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ALGEBRAS OF MATRICES UNDER DEFORMATION

W. STEPHEN PIPER

1. Introduction

The subject of this discussion is families of one-parameter deformations of the associative algebras of $n \times n$ upper triangular real matrices; the purpose is to expand the set of examples of algebraic deformations. Gerstenhaber [1] has given an example of a commutative associative algebra which when deformed is non-commutative. Also, a large class of associative algebras A, namely the class of semi-simple algebras, which includes the algebras of $n \times n$ matrices, has the second Hochschild cohomology group $H^2(A, A)$ equal to zero. These algebras are rigid, meaning that their only deformations are trivial, that is, equivalent to those generated by vector space isomorphisms.

We consider the algebras A_n of $n \times n$ upper triangular real matrices having equal diagonal elements. For any $n \geq 2$, $\dim Z^2(A_n, A_n) > \dim B^2(A_n, A_n)$, and hence $H^2(A_n, A_n) \neq 0$ (§ 4). In the case of $n = 3$, we exhibit 2-cocycles which can not be integrated to a deformation of A_3. Although $H^3(A_2, A_2) \neq 0$, we prove that any infinitesimal deformation f of A_2 and any partial integration of f can be completed to a deformation of A_2. In other words, all obstructions to the integration of f vanish, and as we shall see, with restriction only on the choice of four of the eight coefficients for the cochains involved.

§ 2 presents a brief review of the definitions in algebraic deformation theory, and § 3 introduces the terminology which proves useful in analysis of the deformations of A_n. The existence of non-trivial infinitesimal deformations of A_n is proven in § 4, together with the fact that $H^3(A_n, A_n) \neq 0$. The particular cases of $n = 2$ and 3 are taken up in §§ 5 and 6. Formula 19 and § 7 provide examples of deformations of $A_n, n \geq 2$.

2. Background

We recall from [1] and [2] the principal definitions of algebraic deformation theory. Given an associative algebra A with multiplication denoted by juxtaposition, we define a (one-parameter) deformation of A to be a formal power series,

$$(1) \qquad F_t(\alpha, \beta) = \alpha\beta + f_1(\alpha, \beta)t + f_2(\alpha, \beta)t^2 + \cdots , \qquad \alpha, \beta \in A ,$$

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such that F_t satisfies the law of associativity:

$$(2) \qquad F_t(F_t(\alpha, \beta), \gamma) - F_t(\alpha, F_t(\beta, \gamma)) = 0 , \qquad \alpha, \beta, \gamma \in A .$$

In terms of the Hochschild cohomology of A (with coboundary operator δ), (2) is equivalent to

$$(3) \qquad \begin{aligned} \delta f_1(\alpha, \beta, \gamma) &= 0 , \\ \delta f_r(\alpha, \beta, \gamma) &= \sum_{\substack{p+q=r \\ p,q>0}} f_p(f_q(\alpha, \beta), \gamma) - f_p(\alpha, f_q(\beta, \gamma)) , \end{aligned}$$

or more conveniently,

$$\begin{aligned} \delta f_1(\alpha, \beta, \gamma) &= 0 , \\ \delta f_r(\alpha, \beta, \gamma) &= \sum_{\substack{p+q=r \\ p,q>0}} f_p * f_q(\alpha, \beta, \gamma) , \end{aligned}$$

where $f_p * f_q(\alpha, \beta, \gamma) = f_p(f_q(\alpha, \beta), \gamma) - f_p(\alpha, f_q(\beta, \gamma))$.

Given an associative algebra A and a cocycle $f_1 \in Z^2(A, A)$, one seeks to "integrate" f_1 to a deformation F_t, i.e., to obtain 2-cochains $f_2, f_3, \cdots$ satisfying (3). Having obtained $f_2, f_3, \cdots, f_{r-1}$ satisfying (3), we say that f_1 is integrated up to the r^{th}-stage. The 3-cochain

$$\omega_r = \sum_{\substack{p+q=r \\ p,q>0}} f_p * f_q$$

is called an r^{th}-obstruction to the integration of f_1. The obstruction is said to vanish if ω_r is cohomologous to zero. Gerstenhaber [1] has shown that $\omega_r \in Z^3(A, A)$, and the question of integration is then to find $f_r \in C^2(A, A)$ such that $\delta f_r = \omega_r$.

3. The algebras A_n

Denote by A_n, for fixed $n \geq 2$, the algebra over the real numbers of $n \times n$ upper triangular matrices which have equal diagonal elements. Thus A_n is a subalgebra of the algebra of all $n \times n$ upper triangular real matrices. While this latter algebra, being semi-simple, has second Hochschild cohomology equal to zero, the algebra A_n does not. As a vector space over $\boldsymbol{R}$, A_n has a canonical basis

$$\{\varepsilon_1, \cdots, \varepsilon_v\} , \qquad \dim A_n = v = 1 + n(n-1)/2 ,$$

where ε_1 is the $n \times n$ identity matrix, and the remaining ε_i each have a single non-zero entry (specifically, 1) above the diagonal. It is convenient to express the product of elements of A_n in terms of this basis. In particular,

$$(4) \qquad \varepsilon_i \varepsilon_j = \sum_k e_{ijk} \varepsilon_k \ .$$

Here and subsequently all summation is over the index set of the basis (i.e., from $k = 1$ to $k = v$).

