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KÄHLER SURFACES OF NONNEGATIVE CURVATURE

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1. A well-known theorem of Andreotti and Frankel [4] asserts that any compact Kähler surface of positive sectional curvature is biholomorphically equivalent to the complex projective plane. In this paper we investigate compact complex analytic surfaces which carry a Kähler metric of nonnegative curvature. Our basic assumption is ostensibly weaker than that of nonnegative sectional curvature, and invokes the notion of *holomorphic bisectional curvature* recently introduced by Goldberg and Kobayashi [5]. If p and p' are planes in the tangent space of a Kähler manifold each invariant with respect to the almost-complex structure tensor J, then the *(holomorphic) bisectional curvature* $H(p, p')$ is defined by

$$H(p,p') = R(X, JX, Y, JY) \,, \tag{1}$$

where R is the Riemann curvature tensor, and X and Y are unit vectors in the planes p and p'. From Bianchi's identity,

$$H(p,p') = R(X, Y, X, Y) + R(X, JY, X, JY) \,, \tag{2}$$

so that $H(p, p')$ is the sum of two sectional curvatures. It follows that nonnegative sectional curvature at a point implies nonnegative bisectional curvature at that point. (For complex dimension 2, it follows from the results of this paper that everywhere nonnegative bisectional curvature is equivalent to everywhere nonnegative sectional curvature.) With this definition, we may state our main result.

Theorem. *Let M be a compact Kähler surface whose holomorphic bisectional curvature is everywhere nonnegative. Then one of the following holds:*

(i) *M is biholomorphically equivalent to the complex projective plane $\boldsymbol{P}^2$.*

(ii) *M is biholomorphically equivalent to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, and the metric is a product of metrics of nonnegative curvature.*

(iii) *M is flat.*

(iv) *M is a ruled surface* (i.e., *$\boldsymbol{P}^1$-bundle*) *over an elliptic curve. In this case, the universal covering space of M is $\boldsymbol{C} \times \boldsymbol{P}^1$ endowed with the product of the flat metric on $\boldsymbol{C}$ and a metric of nonnegative curvature on $\boldsymbol{P}^1$.*

In § 3 we show that if the Ricci tensor is nondegenerate at any point, then

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(i) or (ii) holds, and we obtain the Andreotti-Frankel theorem as a corollary. In § 4 we show that if the Ricci tensor is everywhere degenerate but is nonzero at some point, then (iv) holds. For this case we describe those surfaces which occur by explicit construction as quotient spaces of $\boldsymbol{C} \times \boldsymbol{P}^1$ and also in terms of Atiyah's classification of ruled surfaces over a torus [2].

2. We begin by fixing some notation and pointing out some obvious but useful facts. First, by taking $X = Y$ in (1), it is apparent that nonnegative (resp. positive) bisectional curvature at a point implies nonnegative (resp. positive) holomorphic curvature at that point. Secondly, let S denote the Ricci tensor on M, suppose X, JX, Y, JY are orthonormal unit vectors at a point of M, and write $K(X, Y) = R(X, Y, X, Y)$. Then

$$S(X,X) = K(X,JX) + K(X,Y) + K(X,JY) \,, \tag{3}$$

and our assumption on the curvature implies that S is positive semi-definite, and

$$S(X,X) \geq K(X,JX) \tag{4}$$

for any unit vector X.

Since $S(JX, JX) = S(X, X)$, the eigenvalues of S occur in pairs, and the proof of the theorem separates naturally into three cases corresponding to the maximum rank of S on M. The first case is easily disposed of by the following:

Proposition 1. *Let M be a Kähler surface of nonnegative bisectional curvature. If the Ricci tensor of M is identically zero, then M is flat.*

Proof. It is easily seen that all holomorphic curvatures of M are zero. A simple algebraic argument then shows that M is flat (see, for example, [7, Vol. I, p. 167]).

From now on, we assume that the Ricci tensor is not identically zero.

Lemma 1. (i) *If the Ricci tensor is nondegenerate at any point of M, there are no holomorphic 1-forms.*

(ii) *If the Ricci tensor is nonzero at any point of M, there are no holomorphic 2-forms.*

Proof. The real and imaginary points β, γ of a holomorphic form α of degree p are harmonic, and so β (and γ) satisfies the Bochner-Myers formula [3]

$$\int_M \{pF_p(\beta) + g(\nabla\beta, \nabla\beta)\} dV = 0 \tag{5}$$

(and F_p we introduce below). What we show is that if $p = 1$ (resp. $p = 2$) then $F_1(\beta) \geq 0$ (resp. $F_2(\beta) \geq 0$) and not identically zero unless β is zero whenever $\operatorname{rank}_C S = 2$ (resp. $\operatorname{rank}_C S \geq 1$). But then, by (5), β is parallel and so must be identically zero. Let $\{x^1, \bar{x}^1, x^2, \bar{x}^2\}$ be normal coordinates in a neighbourhood of $p \in M$ associated with an orthonormal basis $\{e_1, e_{\bar{1}}, e_2, e_{\bar{2}}\}$ at p where $e_{\bar{i}} = Je_i$. In the summands which follow $i, j, k, l \in \{1, \bar{1}, 2, \bar{2}\}$ and whenever the suffix $\bar{\bar{i}}$ occurs we interpret this as i with a sign change in that term.

