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TWO CLASSES OF CLASSICAL SUBGROUPS OF Diff(M)

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Introduction

Sometime ago in a letter J. Eells asked us whether it was possible to give a differential structure to the automorphisms of a G-structure similar to the one for the group of diffeomorphisms. At this time the author does not know whether it is possible to give a local modelling of the group of automorphisms $D_G(M)$ of an arbitrary G-structure on a compact manifold M, although many of the formal properties of a manifold are satisfied for $D_G(M)$. The purpose of this note is to give a manifold structure to $D_G(M)$ in two cases:

(i) when the Lie algebra of G is closed under matrix multiplication, and (ii) it contains the case when G is elliptic in the sense of Spencer [11].

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1. Analysis in topological vector spaces

All topological vector spaces appearing in this paper are Hausdorff complete locally convex topological vector spaces over the real numbers R; continuous functions will be called C^0 functions when convenient.

Definition 1. Let $U \subset E$, $V \subset F$ be open sets in topological vector spaces E and F, and suppose that G is a third topological vector space. A function $f: U \times V \to G$ is n times differentiable at $(\xi, \eta) \in U \times V$ in the first (resp. second) variable, if f is $n-1$ times differentiable in the first (resp. second) variable at (ξ, η), and there exists a continuous symmetric n-multilinear function

$$(\partial^u f/\partial x^n)(\xi, \eta) : \underbrace{E \times \cdots \times E}_{n\text{-times}} \to G$$

$$(\text{resp. } (\partial^n f/\partial y^n)(\xi, \eta) : \underbrace{F \times \cdots \times F}_{n\text{-times}} \to G)$$

such that

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$$F(v) = f(\xi + v, \eta) - f(\xi, \eta) - (\partial f/\partial x)(\xi, \eta)(v) - \cdots - (1/n!)(\partial^n f/\partial x^n)(\xi, \eta)(v, \cdots, v)$$

$$(\text{resp. } G(v) = f(\xi, \eta + v) - (\partial f/\partial y)(\xi, \eta)(v) - \cdots - (1/n!)(\partial^n f/\partial y^n)(\xi, \eta)(v, \cdots, v))$$

has the property that

$$\begin{aligned} \varphi(t, v) &= F(tv)/t^n , \qquad t \neq 0 , \\ &= 0 , \qquad t = 0 \end{aligned}$$

$$(\text{resp. } \gamma(t, v) = G(tv)/t^n, t \neq 0, \gamma(t, v) = 0, t = 0)$$

is continuous on $R \times E$ (resp. $R \times F$) at $(0, v)$, $v \in E$ (resp. $v \in F$).

Throughout this paper when we speak of derivative and differentiability it will be with respect to the above definition. Setting $F = \{0\}$ we find the definition of an n times differentiable function $f : U \to G$. It is obvious how to generalize the above definition to any number of variables.

Definition. f is said to be C^n in the first (resp. 2nd) variable if f is n times differentiable at each $(x, y) \in U \times V$, and

$$\partial^m f/\partial x^m \ (\text{resp. } \partial^m f/\partial y^m)$$

defines a continuous function

$$\begin{aligned} &U \times V \times E \times \cdots \times E \to G \\ &(\text{resp. } U \times V \times F \times \cdots \times F \to G) \qquad \text{for } 0 \leq m \leq n . \end{aligned}$$

The following four propositions are easy to prove, but useful to state.

Proposition 1. *Let E and F be Banach spaces, and U an open subset of E. If $f : U \to F$ is C^n in the above sense, then f is C^{n-1} in the Fréchet sense.*

Proof. Note that C^0 in the above sense and C^0 in the Fréchet sense are the same, namely, continuous. By definition C^1 in the above sense implies C^0 in the Fréchet sense. Suppose it has been established that C^k in the above sense implies C^{k-1} in the Fréchet sense for $k < n$, and suppose $f : U \to F$ is C^n in the above sense. As $Df : U \times E \times \cdots \times E \to F$ is continuous at $x_0 \in U$, it follows that there exist an open neighborhood U_0 of x_0 in U and a positive constant K so that $|D^n f(U_0)| < K$, where $D^n f(x)$ in $L_s^n(E, F)$ is the map induced by fixing x from $D^n f$. Thus for y in U_0 we have

$$\begin{aligned} &|D^{n-1}f(y)(\alpha_1, \cdots, \alpha_n) - D^{n-1}f(x_0)(\alpha_1, \cdots, \alpha_n)| \\ &\quad < \left|\int_0^1 D^n f(x_0 + t(y - x_0), (y - x_0), \alpha_1, \cdots, \alpha_{n-1})dt\right| \\ &\quad < \int_0^1 |D^n f(x_0 + t(y - x_0), (y - x_0), \alpha_1, \cdots, \alpha_{n-1})| dt \\ &\quad < K|y - x_0||\alpha_1| \cdots |\alpha_{n-1}| . \end{aligned}$$

