

COMPLEX LAPLACIANS ON ALMOST-HERMITIAN MANIFOLDS

CHUAN-CHIH HSIUNG & JOHN J. LEVKO III

Introduction

In [2] Hsiung (i) defined a new complex Laplacian \square_2 for an almost-Hermitian structure, which is different from the one, denoted by \square_1 , given by Kodaira and Spencer [3], (ii) verified for \square_2 the well-known conjecture that if $\square_2 = \Delta/2$ for all 0- and 1-forms, where Δ is the real Laplacian, then the structure is Kählerian, (iii) studied the conditions for \square_2 to be real for all 0- and 1-forms. Very recently, Ogawa [5] continued Hsiung's work to show that if either \square_2 or \square_1 is real for all 0- and 1-forms, then the structure is Kählerian.

The purpose of this paper is to introduce three more complex Laplacians \square_3 , \square_4 , \square_5 for an almost-Hermitian structure and to study the conditions for these Laplacians to be real, together with some relationships among all \square 's. We shall continue to use Hsiung's method [2] which is somewhat different from Ogawa's, and also for completeness we shall reprove Ogawa's result here.

§ 1 contains fundamental notation and real operators on a Riemannian manifold. In § 2 we define various almost-Hermitian structures first and then some complex operators for an almost Hermitian structure leading to the complex Laplacians $\square_i, i = 1, \dots, 5$. Some conditions for the tensor of an almost-Hermitian structure to be Kählerian are also given for use in the proofs of our main theorems. § 3 is devoted to the computation of $\square_i \xi$ and $\square_i \eta, i = 1, \dots, 5$, for any 0-form ξ and 1-form η on an almost-Hermitian manifold. In § 4 we show that for an almost-Hermitian structure if the complex Laplacian $\square_i, i = 1, 2$ or 4 is real with respect to all 0- and 1-forms, then the structure is Kählerian. In § 5 we obtain the following relationships among the \square 's: If for an almost-Hermitian structure the relation $\text{Im } \square_1 = \text{Im } \square_i (i = 2 \text{ or } 4)$ or $\text{Im } \square_2 = \text{Im } \square_j (j = 4 \text{ or } 5)$ holds for all 0- and 1-forms, where Im denotes the imaginary part, then the structure is Kählerian.

Throughout this paper, the dimension of a manifold M^n is understood to be $n \geq 2$, and all forms and structures are of class at least C^2 .

1. Notation and real operators

Let M^n be a Riemannian manifold of dimension $n(\geq 2)$, $\|g_{ij}\|$ with $g_{ij} = g_{ji}$ be the matrix of the positive definite metric of M^n , and $\|g^{ij}\|$ be the inverse matrix of $\|g_{ij}\|$. Throughout this paper all Latin indices take the values $1, \dots, n$ unless stated otherwise. We shall follow the usual tensor convention that indices can be raised and lowered by using g^{ij} and g_{ij} respectively, and also that when a Latin letter appears in any term as a subscript and superscript, it is understood that this letter is summed over its range. Moreover, if we multiply, for example, the components a_{ij} of a tensor of type $(0, 2)$ by the components b^{jk} of a tensor of type $(2, 0)$, it will always be understood that j is to be summed.

Let \mathcal{N} be the set $\{1, \dots, n\}$ of positive integers less than or equal to n , and let $I(p)$ denote an ordered subset $\{i_1, \dots, i_p\}$ of the set \mathcal{N} for $p \leq n$. If the elements i_1, \dots, i_p are in the natural order, that is, if $i_1 < \dots < i_p$, then the ordered set $I(p)$ is denoted by $I_0(p)$. Furthermore, denote the nondecreasingly ordered p -tuple having the same elements as $I(p)$ by $\langle I(p) \rangle$, and let $I(p; \hat{s} | j)$ be the ordered set $I(p)$ with the s -th element i_s replaced by another element j of \mathcal{N} , which may or may not belong to $I(p)$. We shall use these notations for indices throughout this paper. When more than one set of indices is needed at one time, we may use other capital letters such as J, K, L, \dots in addition to I .

At first we define

$$(1.1) \quad \varepsilon_{K(p)}^{J(p)} = \begin{cases} 0, & \text{if } \langle J(p) \rangle \neq \langle K(p) \rangle, \\ 0, & \text{if } J(p) \text{ or } K(p) \text{ contains repeated integers,} \\ +1 \text{ or } -1, & \text{if the permutation taking } J(p) \text{ into } K(p) \text{ is} \\ & \text{even or odd.} \end{cases}$$

By counting the number of terms it is easy to verify that

$$(1.2) \quad \varepsilon_{1 \dots n}^{I(p)J(n-p)} \varepsilon_{I(p)K(n-p)}^{1 \dots n} = p! \varepsilon_{K(n-p)}^{J(n-p)},$$

$$(1.3) \quad \varepsilon_{K(p+q)}^{I(p)J(q)} \varepsilon_{I(p)}^{L(p)} = p! \varepsilon_{K(p+q)}^{L(p)J(q)}.$$