(i) In this case we have clearly

$$F_1(\beta) = \Sigma S_{ij}\beta_i\beta_j \geq 0 ,$$

and $F_1(\beta) > 0$ wherever S is nondegenerate unless β is zero.

(ii) The condition that α be of type $(2, 0)$ may be written in terms of its components

$$\beta_{kl} + \beta_{\bar{k}\bar{l}} = 0 \tag{6}$$

with the same equation for γ. Now at p we have

$$F_2(\beta) = \Sigma S_{ij}\beta_{ik}\beta_{jl} - \tfrac{1}{2}\Sigma R_{ijkl}\beta_{ij}\beta_{kl} ,$$

and the curvature terms may be collected in groups of four as follows:

$$\begin{aligned} & R_{ijkl}\beta_{ij}\beta_{kl} + R_{\bar{i}\bar{j}kl}\beta_{\bar{i}\bar{j}}\beta_{kl} + R_{ij\bar{k}\bar{l}}\beta_{ij}\beta_{\bar{k}\bar{l}} + R_{\bar{i}\bar{j}\bar{k}\bar{l}}\beta_{\bar{i}\bar{j}}\beta_{\bar{k}\bar{l}} \\ &\quad = R_{ijkl}[\beta_{ij}\beta_{kl} + \beta_{\bar{i}\bar{j}}\beta_{kl} + \beta_{ij}\beta_{\bar{k}\bar{l}} + \beta_{\bar{i}\bar{j}}\beta_{\bar{k}\bar{l}}] \\ &\quad = R_{ijkl}[\beta_{ij} + \beta_{\bar{i}\bar{j}}][\beta_{kl} + \beta_{\bar{k}\bar{l}}] = 0 \qquad \text{by (6)} . \end{aligned}$$

Thus $F_2(\beta) = \Sigma S_{ij}\beta_{ik}\beta_{jl}$, and since the original frame at p might have been chosen to diagonalize S this expression for $F_2(\beta)$ may be simplified to

$$F_2(\beta) = s(\beta_{12}^2 + \beta_{1\bar{2}}^2) ,$$

by using (6) several times, where s denotes the scalar curvature of M. Hence $F_2(\beta) \geq 0$ and $F_2(\beta) > 0$ wherever $s > 0$ unless β is zero there. This concludes the proof of Lemma 1.

Lemma 2. *Let M be a compact Kähler surface of nonnegative bisectional curvature, whose Ricci tensor is not identically zero. Then M is algebraic.*

Proof. Let $h^{p,q}$ denote the dimension of the space of harmonic (p, q)-forms. Since M is Kählerian, it follows from Lemma 1 that $h^{0,2} = h^{2,0} = 0$. Moreover, the first Betti number of M is even. It now follows from [10, Theorem 10] that M is algebraic.

Next let k be the canonical bundle of M, let k^m be the m-fold tensor product, and let $\mathcal{O}(k^m)$ denote the sheaf of germs of holomorphic sections of k^m. We recall that the *plurigenera* of M are defined to be the integers

$$P_m = \dim H^0(M, \mathcal{O}(k^m)) , \qquad \text{for m} = 1, 2, \cdots .$$

We also need the following facts about the Chern classes $c(k^m)$, for which we refer to [7, Vol. II], [8]. First, $c(k) = -c_1$, where c_1 is the first Chern class of M. Secondly, c_1 is represented by the exterior 2-form $\sigma/(2\pi)$, where $\sigma(X, Y) = S(JX, Y)$ (see, for example, [8]). It follows that $c(k^m)$ is represented by the form $-m\sigma/(2\pi)$.

Lemma 3. *Let M be a compact Kähler surface of nonnegative bisectional curvature, whose Ricci tensor is not identically zero. Then $P_m = 0$ for $m \geq 1$.*

Proof. We first show that σ is not exact. Suppose $\sigma = d\mu$ for some 1-form μ. Let φ denote the *fundamental form* of M, defined by $\varphi(X, Y) = \frac{1}{2}g(JX, Y)$ where g is the metric tensor. Then

$$0 = \int_M d(\mu \wedge \varphi) = \int_M \sigma \wedge \varphi .$$

On the other hand, $\sigma \wedge \varphi = csdV$, where c is a positive constant, s the scalar curvature, and dV the volume element of M. Since s is nonnegative and not identically zero, we have

$$\int_M \sigma \wedge \varphi \neq 0 ,$$

which contradicts the assumption that $\sigma = d\mu$.