Proposition 2. *Let E and G be complete locally convex topological vector spaces, and suppose that F is a closed subspace of G and that U is an open subset of E. A function* $f : U \to F$ *is* C^n *if and only if* $i \circ f$ *is* C^n, *where* $i : F \to G$ *is the canonical injection.*

Our proof makes use of the following

Lemma. *Let E and F be topological vector spaces, and suppose* $U \subset E$ *be an open convex subset. If* $f : U \to F$ *is* C^r, *and* $D^r f : U \times E \times \cdots \times E \to F$ *is* C^s *in the first variable, then f is* C^{r+s}.

Proof. Let $A_{s,r}(x, \alpha_1, \cdots, \alpha_s, \beta_1, \cdots, \beta_r)$ be the symmetrization of

$$\frac{(r+s)!}{(s+1)!r!}(\partial^s/\partial x^s)D^n f(x, \alpha_1, \cdots, \alpha_s; \beta_1, \cdots, \beta_r) .$$

Now

$$0 = f(x+th) - f(x) - Df(x, th) - \cdots - (1/(r-1)!)D^{r-1}f(x, th, \cdots, th) - \frac{1}{r!}\int_0^1 D^r f(x + \tau th, th, \cdots, th)d\tau$$

$$= f(x+th) - f(x) - Df(x, th) - \cdots - (1/(r-1)!)D^{r-1}f(x, th, \cdots, th)$$
$$- \frac{1}{r!}\int_0^1 [D^r f(x, th, \cdots, th) + (\partial/\partial x)D^r f(x, th, \cdots, th, t\tau h)$$
$$+ \cdots + \frac{1}{(s-1)!}(\partial^{s-1}/\partial x^{s-1})D^r f(x, th, \cdots, th, t\tau h, \cdots, t\tau h)$$
$$+ \frac{1}{s!}\int_0^1 (\partial^s/\partial x^s)D^r f(x + \sigma\tau th, th, \cdots, th, t\tau h, \cdots, t\tau h)d\sigma]dt ,$$

$$\tau = f(x+th) - f(x) - Df(x, th) - \cdots - \frac{1}{(r-1)!}D^{r-1}(fx, th, \cdots, th)$$
$$- \frac{1}{r!2}(\partial/\partial x)D^r f(x, th, \cdots, th) - \cdots$$
$$- \frac{1}{r!s!}(\partial^{s-1}/\partial x^{s-1})D^r f(x, th, \cdots, th)$$
$$- \frac{1}{r!s!}\int_0^1\int_0^1 (\partial^s/\partial x^s)D^r f(x + \sigma\tau th, th, \cdots, th, t\tau h, \cdots, t\tau h)d\sigma d\tau .$$

If we subtract the last expression divided by t^{r+s} from

$$\{\mathrm{f}(x+th) - f(x) - A_{0,1}(x, th) - \cdots - \frac{1}{r!}A_{0,r}(x, th, \cdots, th)$$
$$- \frac{1}{(r+1)!}A_{1,r}(x, th, \cdots, th) - \cdots - \frac{1}{(r+s)!}A_{s,r}(x, th, \cdots, th)\}/t^{r+s},$$

we obtain

$$\int_0^1\int_0^1\Big[\frac{1}{(sh)!r!}(\partial^s/\partial x^s)D^rf(x, th, \cdots, th)$$
$$- \frac{1}{s!r!}(\partial^s/\partial x^s)D^rf(x + \sigma\tau th, th, \cdots, th, t\tau h, \cdots, t\tau h)\Big]d\sigma d\tau/t^{r+s}$$
$$= \int_0^1\int_0^1\Big[\frac{1}{(s+1)!r!}(\partial^s/\partial x^s)D^rf(x, h, \cdots, h)$$
$$- \frac{1}{s!r!}(\partial^s/\partial x^s)D^rf(x + \sigma\tau th, h, \cdots, h, th, \cdots, th)\Big]d\sigma d\tau$$
$$= \phi(t, x, h) \ .$$

ϕ is a continuous function so that $\phi(0, x, h) = 0$.