One and 2-cochains g and f can be expressed as:

$$(5) \qquad g(\varepsilon_i) = \sum_k b_{ik} \varepsilon_k \ , \qquad f(\varepsilon_i, \varepsilon_j) = \sum_m a_{ijm} \varepsilon_m \ .$$

One ascertains that $B^2(A_n, A_n)$ consists of elements of the form:

$$(6) \qquad f(\varepsilon_i, \varepsilon_j) = \delta g(\varepsilon_i, \varepsilon_j) = \sum_{p,k} (e_{ikp} b_{jk} - e_{ijk} b_{kp} + e_{kjp} b_{ik}) \varepsilon_p \ .$$

The requirement that $f \in C^2(A_n, A_n)$ be a cocycle imposes restrictions on the coefficients a_{ijm} in the expression (5). In particular, $\delta f(\varepsilon_i, \varepsilon_j, \varepsilon_k) = 0$ for all i, j, k implies that

$$\sum_{m,p} (e_{ipm} a_{jkp} - e_{pkm} a_{ijp} - e_{ijp} a_{pkm} + e_{jkp} a_{ipm}) \varepsilon_m = 0 \ ,$$

and, by the linear independence of the ε_m, that

$$(7) \qquad \sum_p (e_{ipm} a_{jkp} - e_{pkm} a_{ijp} - e_{ijp} a_{pkm} + e_{jkp} a_{ipm}) = 0 \ ,$$

for each $m = 1, \cdots, v$.

The general form of $h(\varepsilon_i, \varepsilon_j, \varepsilon_k) \in B^3(A_n, A_n)$ is obtained from consideration of $f(\varepsilon_i, \varepsilon_j) = \sum_m a_{ijm} \varepsilon_m$. Then

$$(8) \qquad \delta f(\varepsilon_i, \varepsilon_j, \varepsilon_k) = \sum_{m,p} (e_{imp} a_{jkm} - e_{mkp} a_{ijm} - e_{ijm} a_{mkp} + e_{jkm} a_{imp}) \varepsilon_p \ .$$

Let the deformation cochains $f_p, p = 0, 1, 2, \cdots$, of the algebra A_n be given by

$$f_p(\varepsilon_i, \varepsilon_j) = \sum_k c^p_{ijk} \varepsilon_k \ ,$$

where, of course,

$$f_0(\varepsilon_i, \varepsilon_j) = \varepsilon_i \varepsilon_j = \sum_k e_{ijk} \varepsilon_k \ ,$$

and f_1 is a cocycle. With this notation and the assumption that

$$\delta f_s = \sum_{\substack{p+q=s \\ p,q>0}} f_p * f_q \ , \qquad s = 2, \cdots, r-1 \ ,$$

the r^{th}-obstruction,

$$\omega_r = \sum_{\substack{p+q=r \\ p,q>0}} f_p * f_q \ , \tag{9}$$

can be expressed as

$$\begin{aligned} &\omega_r(\varepsilon_i, \varepsilon_j, \varepsilon_k) \\ &\quad = \sum_{\substack{p+q=r \\ p,q>0}} [f_p(c^q_{ij1}\varepsilon_1 + \cdots + c^q_{ijv}\varepsilon_v, \varepsilon_k) - f_p(\varepsilon_i, c^q_{jk1}\varepsilon_1 + \cdots + c^q_{jkv}\varepsilon_v)] \\ &\quad = \sum_{\substack{p+q=r \\ p,q>0}} [(c^q_{ij1}c^p_{1k1} + \cdots + c^q_{ijv}c^p_{vk1} - c^q_{jk1}c^p_{i11} - \cdots - c^q_{jkv}c^p_{iv1})\varepsilon_1 \\ &\qquad + (c^q_{ij1}c^p_{1k2} + \cdots + c^q_{ijv}c^p_{vk2} - c^q_{jk1}c^p_{i12} - \cdots - c^q_{jkv}c^p_{iv2})\varepsilon_2 \\ &\qquad + \cdots \\ &\qquad + (c^q_{ij1}c^p_{1kv} + \cdots + c^q_{ijv}c^p_{ikv} - c^q_{jk1}c^p_{i1v} - \cdots - c^q_{jkv}c^p_{ivv})\varepsilon_v] \end{aligned} \tag{10}$$

4. Existence of infinitesimal deformations

The result of this section is the statement that for each $n \geq 2, H^2(A_n, A_n) \neq 0$ and $H^3(A_n, A_n) \neq 0$. Thus, there are non-trivial infinitesimal deformations, and so possibly deformations. Further, obstructions do not necessarily vanish. We shall see in § 5 that for $n = 2$ all obstructions do vanish, and one has actual deformations. For $n = 3$, a non-vanishing primary obstruction will be exhibited (§ 6).