Now suppose $P_m > 0$ for some m. Since $c_1 \neq 0$, the bundle k^m is nontrivial and is therefore represented by a divisor $\Sigma r_\alpha C_\alpha$, where each r_α is a positive integer and each C_α is an irreducible curve. Let C be any irreducible curve in M such that $C \neq C_\alpha$ for any α but $C \cdot C_\alpha > 0$ for some α. (Since M is algebraic we may, for example, choose C to be a suitable hyperplane section.) Then, on the one hand,

$$C \cdot k^m = \Sigma r_\alpha C_\alpha \cdot C > 0 ,$$

but, on the other hand,

$$C \cdot k^m = -\frac{m}{2\pi} \int_C \sigma \leq 0 .$$

Therefore $P_m = 0$ as asserted.

If C is a nonsingular complex analytic curve lying in M, we denote by K_C its Gaussian curvature, and by K the sectional curvature in M of planes tangent to C. At any point of C these are related by the Gauss equation [12, Corollary 1],

$$K = K_C + 2\{g(AX, X)^2 + g(JAX, X)^2\} , \tag{7}$$

where X is a unit vector tangent to C, g is the metric, and A is the second fundamental form of C; the second fundamental form is defined by

$$-AX = \text{tangential component of } \nabla_X \xi ,$$

where ξ is a field of unit vectors normal to C, and ∇ denotes covariant differentiation in M. We remark that $A \equiv 0$ on C if and only if C is totally geodesic in M.

Combining (4) and (7) we obtain

Lemma 4. *For any nonsingular complex analytic curve C in M and any unit vector X tangent to C,*

$$S(X,X) \geq K \geq K_C .$$

Moreover, $K \equiv K_C$ on C if and only if C is totally geodesic.

It is convenient at this point to recall some notions concerning complex analytic surfaces. If C_j is either a line bundle or a divisor on M for $j = 1, 2$, then the *intersection number* may be defined either as the intersection of the cycles determined by C_1 and C_2 or as the cup product of the corresponding Chern classes. Since M is oriented by its complex structure, we obtain a well-determined integer denoted by $C_1 \cdot C_2$. If C_1 and C_2 are distinct irreducible divisors, their intersection number is greater than or equal to the number of points at which they intersect, with equality holding in the case when every intersection is transversal and occurs at a point of regularity for both curves.

Next let C be any nonsingular connected complex analytic curve in M, and let k denote the canonical bundle of M. Then the genus p of C is determined by the formula

$$k \cdot C + C^2 = 2p - 2 , \tag{8}$$

[9, p. 119]. Using this formula, we prove

Lemma 5. *If C is a nonsingular connected complex analytic curve in M, then $C^2 \geq 0$. Moreover, $C^2 = 0$ if and only if C is totally geodesic.*

Proof. The Chern class $c(k)$ of the canonical bundle k is represented by the exterior 2-form $-\sigma/(2\pi)$, where σ is given by $\sigma(X, Y) = S(JX, Y)$. Let dA be the area element of C. From the Gauss-Bonnet formula, Lemma 4, and formula (8), it follows that

$$\begin{aligned} 2 - 2p &= \frac{1}{2\pi}\int_C K_C dA \leq \frac{1}{2\pi}\int_C K dA \leq \frac{1}{2\pi}\int_C \sigma \\ &= -k \cdot C = 2 - 2p + C^2 . \end{aligned}$$

Consequently $C^2 \geq 0$, with equality holding if and only if $K_C \equiv K$ on C, that is, if and only if C is totally geodesic.

By an *exceptional curve* (of the first kind) on a surface is meant a nonsingular connected curve C of genus 0 such that $C^2 = -1$; it is known that a curve is exceptional if and only if it arises as the result of blowing up a point via a quadric transform. From Lemma 5, we obtain at once

Lemma 6. *M does not contain any exceptional curve.*

3. We now prove

Proposition 2. *Let M be a compact Kähler surface of nonnegative bisectional curvature. If the Ricci tensor is nondegenerate at some point, then M is biholomorphically equivalent to either $\boldsymbol{P}^2$ or $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. In the latter case the metric is a product of metrics of nonnegative curvature.*