Corollary. *$f : U \to F$ is C^{r+s} if and only if f is C^r and $D^rf(x, \alpha_1, \cdots, \alpha_r)$ is C^s in the first variable.*

Definition. Let $\{B_i\}_{i\geq 0}$ be a sequence of Banach spaces so that

(i) B_{i+1} is a subspace of B_i for the underlying vector space structure,

(ii) the injection $k_i^{i+1} : B_{i+1} \to B_i$ induces a continuous function $\{B_i\}_{0\leq i\leq\infty}$ is called a Banach chain where $B_\infty = \bigcap_{i\geq 0} B_i$ is considered to have the inverse limit topology.

Proposition 3. *Let $\{B_i^1\}$ and $\{B_i^2\}$ be Banach chains. Suppose $U \subset B_\infty^1$ is an open set, and $f : U \to B_\infty^2$ is a function so that for every positive integer r there exist a strictly increasing sequence of positive integers k, a monotonically increasing positive integral valued function $\alpha_r(k)$, and a collection of open sets $U_{k,r} \subset B_k^1$ so that $U \subset U_{k,r} \cap B^1$. If f extends to a C^r function $f_{k,r} : U_{k,r} \to B_{\alpha(k)}^2$, then f is a C^∞ function.*

The proof of the above proposition follows from the definitions as does

Proposition 4. *Let E and G be complete locally convex topological vector spaces and $U \subset E$ be open, and suppose F is a closed subspace of G. Then f is C^n if and only if $i \circ f : U \to G$ is C^n, where $i : F \to G$ is the canonical injection.*

2. The automorphisms of two classes of G-structures

We recall that Diff(M), $D(M)$, $D_n(M)$, and $\mathscr{D}_n(M)$ are respectively the group of diffeomorphisms with the C^∞ topology, the connected component of the indentity in Diff(M), the group of C^n diffeomorphisms of M, and the vector space of C^{n-1} right invariant vactor fields on $D_n(M)$, and [5, p. 267] that the tangent space at $f \in$ Diff(M) can be represented by $\mathscr{D}_f(M) = \{\alpha : M \to TM \,|\, \tau \circ \alpha = f\}$. An admissible chart at $f \in$ Diff(M) can be given as follows: From [5], $\exists t > 0$ such that setting

$$S_t = \{\alpha \in \mathscr{D}_f(M) \,|\, |\alpha|_1 < t \quad \text{where} \quad |\alpha|_1 \text{ is the } C^1 \text{ norm}\}$$

and defining $e(\alpha)(x) = \exp_r(\alpha(x))$ where $\exp_r$ is the Riemannian exponential we obtain a chart at f. Multiplication and inversion define smooth maps.

It is now useful to put some properties of the classical subgroups of Diff (M) in our terminology. Let $C^\infty(E)$ be the space of sections of a locally trivial fiber bundle $\pi : E \to M$, where M is compact.

Proposition 1. *$C^\infty(E)$ can be given the structure of a smooth C^∞ manifold in such a way that the tangent space of $C^\infty(E)$ at $s \in C^\infty(E)$ can be represented by the nuclear space of smooth sections of $TF(E) \xrightarrow{\pi \circ p} M$ with the C^∞ topology.*

Using Palais' notion of a bundle spray (see [9]) this proposition can be proved by the same methods used in [5] to show that Diff(M) admits a smooth manifold structure. Hereafter when $C^\infty(E)$ is considered as a manifold it will be with respect to this structure.

Proposition 2. *Suppose $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$, where M is a smooth compact manifold, are smooth fiber bundles. If $f : E_1 \to E_2$ is a bundle homomorphism over M, then $f_* : C^\infty(E_1) \to C^\infty(E_2)$ is a smooth function.*

The following is immediate from the definitions.

Proposition 3. *Let $E = M \times M \xrightarrow{\pi_1} M$ be projection on the first factor, and $J_r \xrightarrow{\alpha_r} M$ be the fiber space of r jets with projection on the source. Then the jet extension map $j_r : C^\infty(E) \to \gamma_r(M) = C^\infty(J_r)$ is smooth.*

Designate by μ_r the fiber space of invertible r-jets of smooth endomorphisms of M. μ_r is an open submanifold of J_r so that $\alpha_r \,|\, \mu_r$ is a principal fibration.