Theorem 1. $H^2(A_n, A_n) \neq 0,\ n \geq 2.$

For the case where n, and hence v, are greater than 2, the proof is given most easily by demonstrating that the following cocycle is not a coboundary:

$$f(\varepsilon_i, \varepsilon_j) = \delta_{iv}\delta_{j2}\varepsilon_n \ , \tag{11}$$

where δ_{ij} is the Kronecker delta, and $\varepsilon_2, \varepsilon_n, \varepsilon_v$ denote the following matrices in the canonical basis $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n, \cdots, \varepsilon_v\}$ of A_n:

ε_2 has a 1 in the 1st row, 2nd column, otherwise zero,

ε_n has a 1 in the 1st row, nth column, otherwise zero,

ε_v has a 1 in the $(n-1)$th row, nth column, otherwise zero.

First, one shows that (11) is a cocycle.

$$\begin{aligned} &\delta f(\varepsilon_i, \varepsilon_j, \varepsilon_k) \\ &\quad = \varepsilon_i(\delta_{jv}\delta_{k2}\varepsilon_n) - \sum_m e_{ijm}\delta_{mv}\delta_{k2}\varepsilon_n + \sum_m e_{jkm}\delta_{iv}\delta_{m2}\varepsilon_n - (\delta_{iv}\delta_{j2}\varepsilon_n)\varepsilon_k \ . \end{aligned} \tag{12}$$

Since

$$\begin{aligned} \varepsilon_i\varepsilon_n &= \varepsilon_n\varepsilon_i = \delta_{i1}\varepsilon_n \ , \qquad 1 \leq i \leq v \ , \\ e_{ijv} &= \delta_{i1}\delta_{jv} + \delta_{iv}\delta_{j1} \ , \end{aligned}$$

and

$$e_{jk2} = \delta_{j1}\delta_{k2} + \delta_{j2}\delta_{k1} ,$$

(12) becomes

$$\delta f(\varepsilon_i, \varepsilon_j, \varepsilon_k) = (\delta_{i1}\delta_{jv}\delta_{k2} - e_{ijv}\delta_{k2} + e_{jk2}\delta_{iv} - \delta_{iv}\delta_{i2}\delta_{k1})\varepsilon_n = 0 .$$

In order that $f(\varepsilon_i, \varepsilon_j)$ be equal to $\delta g(\varepsilon_i, \varepsilon_j)$ for some $g(\varepsilon_i) = \sum_k b_{ik}\varepsilon_k$, the coefficient a_{ijm} of ε_m in (11) must be

$$a_{ijm} = \sum_k (e_{ikm}b_{jk} - e_{ijk}b_{km} + e_{kjm}b_{ik}) . \tag{13}$$

In particular, when $i = v$, $j = 2$, and $m = n$, (13) becomes

$$a_{ijm} = a_{v2n} = \sum_k (e_{vk2}b_{2k} - e_{v2k}b_{kn} + e_{k2n}b_{vk}) .$$

But, $e_{vk2} = e_{v2k} = e_{k2n} = 0$, for all $k, 1 \leq k \leq v$. Therefore, since $a_{v2n} = 1$ in (11), and not 0, $f(\varepsilon_i, \varepsilon_j) = \delta_{iv}\delta_{j2}\varepsilon_n$ is not an element of $B^2(A_n, A_n)$. Hence the cohomology class of f in $H^2(A_n, A_n)$ is non-trivial.

When n, and hence v, equal 2, $f(\varepsilon_i, \varepsilon_j) = \delta_{i2}\delta_{j2}\varepsilon_i$ is a non-cobounding cocycle. The proof is analogous to the preceding general case: (13) becomes for $i = j = 2, m = 1$,

$$\sum_k (e_{2k1}b_{2k} - e_{22k}b_{k1} + e_{k21}b_{2k}) .$$

And,

$$e_{2k1} = e_{k21} = e_{22k} = 0 , \qquad k = 1, 2 .$$

Again, since $a_{221} = 1$ in (11), and not 0, $f(\varepsilon_i, \varepsilon_j) = \delta_{i2}\delta_{j2}\varepsilon_1$ is not an element of $B^2(A_2, A_2)$.

Theorem 2. $H^3(A_n, A_n) \neq 0, n \geq 2$.