Proof. It follows from Lemmas 1 through 6 that M is an algebraic surface without exceptional curves, whose first Betti number is zero, and whose plurigenera P_m vanish for $m \geq 1$. A theorem of Castelnuovo [11, p. 46] asserts that an algebraic surface with $b_1 = 0$ and $P_m = 0$ for $m \geq 1$ is rational, and Andreotti proved in [1] that any rational surface free of exceptional curves is biholomorphically equivalent either to P^2 or to one of the so-called Hirzebruch surfaces $\Sigma_m, m \geq 0$. These surfaces may be described as follows: Consider $\boldsymbol{P}^2$ and $\boldsymbol{P}^1$ with homogeneous coordinates (x_0, x_1, x_2) and (y_1, y_2) respectively, and let Σ_m be the surface in $\boldsymbol{P}^2 \times \boldsymbol{P}^1$ defined by the equation $x_1 y_1{}^m = x_2 y_2{}^m$. Clearly $\Sigma_0 = \boldsymbol{P}^1 \times \boldsymbol{P}^1$; furthermore, it is easy to see that for $m > 0, \Sigma_m$ contains a nonsingular curve C with $C^2 = -m$. (For example, take the curve defined by the equations $x_1 = x_2 = 0$.) It follows from Lemma 5 that $M \neq \Sigma_m$ for $m > 0$; hence M is biholomorphically equivalent to $\boldsymbol{P}^2$ or $\boldsymbol{P}^1 \times \boldsymbol{P}^1$.

It remains to show that the metric is a product in the case $M = \boldsymbol{P}^1 \times \boldsymbol{P}^1$. For any point $(p, q) \in \boldsymbol{P}^1 \times \boldsymbol{P}^1$ we write $C_p = \{p\} \times \boldsymbol{P}^1$ and $D_q = \boldsymbol{P}^1 \times \{q\}$. Letting z_1 and z_2 be (nonhomogeneous) coordinates in each factor, we write the fundamental form as $\varphi = \Sigma g_{ij} dz_i \wedge d\bar{z}_j$, and let $|g|$ denote the determinant. For any (p, q) the function $(|g|/g_{22})(p, z_2)$ is well-defined on all of C_p, i.e., is invariant under a change of coordinate from z_2 to w_2 in C_p. Similarly, $(|g|/g_{11})(z_1, q)$ is well-defined on all of D_q. In addition, each of these functions is bounded away from zero. From the proof of Lemma 5 we see that

$$\frac{\partial^2}{\partial_{z_1}\partial_{\bar{z}_1}} \log g_{11} = \frac{\partial^2}{\partial_{z_1}\partial_{\bar{z}_1}} \log |g|$$

on D_q. Thus $\log (|g|/g_{11})$ is harmonic on D_q so that $|g|/g_{11}$ is constant on D_q. Similarly, $|g|/g_{22}$ is constant on C_p. We write $|g|/g_{22} = f(z_1)$ and $|g|/g_{11} = h(z_2)$. For convenience let $C'_p = C_p - \{\infty\}$, $D'_q = D_q - \{\infty\}$, and $M' = M - (C_\infty \cup D_\infty)$, so that (z_1, z_2) is a set of global coordinates on M', and the functions $f, h, |g|$, etc., are defined throughout.

For a fixed $(p, q), C_p$ and D_q generate $H_2(M, \boldsymbol{R})$, and $C_p{}^2 = D_q{}^2 = 0$, $C_p \cdot D_q = 1$. In terms of cohomology, the fundamental form may be represented as $\varphi \sim aC_p + bD_q$, where $a, b \in \boldsymbol{R}$, and $\sim$ denotes Poincaré duality. We thus obtain

$$2ab = \int_M \varphi \wedge \varphi = \int_{M'} 2\,|g|\, dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2\,, \tag{9}$$

$$a = \int_{D_q} \varphi = \int_{D'_q} g_{11} dz_1 \wedge d\bar{z}_1\,, \tag{10}$$

$$b = \int_{C_p} \varphi = \int_{C'_p} g_{22} dz_2 \wedge d\bar{z}_2 \; ; \tag{11}$$

and since $|g| = f(z_1)g_{22}$, (9) and (11) yield

$$ab = \int_{M'} f(z_1)g_{22} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 = b \int_{D'_q} f(z_1) dz_1 \wedge d\bar{z}_1$$

and hence

$$a = \int_{D'_q} f(z_1) dz_1 \wedge d\bar{z}_1 \; .$$

Similarly,

$$b = \int_{C'_p} h(z_2) dz_2 \wedge d\bar{z}_2 \; .$$

Finally, we note that

$$f(z_1)h(z_2) = \frac{|g|^2}{g_{11}g_{22}} \leq |g| \; ,$$

so that

$$\begin{aligned} ab &= \int_{M'} f(z_1)h(z_2) dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \; . \\ &= \int_{M'} \frac{|g|^2}{g_{11}g_{22}} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \leq \int_M |g| \, dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \\ &= ab \; . \end{aligned}$$

It follows that $|g| = g_{11}g_{22}$, i.e., $g_{12} = g_{21} = 0$, and $g_{11} = f(z_1)$, $g_{22} = h(z_2)$, which completes the proof.