Definition. Let $\pi : E \to M$ be an arbitrary smooth locally trivial fibration. A Lie differential operator of order r on Diff (M) is a function $D = f_* \circ j_r$, where $j_r :$ Diff $(M) \to \gamma_r$ is the canonical map, and $f : \gamma_r \to E_2$ is a smooth morphisms of fiber bundles over M; so that

(i) $D^{-1}(D(e)) = G$ is a subgroup of Diff (M),

(ii) $D(gh) = D(h)$ for $g \in G$.

When $D :$ Diff $(M) \to C^\infty(E)$ is a Lie differential operator, $D^{-1}(D(e))$ is called a classical subgroup of Diff (M). Note that a Lie differential operator defines a smooth function $D :$ Diff $(M) \to C^\infty(E)$.

Proposition 4. *Let $D^{-1}(D(e)) = G$ be a classical subgroup of* Diff (M), *suppose* $\exp : \mathscr{D}(M) \to$ Diff (M) *is the Lie exponential, and let $g = \{g \in \mathscr{D}(M) \,|\, T_h D(R_h(g)) = 0$ for all $h \in$* Diff (M), *where R_h is induced by right multiplication by $h\}$. Then* $\exp(tX) \in G$ *for every t if and only if $X \in g$.*

Proof. Suppose $\exp(tX) \in G$ for all t. Then

$$T_h D(R_h X) = \left(\frac{d}{dt}\right)_{t=0} D(\exp(tX)h) = 0 ,$$

since D is constant on Gh.

Now for $X \in g$ set $f(t) = D(\exp(tX))$, so that

$$f'(t) = T_{\exp(tX)} D(X(\exp(tX)) = T_{\exp(tX)} D(R_{\exp(tX)} X) = 0 .$$

Thus $f(t) = f(0) = D(e)$ and $f(t) \in G$.

Proposition 5. *Under the hypotheses of Proposition* 9, *g is a Lie subalgebra of $\mathscr{D}(M)$.*

Proof. Since g is obviously a vector space, we need only to show that g is closed under the bracket operation for vector fields. Let $X, Y \in g$, and suppose φ_t generates X (i.e., $T_{t=0}\varphi_t = X$) and ψ_t generates Y. Then we have

$$\begin{aligned} T_h D(R_h[X, Y]) &= T_h D\left(\lim_{t\to 0} \frac{1}{t}(R_h X - R_h(ad\, \psi_t X))\right) \\ &= \lim_{t\to 0} \frac{1}{t}\{T_{t=0}(D(\psi_t \circ h)) - T_{t=0}(D(\psi_t \psi_t^{-1} h))\} \\ &= \lim_{t\to 0} \frac{1}{t}\{T_{t=0} D(h) - T_{t=0} D(h)\} \\ &= 0 . \end{aligned}$$

Hence $[X, Y] \in g$. q.e.d.

g is called the Lie algebra of G.

A subgroup of the full linear group $GL(n)$ will be called locally convex when it is locally convex for the canonical vector space structure on $M(n)$.

Proposition 6. *Let G be a Lie subgroup of $GL(n)$. If its Lie algebra $g \subset M(n)$ is closed under matrix multiplication, then G is a locally convex open subset of $I + g$.*

Theorem. *Let M be a compact smooth manifold, without boundary, of dimension n, and let G be a subgroup of $GL(n)$ whose Lie algebra is closed under matrix multiplication. Suppose the group of the tangent bundle of M can be reduced to G (i.e., M admits a G-structure). Then the automorphisms of the G-structure $D_G(M)$ admits a manifold structure locally diffeomorphic to its tangent space at a point, and $f: U \to D_G(M)$ is smooth if and only if $i \circ f: U \to \mathrm{Diff}(M)$ is smooth, where $i: D_G(M) \to \mathrm{Diff}(M)$ is the canonical homomorphism.*

Proof. Choose a G connection on M, and let $\exp_G: TM \to M$ be the exponential map associated with this G-structure. $\widetilde{\exp}_G: \mathscr{D}(M) \to F(M, M)$ given by $\widetilde{\exp}_G(\alpha)(x) = \widetilde{\exp}_G \circ \alpha(x)$ is such that there exists a real number $t > 0$ so that $\widetilde{\exp}_G | S_t(0)$ is a diffeomorphism onto an open neighborhood of the identity in $\mathrm{Diff}(M)$.