Analogously to the preceding, one demonstrates that

$$g(\varepsilon_i, \varepsilon_j, \varepsilon_k) = \delta_{in}\delta_{jn}\delta_{kn}\varepsilon_n \tag{14}$$

is a non-cobounding cocycle. Since $e_{ijn} = \delta_{i1}\delta_{jn} + \delta_{in}\delta_{j1}$, we have

$$\delta g(\varepsilon_i, \varepsilon_j, \varepsilon_k, \varepsilon_m) = \varepsilon_i(\delta_{jn}\delta_{kn}\delta_{mn}\varepsilon_n) - e_{ijn}\delta_{kn}\delta_{mn}\varepsilon_n + \delta_{in}e_{jkn}\delta_{mn}\varepsilon_n - \delta_{in}\delta_{jn}e_{kmn}\varepsilon_n + \delta_{in}\delta_{jn}\delta_{kn}\varepsilon_n\varepsilon_m = 0 .$$

Suppose $g = \sum_m c_{ijkm}\varepsilon_m \in B^3(A_n, A_n)$. Then one shows that $c_{nnnn} = 0$, whereas for the $g \in Z^3(A_n, A_n)$ defined by (14) above, $c_{nnnn} = 1$.

Consider $f(\varepsilon_i, \varepsilon_j) = \sum_m a_{ijm}\varepsilon_m$ such that $\delta f = g$. The ε_n-coefficient of $\delta f(\varepsilon_i, \varepsilon_j, \varepsilon_k)$ is

$$\sum_m e_{imn}a_{jkm} - e_{mkn}a_{ijm} - e_{ijm}a_{mkn} + e_{jkm}a_{imn} .$$

Setting $i = j = k = n$, we get

$$c_{nnnn} = \sum_m e_{nmn}a_{nnm} - e_{mnn}a_{nnm} - e_{nnm}a_{mnn} + e_{nnm}a_{nmn} = 0 ,$$

since $e_{nnm} = 0$ for all m, and $e_{nmn} = e_{mnn} = \delta_{m1}$.

5. Deformations of A_2

The algebra A_2 of 2×2 upper triangular matrices with equal diagonal elements, considered as a vector space over $\boldsymbol{R}$, has a canonical basis

$$\left\{\varepsilon_1 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \varepsilon_2 = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\right\} .$$

The coefficients e_{ijk} in (4) can be conveniently expressed in matrix form:

$$(15) \qquad e_{ij1} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} , \qquad e_{ij2} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} .$$

In order that

$$(16) \qquad f(\varepsilon_i, \varepsilon_j) = \sum_m a_{ijm}\varepsilon_m$$

be an element of $B^2(A_2, A_2)$, the coefficients a_{ijm} must satisfy

$$(17) \qquad a_{ij1} = \begin{pmatrix}b_{11} & 0\\0 & 0\end{pmatrix} , \qquad a_{ij2} = \begin{pmatrix}b_{12} & b_{11}\\b_{11} & b_{22}\end{pmatrix} ,$$

from which we conclude $\dim B^2(A_2, A_2) = 3$.

In order that $f \in C^2(A_2, A_2)$ be a cocycle, its coefficients in (16) must satisfy

$$(18) \qquad a_{ij1} = \begin{pmatrix}a_{111} & 0\\0 & a_{221}\end{pmatrix} , \qquad a_{ij2} = \begin{pmatrix}a_{112} & a_{111}\\a_{111} & a_{222}\end{pmatrix} ,$$

from which we conclude $\dim Z^2(A_2, A_2) = 4$. Therefore $\dim H^2(A_2, A_2) = 1$. A generator for $H^2(A_2, A_2)$ is the cohomology class of the cocycle $f(\varepsilon_i, \varepsilon_j) = \delta_{i2}\delta_{j2}\varepsilon_1$.

Using the cocycle $f_1(\varepsilon_i, \varepsilon_j) = \delta_{i2}\delta_{j2}\varepsilon_1$, we proceed to deform A_2. The primary obstruction $\omega_2 = f_1 * f_1$ is equal to zero for our choice of f_1. Hence f_2 can be any cocycle, the zero cocycle, for instance. Letting $f_r = 0$, $r > 1$, we have

(19) $$F_t(\alpha, \beta) = \alpha\beta + \sum_{r=1}^{\infty} f_r(\alpha, \beta)t^r = \alpha\beta + f_1(\alpha, \beta)t ,$$

a non-trivial deformation of the original multiplication of A_2. If

$$\alpha = \begin{pmatrix} a & a_1 \\ 0 & a \end{pmatrix} , \qquad \beta = \begin{pmatrix} b & b_1 \\ 0 & b \end{pmatrix} ,$$

then

$$F_t(\alpha, \beta) = \begin{pmatrix} ab + a_1b_1t & ab_1 + a_1b \\ 0 & ab + a_1b_1t \end{pmatrix} .$$

On the other hand, suppose for f_2, one chooses a non-zero cocycle. The question is then whether

$$\alpha\beta + f_1(\alpha, \beta)t + f_2(\alpha, \beta)t^2$$

can be extended to a deformation of A_2, or if f_1 is an arbitrary cocycle, whether there even exists an f_2 whose coboundary equals $f_1 * f_1$. More generally, one asks what restrictions, if any, are needed on the f_i in order that the partial integration of f_1,

(20) $$\alpha\beta + f_1(\alpha, \beta)t + f_2(\alpha, \beta)t^2 + \cdots + f_r(\alpha, \beta)t^r ,$$

be extendible to a deformation of A_2.