Corollary 1. *If the bisectional curvature of M is everywhere nonnegative and positive at some point, then M is biholomorphically equivalent to $\boldsymbol{P}^2$.*

Proof. Since the Ricci tensor is positive definite at some point, it follows that M is biholomorphically equivalent to either $\boldsymbol{P}^2$ or $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. In the latter case, however, there is at every point a pair of vectors X, Y such that $K(X, Y) + K(X, JY) = 0$. In the notation of the proof of Proposition 2, it suffices to take X tangent to C_p and Y tangent to D_q.

In particular, Corollary 1 contains the theorem of Andreotti and Frankel, as well as a theorem of Goldberg and Kobayashi [5] to the effect that any

Kähler surface of positive bisectional curvature is biholomorphically equivalent to $\boldsymbol{P}^2$.

4. We next consider the case when the Ricci tensor is everywhere degenerate but is nonzero at some point.

Proposition 3. *Suppose that M is a compact Kähler surface of nonnegative bisectional curvature whose Ricci tensor is everywhere degenerate but not identically zero. Then M is a ruled surface* (i.e., *$\boldsymbol{P}^1$-bundle) over an elliptic curve.*

Proof. Again using Lemmas 1 through 6, we may apply a theorem of Enriques [11, Chap. IV, Theorem 13] to the effect that an algebraic surface with $P_{12} = 0$ is birationally equivalent to $B \times \boldsymbol{P}^1$, where B is an algebraic curve whose genus equals the irregularity q of the surface. (We recall that the *irregularity* $q = \frac{1}{2}b_1$ of an algebraic surface is equal to the number of linearly independent holomorphic 1-forms.)

The assumption on the Ricci tensor implies that $c_1^2 = 0$. By Lemma 1, moreover, $q \leq 1$. Since, by Lemma 6, M has no exceptional curve, it follows that M must be either $\boldsymbol{P}^2$, one of the surfaces Σ_m, or a $\boldsymbol{P}^1$-bundle over an elliptic curve. The first two possibilities are not compatible with the requirement $c_1^2 = 0$; hence M is as asserted.

Examples of surfaces satisfying the hypotheses of Proposition 3 may be constructed as follows. Choose a complex number α with $\operatorname{Im} \alpha > 0$, and let T be the torus with periods 1 and α, i.e., $T = \boldsymbol{C}/\Gamma$ where $\Gamma = \{m + n\alpha \colon m, n \in \boldsymbol{Z}\}$. Now consider the space $\boldsymbol{C} \times \boldsymbol{P}^1$ endowed with the product of the flat metric on $\boldsymbol{C}$ and the standard metric of curvature 1 on $\boldsymbol{P}^1$. For any pair of numbers θ_1, θ_2 with $0 \leq \theta_j < 2\pi$, let $G = \{g_{mn} \colon m, n \in \mathrm{Z}\}$ be the group of holomorphic isometries defined as follows:

$$g_{mn} \colon (z, w) \longrightarrow (z + m + n\alpha, e^{i(m\theta_1 + n\theta_2)}w) ,$$

where w is a nonhomogeneous coordinate on $\boldsymbol{P}^1$. It is clear that the surface $S(\alpha, \theta_1, \theta_2) = \boldsymbol{C} \times \boldsymbol{P}^1/G$ is a ruled surface over T and satisfies the hypotheses of Proposition 3.

Another class of examples is the following. Considering $\boldsymbol{P}^1$ as the unit sphere $S^2 \subset \boldsymbol{R}^3 = \{(x^1, x^2, x^3)\}$, we let σ_j denote rotation through angle π around the x^j-axis. Each σ_j is a holomorphic isometry of $\boldsymbol{P}^1$, so that $G = \{g_{mn}\}$ acts on $\boldsymbol{C} \times \boldsymbol{P}^1$ as a group of holomorphic isometries by

$$g_{mn} \colon (z, w) \longrightarrow (z + m + n\alpha, \sigma_2{}^m \sigma_3{}^n(w)) .$$

The surface $S(\alpha) = \boldsymbol{C} \times \boldsymbol{P}^1/G$ is a compact Kähler surface of nonnegative curvature and is ruled over T.

In the other direction we prove

Proposition 4. *Suppose that M is a compact Kähler surface of nonnegative curvature whose Ricci tensor is everywhere degenerate but not identically zero.*

Then M is biholomorphically equivalent to one of the surfaces $S(\alpha, \theta_1, \theta_2)$ or $S(\alpha)$.