Cover M by normal coordinate neighborhoods $\{U_i\}$ with respect to the given G-connection, and consider X in the Lie algebra $\mathscr{G}$ of $D_G(M)$. We shall prove that $\widetilde{\exp}_G(X) \in D_G(M)$ for $|X|_1 < t$. Locally with respect to the normal coordinate $\widetilde{\exp}_G(X)(x)$ can be written as $x + X_x$ for t sufficiently small. Suppose $g_t = \exp_l(tX)$. Then we have

$$\begin{aligned} D_x \widetilde{\exp}_G(X)(x, \alpha) &= D \exp_G \circ D_x D_{t=0} g_t(x, \alpha) \\ &= D \exp_G \circ D_{t=0} D_x g_t(x, \alpha) = D \exp \circ g(x, \alpha) \end{aligned}$$

where $g(x, \alpha) = (x, \alpha, X_x, \gamma(\alpha))$, γ being in the Lie algebra of G. Thus

$$D_x(\widetilde{\exp}_G(X))(x, \alpha) = (x + X_x, \alpha + g_x(\alpha)) ,$$

where g_x is in the Lie algebra of G. For X sufficiently C^1 small, g_x is small and thus $\alpha \to \alpha + g_x(\alpha) \in G$.

Now suppose $(D_x g)(x, \alpha) = (g(x), h(\alpha))$ where $h \in G$ and $g \in \mathrm{Diff}(M)$; $h \in GL(n)$ is given by the connection on M. Now $\widetilde{\exp}_G^{-1}(g)(x) = (x, g(x) - x)$.

Consider $h_t(x) = x + t(g(x) - x)$ so that $D_x h_t(x, \alpha) = (h_t(x), \alpha + t\gamma(\alpha))$ where γ is in the Lie algebra of G. Thus $D_x h_t(x, \alpha) = (h_t(x), g(\alpha))$ where $g \in G$, and $H(t) = h_t$ is a smooth arc in $\mathrm{Diff}(M)$ so that $H(-1, 1) \subset D_G(M)$. Hence $D_{t=0} H = \{x \to (x, g(x)) - x\} \in T_e D_G(M)$.

Similarly, $\exp_G : \{X \cdot g \mid g \in D_G(M),\ X \in g,$ and $|X|_1 < t\} \to D_G(M)$ maps diffeomorphically onto a neighborhood of g. By the same procedure as in [5] one obtains that $D_G(M)$ is a manifold where multiplication defines a smooth function and $g \to g^{-1}$ is smooth.

The final statement of the theorem follows from Proposition 2, § 1.

Corollary (*see* [12]). *The automorphisms of a multifoliate structure on a compact manifold satisfy the conclusions of the above theorem.*

Definition 1. A chain of Hilbert spaces $\{H_i\}_{0<i<\infty}$ is a chain of Banach spaces where the H_i are Hilbertable spaces.

It is classical that a nuclear space can be given as the H_∞ in a chain of Hilbert spaces.

In the category of chains of Hilbert spaces as in the category of chains of Banach spaces (see [6]), a mapping $f : U \to H^2_\infty$, $U \subset H^1_\infty$ being open, is said to be C^r when there exists a sequence of integers $k \to \infty$ such that f extends to C^r mappings $f_k : U_k \to H^2_{\lambda(k)}$ where $U_k \subset H^1_k$ is open and $U = H^1_\infty \cap U_k$. Proposition 3 of § 1 states that C^r in the category of Banach or Hilbert chains is a stronger notion than C^r in the category of nuclear spaces in terms of Definition 1, § 1.

We shall now review the Ebin-Omori notion of inverse limit Hilbert manifolds as applied to the group of diffeomorphisms.

Definition 2. A sequence of C^∞ Hilbert manifolds $\{X_r\}$ is called an inverse limit Hilbert system (or an I.L.H. system) when

(i) $X_{r+1} \subset X_r$,

(ii) there is a Hilbert chain $\{H_r\}$ such that for $x \in X_\infty$ there exist charts at x:

$$\varphi_r : U_r \to X \ , \qquad U_r \subset H_r \text{ being open.}$$

An I.L.H. system $\{X_r\}$ is called an inverse limit Hilbert system of groups (or an I.L.H.G. system) when X_{r+1} is a subgroup of X_r and multiplication and inversion define smooth maps in the category of Banach chains.