From (8) we conclude that $B^3(A_2, A_2)$ consists of cochains whose coefficients c_{ijkm} satisfy

(21) $$c_{ij11} = \begin{pmatrix} 0 & 0 \\ -a_{211} & 0 \end{pmatrix} , \qquad c_{ij12} = \begin{pmatrix} 0 & 0 \\ a_{111} - a_{212} & a_{211} \end{pmatrix} .$$
$$c_{ij21} = \begin{pmatrix} a_{121} & 0 \\ 0 & 0 \end{pmatrix} , \qquad c_{ij22} = \begin{pmatrix} a_{122} - a_{111} & -a_{121} \\ a_{121} - a_{211} & 0 \end{pmatrix} ,$$

where the a_{ijm} are the coefficients for some 2-cochain $f(\varepsilon_i, \varepsilon_j) = \sum_m a_{ijm}\varepsilon_m$. Dimension $B^3(A_2, A_2)$ is then four,

In an analogous manner, one can show that for a 3-cochain $h(\varepsilon_i, \varepsilon_j, \varepsilon_k) = \sum_m c_{ijkm}\varepsilon_m$ to be an element of $Z^3(A_2, A_2)$, its coefficients c_{ijkm} must satisfy

(22) $$c_{ij11} = \begin{pmatrix} 0 & 0 \\ c_{2111} & 0 \end{pmatrix} , \qquad c_{ij12} = \begin{pmatrix} 0 & 0 \\ c_{21112} & -c_{2111} \end{pmatrix} ,$$
$$c_{ij21} = \begin{pmatrix} c_{1121} & 0 \\ 0 & 0 \end{pmatrix} , \qquad c_{ij22} = \begin{pmatrix} c_{1122} & -c_{1121} \\ c_{1121} + c_{2111} & c_{2222} \end{pmatrix} .$$

Hence $\dim Z^3(A_2, A_2) = 5$, and $\dim H^3(A_2, A_2) = 1$. A representative of a non-zero class in $H^3(A_2, A_2)$ is $h(\varepsilon_i, \varepsilon_j, \varepsilon_k) = \delta_{i2}\delta_{j2}\delta_{k2}\varepsilon_2$, and all other cocycles are cohomologous to real multiples of this one.

Lemma 1. *If $f_1 \in Z^2(A_2, A_2)$ and $f_2, \cdots, f_r \in C^2(A_2, A_2)$ satisfy*

$$\delta f_s = \sum_{\substack{p+q=s \\ p,q>0}} f_p * f_q , \qquad s = 2, \cdots, r , \tag{23}$$

then for $1 \leq s \leq r$,

$$\text{i)} \quad a^s_{121} = a^s_{211} ,$$

$$\text{ii)} \quad a^s_{122} = a^s_{212} ,$$

where $f_s(\varepsilon_i, \varepsilon_j) = \sum_m a^s_{ijm}\varepsilon_m$.

For $s = 1$ the lemma is a consequence of f_1's being a cocycle. Computing $\delta f_2 = f_1 * f_1$, one notes that $a^2_{121} - a^2_{211} = 0$ by examining the coefficient c_{2122} in (21). Similarly, the sum of coefficients

$$c_{2112} + c_{1122} = a^2_{122} - a^2_{212} = 0 .$$

The proof for general s now proceeds by induction. Let

$$\delta f_s(\varepsilon_i, \varepsilon_j, \varepsilon_k) = \sum_{\substack{p+q=s \\ p,q>0}} f_p * f_q(\varepsilon_i, \varepsilon_j, \varepsilon_k) = \sum_m c^s_{ijkm}\varepsilon_m .$$