Proof. By Proposition 3, M is a ruled surface over a torus T. We have a projection $p: \boldsymbol{C} \to T = \boldsymbol{C}/\Gamma$, where $\Gamma = \{m + n\alpha : m, n \in \mathrm{Z}\}, \alpha \in \boldsymbol{C}, \operatorname{Im} \alpha > 0$. Now let $\tilde{M}$ be the $\boldsymbol{P}^1$-bundle over $\boldsymbol{C}$ induced by M under the map p, so that we have the commutative diagram:

$$\begin{array}{ccc} \tilde{M} & \longrightarrow & M \\ \downarrow & & \downarrow \\ \boldsymbol{C} & \xrightarrow[p]{} & T \end{array}$$

It is clear that $\tilde{M}$ is the universal covering space of M, and that under the induced metric $\tilde{M}$ satisfies the same curvature assumptions, and $M = \tilde{M}/G$ where G is a group of holomorphic isometries of $\tilde{M}$.

Lemma 7. *$\tilde{M} = \boldsymbol{C} \times \boldsymbol{P}^1$ metrized by the product of the flat metric on $\boldsymbol{C}$ and a metric of nonnegative curvature on $\boldsymbol{P}^1$.*

Proof. M has at least two linearly independent harmonic 1-forms ψ_1 and ψ_2 to which we may apply the Bochner-Myers formula [3], so that

$$0 = (\varDelta\psi_j, \psi_j) = \int_M S(\psi_j^*, \psi_j^*)dV + (\nabla\psi_j, \nabla\psi_j) ,$$

where $\varDelta$ is the Laplacian, ∇ is covariant differentiation, ψ^* is the vector field dual to ψ, and $(\,,)$ denotes the global inner product on M. Since the Ricci tensor S is positive semi-definite, it follows that $S(\psi_j^*, \psi_j^*) \equiv 0$ and $\nabla\psi_j = 0$. Thus the distribution spanned by the vector fields ψ_1^*, ψ_2^*, on M is parallel and therefore invariant under the action of the restricted holonomy group of M. It follows at once that $\tilde{M}$ is reducible, and the de Rham decomposition theorem for Kähler manifolds [7, Vol. II] asserts that $\tilde{M}$ is a product of Kähler manifolds. It is easily seen that the decomposition is the one indicated above.

For the group G we have

Lemma 8. *There exist holomorphic isometries σ and τ of $\boldsymbol{P}^1$ (in the metric induced from $\tilde{M}$) such that $\sigma\tau = \tau\sigma$ and $G = \{\gamma_{mn} : m, n \in \boldsymbol{Z}\}$, where*

$$\gamma_{mn} : (z, w) \longrightarrow (z + m + n\alpha, \sigma^m\tau^n w) .$$

Proof. Let γ be any element of G. We may choose coodinates such that

$$\gamma : (z_1, z_2) \longrightarrow (w_1, w_2) = (z_1 + m + n\alpha, f(z_1, z_2)) .$$

Here z_1 and w_1 are coodinates in $\boldsymbol{C}$, and z_2 and w_2 are local coordinates in $\boldsymbol{P}^1$. We claim that f is, in fact, a function of z_2 alone. For, if we write the fundamental form in coodinates as $\varphi = \varphi_{11}dw_1 \wedge d\bar{w}_1 + \varphi_{22}dw_2 \wedge d\bar{w}_2$, then

$$\begin{aligned}\gamma^*\varphi = (\varphi_{11} + \varphi_{22}|\partial w_2/\partial z_1|^2)dz_1 \wedge d\bar{z}_1 + \varphi_{22}(\partial w_2/\partial z_1)(\overline{\partial w_2/\partial z_2})dz_1 \wedge d\bar{z}_2 \\ + \varphi_{22}(\partial w_2/\partial z_2)(\overline{\partial w_2/\partial z_1})dz_2 \wedge d\bar{z}_1 + \varphi_{22}|\partial w_2/\partial z_2|^2 dz_2 \wedge d\bar{z}_2 \,.\end{aligned}$$

Since γ is an isometry, the middle terms vanish; and since $\varphi_{22} \neq 0$ and

$$\frac{\partial(w_1, w_2)}{\partial(z_1, z_2)} = \begin{vmatrix} 1 & 0 \\ \partial w_2/\partial z_1 & \partial w_2/\partial z_2 \end{vmatrix} \neq 0 \,,$$

it follows that $\partial w_2/\partial z_1 = 0$. Hence $w_2 = f(z_2)$. Thus we may write

$$\gamma_{mn}\colon (z_1, z_2) \longrightarrow (z_1 + m + n\alpha, f_{mn}(z_2)) \;;$$

and if we write $\sigma = f_{10}$ and $\tau = f_{01}$, then it is clear that σ and τ are holomorphic isometries of $\boldsymbol{P}^1$, $\sigma\tau = \tau\sigma$, and $f_{mn} = \sigma^m\tau^n$.

