumstances.

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2. Spectrum and semigroup

Lemma 2.1. *Let f be a real valued C^2-function on G/H, which is bounded from above. Then for an arbitrarily fixed point p of G/H and for any* $\varepsilon > 0$,

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there exists a q depending on p such that: (i) $f(q) \geq f(p)$, *and* (ii) $(\Delta f)(q) < \varepsilon$.

Proof. Let $b = \sup f$. Select $\varphi \in \mathscr{B}^2(G/H)$ such that $\varphi \geq 0$, $\|\Delta\varphi\| < \varepsilon$, $\varphi(eH) > 0$ and $\operatorname{supp}\varphi = C$ is compact. There exists $q' \in G/H$ such that:

$$f(q') + \varphi(eH) > b .$$

Let $g' \in G$ such that g' projects to q' under the natural map $G \to G/H$. Set

$$h(x) = f(x) + (L_{g'}\varphi)(x) = f(x) + \varphi(g'^{-1}x) .$$

Now for $x \notin g'^{-1}(C)$, $h(x) \leq b$. Thus h attains its maximum for some $q \in g'^{-1}(C)$. Hence

$$(\Delta h)(q) \leq 0 .$$

So

$$0 \geq (\Delta f)(q) + \Delta(L_{g'}\varphi)(q) = (\Delta f)(q) + (\Delta\varphi)(g'^{-1}q) ,$$

and

$$(\Delta f)(q) \leq -(\Delta\varphi)(g'^{-1}q) \leq \|\Delta\varphi\| < \varepsilon .$$

Proposition 2.1. a) *Suppose $f \in \mathscr{B}^2(G/H)$ and $\Delta f = \lambda f$. If $\lambda > 0$, then $f \equiv 0$.* b) *Suppose Δ is the Laplacian of a G-invariant metric on G/H and $f \in \mathscr{B}^2(G/H, C)$ and $\Delta f = \lambda f$. If* $\operatorname{Re}\lambda > 0$, *then $f \equiv 0$.*

Proof. a) From Lemma 2.1, there exist sequences of points p_n, q_n in G/H and a sequence of $\varepsilon_n > 0$ such that:

(i) $\lim_{n\to\infty} f(q_n) = \sup f$, $\lim_{n\to\infty} f(p_n) = \inf f$,

(ii) $(\Delta f)(q_n) < \varepsilon_n$, $\Delta f(p_n) > -\varepsilon_n$, and

(iii) $\lim_{n\to\infty} \varepsilon_n = 0$.

As $\Delta f = \lambda f$, $\lambda f(q_n) < \varepsilon_n$ and $\lambda f(p_n) > -\varepsilon_n$ and hence $\lambda \sup f \leq 0$ and $\lambda \inf f \geq 0$. As $\lambda > 0$, $0 \geq \inf f \geq \sup f \geq 0$ and $f \equiv 0$.

b) Suppose $f(x) = u(x) + iv(x)$ and let $h(x) = |f(x)|^2$. Then from Proposition 2.1 there exist a sequence of points q_n and a sequence $\varepsilon_n > 0$ such that

(i) $\lim_{n\to\infty} h(q_n) = \sup h$,

(ii) $(\Delta h)(q_n) < \varepsilon_n$, and

(iii) $\lim_{n\to\infty} \varepsilon_n = 0$.

Let $\langle\ ,\ \rangle$ denote the G-invariant Riemannian metric on G/H. Then

$$\begin{aligned}(\Delta h)(x) &= (\Delta f)(x)\bar{f}(x) + f(x)\Delta\bar{f}(x) + 2\langle \operatorname{grad} u, \operatorname{grad} u\rangle_x \\ &\quad + 2\langle \operatorname{grad} v, \operatorname{grad} v\rangle_x \geq (\Delta f)(x)\bar{f}(x) + f(x)\Delta\bar{f}(x) \\ &= 2\operatorname{Re}\lambda\,|f(x)|^2 .\end{aligned}$$

Thus $\varepsilon_n \geq \Delta h(q_n) \geq 2\,\mathrm{Re}\,\lambda|f(q_n)|^2$ and $0 \geq \sup h$. As $h \geq 0$, $h \equiv 0$ and therefore $f \equiv 0$. q.e.d.

This generalizes a result of E. B. Dynkin [1] on symmetric spaces.

Lemma 2.2. a) *If $f \in \mathscr{B}^2(G/H)$ and $\lambda \geq 0$, then $\|(\Delta - \lambda)f\| \geq \lambda\|f\|$.* b) *If $f \in \mathscr{B}^2(G/H, C)$, Δ is the Laplacian of a G-invariant Riemannian metric on G/H, and $\lambda \geq 0$, then $\|(\Delta - \lambda)f\| \geq \lambda\|f\|$.*

Proof. a) Suppose $\|f\| = \sup f$. Select a sequence q_n in G/H and a sequence $\varepsilon_n > 0$ such that:

(i) $\lim_{n\to\infty} f(q_n) = \sup f$,

(ii) $(\Delta f)(q_n) < \varepsilon_n$, and

(iii) $\lim_{n\to\infty} \varepsilon_n = 0$.