Since this is a 3-coboundary, from (21) we have

$$\begin{aligned} a^s_{121} - a^s_{211} &= c_{1121} + c_{2111} \\ &= \sum_{\substack{p+q=s \\ p,q>0}} [c^p_{212}c^q_{111} + c^p_{221}c^q_{112} - c^p_{111}c^q_{121} - c^p_{121}c^q_{122} \\ &\qquad + c^p_{111}c^q_{211} + c^p_{211}c^q_{212} - c^p_{211}c^q_{111} - c^p_{221}c^q_{112}] \\ &= \sum_{\substack{p+q=s \\ p,q>0}} [(c^p_{211}c^q_{212} - c^p_{121}c^q_{122}) + c^q_{111}(c^p_{121} - c^p_{211}) + c^p_{111}(c^q_{211} - c^q_{121})] \\ &= 0 , \end{aligned} \tag{24}$$

by the induction hypothesis, as p and q are less than s. Also,

$$\begin{aligned} a^s_{122} - a^s_{212} &= c_{1122} + c_{2112} \\ &= \sum_{\substack{p+q=s \\ p,q>0}} [c^p_{112}c^q_{211} + c^p_{212}c^q_{212} - c^p_{212}c^q_{111} - c^p_{222}c^q_{112} \\ &\qquad + c^p_{122}c^q_{111} + c^p_{222}c^q_{112} - c^p_{112}c^q_{121} - c^p_{122}c^q_{122}] \\ &= 0 . \end{aligned} \tag{25}$$

We conclude that all obstructions to the integration of infinitesimal deformations of A_2 vanish by letting $r = 1$ in the following theorem.

Theorem 3. *Given* $f_1 \in Z^2(A_2, A_2)$ *and* $f_2, \cdots, f_r \in C^2(A_2, A_2)$ *such that*

$$\delta f_s = \sum_{\substack{p+q=s \\ p,q>0}} f_p * f_q \ , \qquad s = 2, \cdots, r \ , \tag{26}$$

one can extend

$$\alpha\beta + f_1(\alpha, \beta)t + f_2(\alpha, \beta)t^2 + \cdots + f_r(\alpha, \beta)t^r \tag{27}$$

to a deformation $F_t(\alpha, \beta)$ of A_2.

Gerstenhaber [1] has proven that

$$\omega_{r+1} = \sum_{\substack{p+q=r+1 \\ p,q>0}} f_p * f_q \tag{28}$$

is a 3-cocycle. Comparing (21) and (22), we note that a 3-cocycle is a 3-coboundary if the coefficient c_{2222} is zero. Calculating c_{2222} for (28), we have

$$\begin{aligned} c_{2222} &= \sum_{\substack{p+q=r+1 \\ p,q>0}} [c^p_{122}c^q_{221} + c^p_{222}c^q_{222} - c^p_{212}c^q_{221} - c^p_{222}c^q_{222}] \\ &= \sum_{\substack{p+q=r+1 \\ p,q>0}} (c^p_{122} - c^p_{212})c^q_{221} = 0 \ , \end{aligned} \tag{29}$$

by the lemma.

Comparison of (21) and (22), together with (24), (25) and (29), yields the corollary.

Corollary. *With the hypotheses and notation of the theorem, the extendibility of* (27) *to a deformation of* A_2 *is independent of the values of* $a_{111}, a_{112}, a_{221}$, *and* a_{222}, $1 \leq s \leq r$, *and the corresponding coefficients for values of* $s > r$ *may be chosen arbitrarily in integrating* (27).

6. Deformation of A_3

The 4-dimensional algebra A_3, considered as a vector space over $\boldsymbol{R}$, has a canonical basis

$$\left\{ \varepsilon_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \ \varepsilon_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \ \varepsilon_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \ \varepsilon_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\} .$$

The coefficients e_{ijk} in (4) can be expressed in matrix form:

$$e_{ij1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \qquad e_{ij2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{30}$$

$$e_{ij3} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e_{ij4} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In order that

$$f(\varepsilon_i, \varepsilon_j) = \sum_m a_{ijm}\varepsilon_m \tag{31}$$

be an element of $Z^2(A_3, A_3)$, the coefficients a_{ijm} must satisfy

$$\begin{gathered} a_{ij1} = \begin{pmatrix} a_{111} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{241} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad a_{ij2} = \begin{pmatrix} a_{112} & a_{111} & 0 & 0 \\ a_{111} & a_{222} & -a_{241} & a_{242} \\ 0 & -a_{241} & 0 & 0 \\ 0 & a_{422} & 0 & 0 \end{pmatrix}, \\ a_{ij3} = \begin{pmatrix} a_{113} & 0 & a_{111} & a_{112} \\ a_{213} & a_{223} & a_{233} & a_{243} \\ a_{111} & a_{323} & -2a_{241} & a_{343} \\ 0 & a_{423} & a_{422} & a_{443} \end{pmatrix}, \quad a_{ij4} = \begin{pmatrix} a_{213} & 0 & 0 & a_{111} \\ 0 & 0 & 0 & a_{222} - a_{233} \\ 0 & 0 & 0 & -a_{241} \\ a_{111} & a_{323} & -a_{241} & a_{343} + a_{242} \end{pmatrix}. \end{gathered} \tag{32}$$

The dimension of $Z^2(A_3, A_3)$ is 15.