Since σ and τ are holomorphic isometries of $\boldsymbol{P}^1$ (in the induced metric), each has a pair of fixed points. For, if σ has only one fixed point, we way choose a nonhomogeneous coodinate z_2 on $\boldsymbol{P}^1$ so that ∞ is the fixed point. But then σ is an automorphism of $\boldsymbol{C} = \boldsymbol{P}^1 - \{\infty\}$, and so is given by $\sigma(z_2) = az_2 + b$ $(0 \neq a, b \in \boldsymbol{C})$. The fundamental form of the metric on $\boldsymbol{P}^1$ is $ih(z_2)dz_2 \wedge d\bar{z}_2$, where $h(z_2)$ is positive real function on $\boldsymbol{C}$ with $\lim_{z_2\to\infty} h(z_2) = 0$. Since σ is a holomorphic isometry, $h(z_2) = |a|^{2n} h(\sigma^n(z_2))$, and consequently we have the contradiction $h(z_2) = 0$ if $|a| < 1$. If $|a| > 1$, the same argument applied to σ^{-1} again provides a contradiction. However, if $|a| = 1$ then, since $b \neq 0$, $\lim_{n\to\infty} \sigma^n(0) = \infty$, and therefore $h(0) = 0$.

We assume for the moment that σ and τ have the same pair of fixed points p and q, and show that M is biholomorphically equivalent to one of the surfaces $S(\alpha, \theta_1, \theta_2)$. Let ρ be a holomorphic transformation of $\boldsymbol{P}^1$ such that $\rho(p) = 0$ and $\rho(q) = \infty$. Then $\sigma' = \rho\sigma\rho^{-1}$ and $\tau' = \rho\tau\rho^{-1}$ are holomorphic transformations of $\boldsymbol{P}^1$ fixing 0 and ∞. Hence $\sigma'(z_2) = \mu z_2$ and $\tau'(z_2) = \nu z_2$. Since σ' and τ' are isometries with respect to the metric induced by ρ^{-1}, we conclude as in the previous paragraph that $|\mu| = |\nu| = 1$, that is, $\mu = e^{i\theta_1}$ and $\nu = e^{i\theta_2}$. Define $\hat{\rho}\colon \boldsymbol{C} \times \boldsymbol{P}^1 \to \boldsymbol{C} \times \boldsymbol{P}^1$ by $\hat{\rho}(z_1, z_2) = (z_1, \rho(z_2))$. Then $\boldsymbol{C} \times \boldsymbol{P}^1/G = M$ is biholomorphically equivalent to $\boldsymbol{C} \times \boldsymbol{P}^1/\hat{\rho}G\hat{\rho}^{-1} = S(\alpha, \theta_1, \theta_2)$.

When the fixed points p and q of σ do not coincide with the fixed points $\hat{p}$ and $\hat{q}$ of τ, we note that $\tau(p)$ and $\tau(q)$ are also fixed points of σ as follows from the commutativity of σ and τ. Consequently $\tau(p) = q$ and $\tau(q) = p$, and in particular τ^2 fixes $p, q, \hat{p}, \hat{q}$. Therefore $\tau^2 = \sigma^2 =$ identity. Let ρ be a holomorphic transformation of $\boldsymbol{P}^1$ such that $\rho(p) = 0$ and $\rho(q) = \infty$. Then $\sigma' = \rho\sigma\rho^{-1}$ fixes 0 and ∞, so $\sigma'(z) = \mu z$. Since $\sigma^2 =$ identity, it follows that $\mu = -1$. For the automorphism $\tau' = \rho\tau\rho^{-1}$ we have $\tau'(0) = \infty$ and $\tau'(\infty) = 0$, so that $\tau'(z) = \nu/z$. Define $\rho'(z) = \eta\rho(z)$ where $\eta^2\nu = 1$, and we see at once that $\sigma'' = \rho'\sigma(\rho')^{-1}$ and $\tau'' = \rho'\tau(\rho')^{-1}$ are given by $\sigma''(z) = -z, \tau''(z) = 1/z$. Then $\boldsymbol{C} \times \boldsymbol{P}^1/G = M$ is biholomorphically equivalent to the surface $\boldsymbol{C} \times \boldsymbol{P}^1/\hat{\rho}G\hat{\rho}^{-1} = S(\alpha)$, where $\hat{\rho}$ is given by $\hat{\rho}(z_1, z_2) = (z_1, \rho'(z_2))$.