Then $(\Delta - \lambda)f(q_n) \leq \varepsilon_n - \lambda f(q_n)$, and hence

$$\inf (\Delta - \lambda)f \leq -\lambda \sup f = -\lambda\|f\| .$$

Thus $\|(\Delta - \lambda)(f)\| \geq \lambda\|f\|$. If $\|f\| = -\inf f$, the proof is similar. Part b) is proved by setting $h(x) = |f(x)|^2$ and proceeding as above.

Lemma 2.3. *Let v_n, $n \geq 1$, be a sequence of functions in $C^2(G/H)$ converging uniformly on compact subsets to 0, and suppose Δv_n converges uniformly on compact subsets to f. Then $f \equiv 0$.*

Proof. Suppose not. Then we may assume that there is an open set U with compact closure such that $f|U > B > 0$. Without loss of generality we may assume $\sup_{x\in U}|v_n(x)| \leq 1/2^n$.

Let $V \subset U$ be an open set such that $\bar{V} \subset U$, and $\phi \in C_0^\infty(V)$ be such that $\phi \geq 0$, $\sup\phi = 1$, and $|\Delta\phi| \leq C$. Then $v_n + \phi/2^{n-1}$ attains a local maximum at some point $x_n \in V$. Thus

$$\Delta v_n(x_n) + (1/2^{n-1})\Delta\phi(x_n) \leq 0 .$$

Hence

$$\Delta v_n(x_n) \leq (1/2^{n-1})C ,$$

and for all n we has $x_n \in V$ satisfying the above inequality. This contradicts the uniform convergence of Δv_n on $\bar{V}$ to f. q.e.d.

Thus, if g_n, $n \geq 1$, is a sequence in $C^2(G/H)$, g_n converges uniformly on compact sets to g, and Δg_n converges uniformly on compact sets to f, then we may say $\Delta g = f$. From now on we shall identify Δ with this extended operator.

We now consider the problem of solving the equation $(\Delta - \lambda)g = f$ for $f \in \mathscr{B}(G/H)$ and $\lambda > 0$ with $g \in \mathscr{B}(G/H)$.

Proposition 2.2. *Let $f \in \mathscr{B}(G/H)$ and $\lambda > 0$. Then there exists $g \in \mathscr{B}(G/H)$ such that $(\Delta - \lambda)g = f$.*

Proof. Suppose first that f is C^∞. It is clear that we may assume $f \geq 0$.

Put on G/H a complete C^∞-Riemannian metric. Let $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots$ be a sequence of open subsets of G/H with smooth boundaries such that $\bar{\Omega}_n$ is compact, $\bar{\Omega}_n \subset \Omega_{n+1}$, and $G/H = \bigcup_{n=1}^{\infty} \Omega_n$. Let L be the Laplacian with respect to this metric, and $D_t = t(\Delta - \lambda) + (1-t)L$, and consider the elliptic boundary problem:

$D_t u = F$ on $\bar{\Omega}_n$ and $u = f$ on $\partial\Omega_n$ for $F \in C^\infty(\bar{\Omega}_n)$ and $f \in C^\infty(\partial\Omega_n)$. As $\operatorname{ind} L = 0$ [4, p. 264] we see that $\operatorname{ind} D_t = 0$ for all t, and therefore that $\operatorname{ind}(\Delta - \lambda) = 0$. If $u \in C^\infty(\bar{\Omega}_n)$, and $u = 0$ on $\partial\Omega_n$, we have as in Lemma 2.1 that $\sup_{x \in \Omega_n} |(\Delta - \lambda)u(x)| \geq \lambda \sup_{x \in \Omega_n} |u(x)|$. Thus $\ker(\Delta - \lambda) = 0$. Hence we may find a unique $u_n \in C^\infty(\bar{\Omega}_n)$, $u_n = 0$ in $\partial\Omega_n$, such that $(\Delta - \lambda)u_n = f$ in $\bar{\Omega}_n$. Set $u_n = 0$ on the complement of Ω_n.

Consider the functions $v_n = u_{n+1} - u_n$ and u_n. On the complement of Ω_n we have that $v_n \leq 0$ and $u_n \leq 0$. We claim that $v_n \leq 0$ and $u_n \leq 0$ everywhere. Otherwise, we can find points $x_0, y_0 \in \Omega_n$ such that $0 < v_n(x_0) = \sup v_n(x)$, and $0 < u_n(y_0) = \sup u_n(x)$. However, we must have $\Delta v_n(x_0) \leq 0$ and $\Delta u_n(y_0) \leq 0$, but in Ω_n we have that $0 \geq \Delta v_n(x_0) = \lambda v_n(x_0) > 0$ which is a contradiction, and $0 \geq \Delta u_n(y_0) = f(y_0) + \lambda u_n(y_0) > 0$ which is also a contradiction. Thus $u_n \leq 0$ and $u_{n+1} \leq u_n$ for all n.