In order that $f(\varepsilon_i, \varepsilon_j)$ given by (31) be an element of $B^2(A_3, A_3)$, i.e., $f = \delta g$, for some $g(\varepsilon_i) = \sum_j b_{ij}\varepsilon_j \in C^1(A_3, A_3)$, its coefficients a_{ijm} must satisfy

$$\begin{gathered} a_{ij1} = \begin{pmatrix} b_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_{31} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad a_{ij2} = \begin{pmatrix} b_{12} & b_{11} & 0 & 0 \\ b_{11} & 2b_{21} & b_{31} & b_{41} - b_{32} \\ 0 & b_{31} & 0 & 0 \\ 0 & b_{41} & 0 & 0 \end{pmatrix}, \\ a_{ij3} = \begin{pmatrix} b_{13} & 0 & b_{11} & b_{12} \\ b_{14} & b_{24} & b_{34} + b_{21} & b_{44} - b_{33} + b_{22} \\ b_{11} & b_{21} & 2b_{31} & b_{41} + b_{32} \\ 0 & 0 & b_{41} & b_{42} \end{pmatrix}, \\ a_{ij4} = \begin{pmatrix} b_{14} & 0 & 0 & b_{11} \\ 0 & 0 & 0 & b_{21} - b_{34} \\ 0 & 0 & 0 & b_{31} \\ b_{11} & b_{21} & b_{31} & 2b_{41} \end{pmatrix}. \end{gathered} \tag{33}$$

Hence the dimensions of $B^2(A_3, A_3)$ and $H^2(A_3, A_3)$ are, respectively, 12 and 3.

From (8) we conclude that $B^3(A_3, A_3)$ consists of cochains whose coefficients c_{ijkm} satisfy the following constraints, where the a_{ijm} are the coefficients for some 2-cochain given by (31):

$$c_{ij11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -a_{211} & 0 & 0 & -a_{311} \\ -a_{311} & 0 & 0 & 0 \\ -a_{411} & 0 & 0 & 0 \end{pmatrix},$$

$$c_{ij12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{111} - a_{212} & a_{221} & a_{311} & a_{411} - a_{312} \\ - a_{312} & 0 & 0 & 0 \\ - a_{412} & 0 & 0 & 0 \end{pmatrix},$$

$$c_{ij13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{114} - a_{213} & a_{214} & a_{314} & a_{414} - a_{313} \\ a_{111} - a_{313} & a_{211} & a_{311} & a_{411} \\ - a_{413} & 0 & 0 & 0 \end{pmatrix},$$

$$c_{ij14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -a_{214} & 0 & 0 & -a_{314} \\ -a_{314} & 0 & 0 & 0 \\ a_{111} - a_{414} & a_{211} & a_{311} & a_{411} \end{pmatrix},$$

$$c_{ij21} = \begin{pmatrix} a_{121} & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_{321} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$c_{ij22} = \begin{pmatrix} a_{122} - a_{111} & -a_{121} & -a_{131} & -a_{141} \\ a_{121} - a_{211} & 0 & a_{321} - a_{231} & a_{421} - a_{241} - a_{322} \\ -a_{311} & -a_{321} & -a_{331} & -a_{341} \\ -a_{411} & -a_{421} & -a_{431} & -a_{441} \end{pmatrix},$$

$$c_{ij23} = \begin{pmatrix} a_{123} & 0 & 0 & 0 \\ a_{124} & a_{224} & a_{324} & a_{424} - a_{323} \\ a_{121} & a_{221} & a_{321} & a_{421} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$c_{ij24} = \begin{pmatrix} a_{124} & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_{324} \\ 0 & 0 & 0 & 0 \\ a_{121} & a_{221} & a_{321} & a_{421} \end{pmatrix}, \tag{34}$$

$$c_{ij31} = \begin{pmatrix} a_{131} & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_{331} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$c_{ij32} = \begin{pmatrix} a_{132} & 0 & 0 & 0 \\ a_{131} & a_{231} & a_{331} & a_{431} - a_{332} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$c_{ij33} = \begin{pmatrix} a_{133} - a_{111} & -a_{121} & -a_{131} & -a_{141} \\ a_{134} - a_{211} & a_{234} - a_{221} & a_{334} - a_{231} & a_{434} - a_{241} - a_{333} \\ a_{131} - a_{311} & a_{231} - a_{321} & 0 & a_{431} - a_{341} \\ -a_{411} & -a_{421} & -a_{431} & -a_{441} \end{pmatrix},$$

$$c_{ij34} = \begin{pmatrix} a_{134} & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_{334} \\ 0 & 0 & 0 & 0 \\ a_{131} & a_{231} & a_{331} & a_{431} \end{pmatrix},$$

$$c_{ij41} = \begin{pmatrix} a_{141} & a_{131} & 0 & 0 \\ 0 & a_{231} & 0 & -a_{341} \\ 0 & a_{331} & 0 & 0 \\ 0 & a_{431} & 0 & 0 \end{pmatrix},$$