Now let M be a compact smooth (C^∞) manifold, and $\pi : E \to M$ be a Riemannian vector bundle over M. For an integer $s \geq 0$, let $H^s(E)$ be the completion of $j_s(C^\infty(E))$ in the norm involving the integral of the inner product in $J^s(E)$, and set $C^k(E)$ equal to the space of sections of E of class C^k. Then by the Sobolev theorems one has canonically

$$H^{n/2+k+1}(E) \subset C^k(E) \subset H^k(E) \ ,$$

where $n = \dim(M)$. Similarly, when M and N are manifolds and $s > n/2 + 1$, it makes sense to talk of an H^s map from M to N by looking at the mapping locally. So let $H^s(M, N)$ be the space of H^s maps from M to N for $s > \dim M/2 + 1$, and set $D^s(M) = \{H^s(M, M) \cap D_1(M)\}$ for $S > n/2 + 1$.

By the same construction as on p. 433 one may show that $D^s(M)$ is a smooth Hilbert manifold modelled on $H^s(TM)$. Since $D_\infty(M)$ is an inverse limit Banach group (see [10]), it follows from the Sobolev theorems that $D_\infty(M)$ is an inverse limit Hilbert group.

In [8] Omori proved

Theorem 2. *Let M be a compact manifold, and $D : C^\infty(TM) \to C^\infty(E)$ be a linear differential operator of order l. Then there exists a vector bundle over $D^s(M)$, $\varepsilon^s \to D^s(M)$ with fiber at $g \in D^s(M) H^s(E) \circ g$ so that D defines a vector bundle morphism*

$$\begin{array}{ccc} TD^{s+l} & \xrightarrow{\tilde{D}} & s \\ \Big\downarrow{\scriptstyle \pi_{s+l}} & & \Big\downarrow{\scriptstyle \pi_s} \\ D^{s+l} & \xrightarrow{i_s^{s+l}} & D^s \end{array}$$

with $\tilde{D}(\alpha \circ g) = D(\alpha) \circ g$, where $\alpha \in H^{s+l}(TM)$ and $g \in D^{s+l}(M)$.

Definition. A linear differential operator of order $l, D : C^\infty(E_1) \to C^\infty(E_2)$, is called closed when D extends to maps $D^s : H^{s+l}(E_1) \to H^s(E_2)$ with closed range.

Theorem 3. *Let G be a classical subgroup of* Diff (M). *If its Lie algebra g is the kernel of a closed linear differential operator $d : C^\infty(TM) \to C^\infty(E)$, then G contains a closed normal subgroup H_∞ and* $\exp(g) \in H_\infty$, *where $\{H_i\}$ is a* sub-I.L.H.G. *of $\{D^s(M)\}$ with H_s a closed submanifold of $\{D^s(M)\}$.*

Proof. Let K_s be the complement of $d_{s+l}(H^{s+l}(TM))$ in $H^s(E)$. By means of Theorem 2 and Proposition 6 [4, p. 45] we obtain that $\operatorname{Ker}(\tilde{d}_{s+l}) = \operatorname{Ker}(\tilde{d}_{s+l} \oplus \mathrm{id}_K) : U \times H^{s+l}(TM \oplus K \to U \times H^s(E))$ is a closed sub-bundle of $TD^{s+l}(M)$, so there exists a connected subgroup H_{s+l}, which is also a C^∞

manifold, with tangent space at the identity $= g_{s+l} =$ the closure of g in $H^{s+l}(TM)$ (see [6]). From the construction it hence follows that $g = \bigcap_s g_s$.

Now let $D : \text{Diff}(M) \to C^\infty(\xi)$ be the non-linear differential operator of order k which defines $G = D^{-1}(D(e))$. Then D extends to smooth $D_{s+k} : D^{s+k}(M) \to H^s(\xi)$ (see [9, p. 67]). It is easy to see that $H_{s+k} \subset D_{s+k}^{-1}(D(e))$ and that the arc component of $D_{s+k}^{-1}(D(e)) \subset H_{s+k}$; thus H_{s+k} is normal in $D_{s+k}^{-1}(D(e))$. H_{s+k} is locally closed in $D^{s+k}(M)$ and hence closed being a topological subgroup of $D^{s+k}(M)$.

Remark 1. When $TD_{s+k} : TD^{s+k}(M) \to TH^s(\xi)$ are surjective, the D_{s+k} are submersions, $D_{s+k}^{-1}(D(e))$ are submanifolds of $D^{s+k}(M)$, and G itself may be regarded as an I.L.H.G. In this case a mapping $f : U \to G$, U being open in some vector space, is C^n if and only if $f : U \to H_s$ is C^n for all s.

Remark 2. The differential structures of H_∞ and G are locally the same in some sense due to the fact that if U is a convex open set of a topological vector space, and $f : U \to G$ is continuous with $x_0 \in U$, then $f(U) \cdot f(x_0)^{-1} \in H_\infty$.

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