Remark. As the proof of Proposition 4 shows, if we add the assumption that M has constant scalar curvature, then M is, in fact, isometric to one of the surfaces $S(\alpha, \theta_1, \theta_2)$ or $S(\alpha)$.

If M is any ruled surface over an elliptic curve, and there is a holomorphic reduction of the structure group from the projective linear group to the group of nonzero complex numbers, then we say that *M comes from a line bundle.* The following proposition places the above examples within the context of Atiyah's classification of ruled surfaces over an elliptic curve [2].

Proposition 5. *For any given α the class of surfaces $S(\alpha, \theta_1, \theta_2)$ consists of those ruled surfaces over the torus determined by α, which come from a line bundle with Chern class zero.*

Proof. As a $\boldsymbol{P}^1$-bundle, $M = S(\alpha, \theta_1, \theta_2)$ has two disjoint holomorphic sections, i.e., the curves $\boldsymbol{C} \times \{0\}/G$ and $\boldsymbol{C} \times \{\infty\}/G$ (in the notation of the proof of Proposition 4). It follows that M comes from a line bundle, so that we may identify M with an element of $H^1(T, \mathcal{O}^*)$, where $\mathcal{O}^*$ denotes the sheaf of germs of nonvanishing holomorphic functions, and T is the base curve.

Now consider the sequence

$$H^1(T, \Gamma) \to H^1(T, \boldsymbol{C}^*) \to H^1(T, \mathcal{O}^*)$$

induced by inclusion, where $\boldsymbol{C}^*$ (resp. Γ) is the constant sheaf of nonzero complex numbers (resp. complex numbers of absolute value 1). In the canonical metric on M all fibers are isometric, and therefore the corresponding line bundle is in the image of $H^1(T, \Gamma)$. Conversely, for any line bundle in the image of Γ, we can metrize the corresponding $\boldsymbol{P}^1$-bundle so as to have nonnegative curvature, thus obtaining one of the surfaces $S(\alpha, \theta_1, \theta_2)$. In other words, the class of surfaces $S(\alpha, \theta_1, \theta_2)$ consists of all surfaces coming from line bundles in the image of $H^1(T, \Gamma)$. It is well known (see, for example, [6, p. 134]) that any element in the image of $H^1(T, \Gamma)$—in fact, any element in the image of $H^1(T, \boldsymbol{C}^*)$—has Chern class zero. For the converse, we consider the commutative diagram:

$$\begin{array}{ccccccccccc}
0 \to H^0(T, d\mathcal{O}) & \xrightarrow{\Delta_1} & H^1(T, \boldsymbol{C}) & \xrightarrow{j_1} & H^1(T, \mathcal{O}) & \xrightarrow{d_1} & H^1(T, d\mathcal{O}) & \xrightarrow{\Delta_2} & H^2(T, \boldsymbol{C}) \to 0 \\
& & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} & & & & \\
& & H^1(T, \boldsymbol{C}^*) & \longrightarrow & H^1(T, \mathcal{O}^*) & & & & \\
& & & & \downarrow{\scriptstyle c} & & & & \\
& & & & H^2(T, \boldsymbol{Z}) & & & &
\end{array}$$

Here the top row comes from the exact sequence of sheaves $0 \to \boldsymbol{C} \to \mathcal{O} \to d\mathcal{O} \to 0$, and ε comes from the map $f \to \exp(2\pi i f)$. Obviously those elements of $H^1(T, \mathcal{O}^*)$ with zero Chern class are in the image of ε. Futhermore, $H^1(T, d\mathcal{O})$

and $H^2(T, C)$ are both isomorphic to C; hence $d_1 = 0$, and j_1 is surjective. An examination of the map Δ_1 shows that any element of $H^1(T, C)$ is represented, modulo the image of Δ_1, by a cocycle $\{r_{\alpha\beta}\}$ where $r_{\alpha\beta}$ are real constants. Thus any element of $H^1(T, \mathcal{O}^*)$ with zero Chern class is represented by a cocycle of the form $\{\exp 2\pi i r_{\alpha\beta}\}$, i.e., is an element of $H^1(T, \Gamma)$ as asserted.

5. It should be remarked that we have recently found a partial generalization of this result to higher dimensions:

Theorem. *Let M be an n-dimensional compact Kähler manifold of non-negative bisectional curvature, and r the maximal rank of the Ricci tensor. Then there exist a flat manifold N of dimension $(n - r)$ and a holomorphic fibering $\pi: M \to N$, such that the metric on M is locally a product compatible with the fibering. Moreover, the Ricci tensor of the fiber F has maximal rank r, and under the de Rham decomposition, $F = F_1 \times \cdots \times F_q$, each F_j is simply connected and has second Betti number equal to one.*

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