$$c_{ij42} = \begin{pmatrix} a_{142} & a_{132} & 0 & 0 \\ a_{141} & a_{241} + a_{232} & a_{341} & a_{441} - a_{342} \\ 0 & a_{332} & 0 & 0 \\ 0 & a_{432} & 0 & 0 \end{pmatrix},$$

$$c_{ij43} = \begin{pmatrix} a_{143} - a_{112} & a_{133} - a_{122} & -a_{132} & -a_{142} \\ a_{144} - a_{212} & a_{244} - a_{222} + a_{233} & a_{344} - a_{232} & a_{444} - a_{242} - a_{343} \\ a_{141} - a_{312} & a_{241} - a_{322} + a_{333} & a_{341} - a_{332} & a_{441} - a_{342} \\ -a_{412} & a_{433} - a_{422} & -a_{432} & -a_{442} \end{pmatrix},$$

$$c_{ij44} = \begin{pmatrix} a_{144} - a_{111} & a_{134} - a_{121} & -a_{131} & -a_{141} \\ -a_{211} & a_{234} - a_{221} & -a_{231} & -a_{241} - a_{344} \\ -a_{311} & a_{334} - a_{321} & -a_{331} & -a_{241} \\ a_{141} - a_{441} & a_{434} - a_{421} + a_{241} & -a_{431} + a_{341} & 0 \end{pmatrix}.$$

In the algebra A_2, we found that all obstructions to integration of infinitesimal and partial deformations vanished (Theorem 3). For A_3, we have the contrary result.

Lemma 2. *The infinitesimal deformation*

$$f(\varepsilon_i, \varepsilon_j) = (\delta_{i3} + \delta_{i4})\delta_{j2}\varepsilon_3 + \delta_{i4}\delta_{j2}\varepsilon_4 = \sum_m a_{ijm}\varepsilon_m , \tag{35}$$

where

$$a_{ijm} = (\delta_{i3} + \delta_{i4})\delta_{j2}\delta_{m3} + \delta_{i4}\delta_{j2}\delta_{m4}$$

is not integrable.

Comparing (32) and (33) we note that $f(\varepsilon_i, \varepsilon_j)$ in (35) is a cocycle but not a coboundary. The primary obstruction

$$\omega_2(\varepsilon_i, \varepsilon_j, \varepsilon_k) = f(f(\varepsilon_i, \varepsilon_j), \varepsilon_k) - f(\varepsilon_i, f(\varepsilon_j, \varepsilon_k)) = \sum_m c_{ijkm}\varepsilon_m \tag{36}$$

has as a coefficient:

$$c_{4223} = (2a_{323} - a_{222})a_{423} = 2\ . \tag{37}$$

From (34) any 3-coboundary $\sum_m c_{ijkm}\varepsilon_m \in B^3(A_3, A_3)$ must have $c_{4223} = 0$. Therefore (36) is an actual obstruction, and the 2-cocycle (35) is not integrable.

More generally, in order that the primary obstruction to the integration of the cocycle $f(\varepsilon_i, \varepsilon_j) = \sum_m a_{ijm}\varepsilon_m$ be cohomologous to zero (i.e. vanish), the following relations must be satisfied by the a_{ijm}:

$$a_{423}(2a_{323} - a_{222}) = 0\ , \qquad a_{423}(a_{343} + a_{242} - 2a_{422}) = 0\ . \tag{38}$$

7. Existence of deformation of A_n

The existence of deformations of the algebras $A_n, n > 2$, is demonstrated by consideration of the non-cobounding 2-cocycle,

$$f_1(\varepsilon_i, \varepsilon_j) = \sum_m a^1_{ijm}\varepsilon_m = \delta_{iv}\delta_{j2}\varepsilon_n\ , \tag{39}$$

(cf. (11)). The primary obstruction of (39) is

$$\begin{aligned} f_1 * f_1(\varepsilon_i, \varepsilon_j, \varepsilon_k) &= f_1(\delta_{iv}\delta_{j2}\varepsilon_n, \varepsilon_k) - f_1(\varepsilon_i, \delta_{jv}\delta_{k2}\varepsilon_n) \\ &= (\delta_{iv}\delta_{j2}\delta_{nv}\delta_{k2} - \delta_{jv}\delta_{k2}\delta_{iv}\delta_{n2})\varepsilon_n = 0\ , \end{aligned}$$

since $n \neq v, n \neq 2$. Therefore, in particular, choosing $f_s = 0, s \geq 2$, we have the deformation of A_n,

$$F_t(\varepsilon_i, \varepsilon_j) = \varepsilon_i\varepsilon_j + \delta_{iv}\delta_{j2}\varepsilon_n t\ , \qquad n > 2\ .$$

The similar deformation of A_2 was given in § 5.

References

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