Since the u_n's form a bounded monotone sequence of functions, $\lim_{n\to\infty} u_n = g$ exists in the distribution topology on G/H. Hence $(\Delta - \lambda)g = f$ as distributions. As f is C^∞ and Δ has C^∞-coefficients, we have that g is C^∞ and $(\Delta - \lambda)g = f$ as functions. Moreover, as $\|u_n\| \leq \|f\|/\lambda$ for all n, $\|g\| \leq \|f\|/\lambda$.

Suppose now only that $f \in \mathscr{B}(G/H)$. Select a sequence $f_n \in \mathscr{B}(G/H)$, $n \geq 1$, such that f_n is C^∞ for all n and f_n converges to f in $\mathscr{B}(G/H)$. Then there exists a sequence $g_n \in \mathscr{B}(G/H)$, $n \geq 1$, such that g_n is C^∞ for all n and $(\Delta - \lambda)g_n = f_n$. By Lemmas 2.2 and 2.3, g_n converges to a g in $\mathscr{B}(G/H)$, where $(\Delta - \lambda)g = f$.

Remark. The proof of the above proposition is a simplification of the original proof. Its improvement rests on an observation which was pointed out to the author by the referee.

We may now apply the Hille-Yosida theorem to obtain that for $\lambda > 0$, $(1 - \Delta/\lambda)^{-1}$ is a continuous operator of norm 1 on $\mathscr{B}(G/H)$,

$$T_t = \exp t\Delta = \lim_{n\to\infty} (1 - t\Delta/n)^{-n}$$

defines a continuous operator of norm 1 on $\mathscr{B}(G/H)$ which commutes with the left action of G, and finally, for $f \in \mathscr{B}^2(G/H)$,

$$\lim_{t\to 0} \frac{(T_t - 1)}{t} f(x) = (\Delta f)(x) .$$

Let $\Phi_t(f) = (T_t f)(e)$. Then Φ_t is a continuous functional of norm 1 on

$\mathscr{B}(G/H)$, and thus defines a measure on $\mathscr{B}(G/H)$. Note that $\Phi_t(c) = c$ for a constant c.

Lemma 2.4. *If $f \in \mathscr{B}(G/H)$ and $f \geq 0$, then $\Phi_t(f) \geq 0$.*

Proof. Suppose $\Phi_t(f) = -d < 0$ and let $c = \|f\|$. Then $c - f \geq 0$ and $\|c - f\| \leq c$, but $\Phi_t(c - f) = c + d > c$ which is a contradiction. q.e.d.

Thus

$$\Phi_t(f) = \int_{\beta(G/H)} f(x)dp_t(x) ,$$

where p_t is a probability measure on $\beta(G/H)$.

Now as $T_t(L_g f) = L_g(T_t f)$, we have that

$$T_t f(g) = \Phi_t(L_{g^{-1}} f) = \int_{\beta(G/H)} (L_{g^{-1}} f)(x)dp_t(x) = \int_{\beta(G/H)} f(gx)dp_t(x) .$$

From $T_t \cdot T_s f = T_{t+s} f$, it follows that

$$\begin{aligned} T_{t+s}(f)(g) &= \int_{\beta(G/H)} f(gx)dp_{t+s}(x) = T_t(T_s f)(g) \\ &= \int_{\beta(G/H)} (T_s f)(gy)dp_t(y) = \int_{\beta(G/H)} \int_{\beta(G/H)} f(gyx)dp_s(x)dp_t(y) \end{aligned}$$

Hence $p_t * p_s = p_{t+s}$, and the p_t's form a semigroup of probability measures on $\beta(G/H)$.

We now summarize our results.

Theorem 2.1. *The T_t's for $t > 0$ form a semigroup of continuous operators on $\mathscr{B}(G/H)$, which commute with the left action of G, and determine probability measures p_t for $t > 0$ on $\beta(G/H)$, which form a semigroup under convolution. Moreover, if $f \in \mathscr{B}(G/H)$, then*

$$T_t(f)(g) = \int_{\beta(G/H)} f(gx)dp_t(x) .$$

References

[1] R. Courant & D. Hilbert, *Methods of mathematical physics,* Vol. II, Interscience, New York, 1962.

[2] E. B. Dynkin, *Brownian motion in certain symmetric spaces and nonnegative eigenfunctions of the Laplace-Beltrami operator,* Amer. Math. Soc. Transl. (2) **72** (1968) 203–228.

[3] S. Helgason, *Differential geometry and symmetric spaces,* Academic Press, New York, 1962.

[4] L. Hörmander, *Linear partial differential operators,* Academic Press, New York, 1963.

[5] G. A. Hunt, *Semi-groups of measures of Lie groups.* Trans. Amer. Math. Soc. **81** (1956) 264–293.

[6] H. Omori, *Isometric immersions of Riemannian manifolds,* J. Math. Soc. Japan **19** (1967) 205–214.

[7] R. Palais et al., *Seminar on the Atiyah-Singer index theorem,* Annals of Math. Studies, No. 57, Princeton University Press, Princeton, 1965.

[8] F. Reisz & B. Sz-Nagy, *Functional analysis*, Ungar, New York, 1955.

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