On the manifold M^n , let \mathcal{V} denote the covariant derivation with respect to the affine connection Γ , with components Γ^i_{jk} in local coordinates x^1, \dots, x^n , of the Riemannian metric g , and let ϕ be a differential form of degree p given by

$$(1.4) \quad \phi = \frac{1}{p!} \phi_{I(p)} dx^{I(p)} = \phi_{I_0(p)} dx^{I_0(p)},$$

where $\phi_{I(p)}$ is a skew-symmetric tensor of type $(0, p)$, and we have placed

$$(1.5) \quad dx^{I(p)} = dx^{i_1} \wedge \dots \wedge dx^{i_p}.$$

Then we have

$$(1.6) \quad d\phi = (d\phi)_{I_0(p+1)} dx^{I_0(p+1)},$$

where

$$(1.7) \quad (d\phi)_{I(p+1)} = \frac{1}{p!} \varepsilon_{I(p+1)}^{k_1 \dots k_p} \nabla_{k_1} \phi_{J(p)}.$$

Denote

$$(1.8) \quad e_{I(n)} = \varepsilon_{I(n)}^{1 \dots n} (\det(g_{ij}))^{1/2}.$$

Then by using orthonormal local coordinates x^1, \dots, x^n and relation (1.2) we can easily obtain

$$(1.9) \quad e_{I(p)K(n-p)} e^{I(p)J(n-p)} = p! \varepsilon_{K(n-p)}^{J(n-p)}.$$

The dual operator $*$ is defined by (see, for instance, [6])

$$(1.10) \quad * \phi = (* \phi)_{I_0(n-p)} dx^{I_0(n-p)},$$

where

$$(1.11) \quad (* \phi)_{I(n-p)} = \frac{1}{p!} e_{J(p)I(n-p)} \phi^{J(p)}.$$

From (1.10), (1.11) it follows that for the scalar 1

$$(1.12) \quad * 1 = (\det(g_{ij}))^{1/2} dx^1 \wedge \dots \wedge dx^n,$$

which is just the element of area of the manifold M^n . By using orthonormal local coordinates x^1, \dots, x^n we can easily verify that

$$(1.13) \quad ** \phi = (-1)^{p(n-p)} \phi.$$

Denote the inverse operator of $*$ by $*^{-1}$. Then from (1.13) it is seen that on forms of degree p

$$(1.14) \quad *^{-1} = (-1)^{p(n-p)} *.$$

The codifferential operator δ is defined by

$$(1.15) \quad \delta \phi = (-1)^{p+n+1} *^{-1} d * \phi.$$

Making use of (1.6), (1.7), (1.10), (1.11) we obtain immediately

$$(1.16) \quad \delta \phi = (\delta \phi)_{I_0(p-1)} dx^{I_0(p-1)},$$

where

$$(1.17) \quad (\delta\phi)_{I(p-1)} = -\nabla_j \phi^j_{I(p-1)} .$$

For a form ϕ of degree p defined by (1.4) we can obtain

$$(1.18) \quad p! (\Delta\phi)_{I(p)} = -\nabla^j \nabla_j \phi_{I(p)} + \sum_{s=1}^p \phi_{I(p;\hat{s}|a)} R^a_{i_s} + \sum_{\substack{1, \dots, p \\ s < t}} \phi_{I(p;\hat{s}|a, \hat{t}|b)} R^{ab}_{i_s i_t} ,$$

where Δ is the Laplace-Beltrami operator defined by

$$(1.19) \quad \Delta = \delta d + d\delta ,$$

and

$$(1.20) \quad \nabla^j = g^{jk} \nabla_k ,$$

$$(1.21) \quad R^i_{jkl} = \partial \Gamma^i_{jk} / \partial x^l - \partial \Gamma^i_{jl} / \partial x^k + \Gamma^h_{jk} \Gamma^i_{hl} - \Gamma^h_{jl} \Gamma^i_{hk} ,$$

$$(1.22) \quad R_{jk} = R^s_{jks} .$$

2. Complex structures and operators

On a Riemannian manifold M^n with metric tensor g_{ij} , if there exists a tensor F_i^j of type (1,1) satisfying

$$(2.1) \quad F_i^j F_j^k = -\varepsilon_i^k ,$$

then F_i^j is said to define an almost-complex structure on the manifold M^n , and the manifold M^n is called an almost-complex manifold. From (2.1) it follows that the almost-complex structure F_i^j induces an automorphism J of the tangent space of the manifold M^n at each point with $J^2 = -I, I$ being the identity operator, such that, for any tangent vector v^k ,

$$(2.2) \quad J: v^k \rightarrow F_i^k v^i .$$

If an almost-complex structure F_i^j further satisfies

$$(2.3) \quad g_{ij} F_h^i F_k^j = g_{hk} ,$$

then F_i^j is said to define an almost-Hermitian structure on the manifold M^n , and the manifold M^n is called an almost-Hermitian manifold. From (2.1), (2.3) it follows that the tensor F_{ij} of type (0,2) defined by

$$(2.4) \quad F_{ij} = g_{jk} F_i^k$$

is skew-symmetric. Thus on an almost-Hermitian manifold we have the associated differential form

$$(2.5) \quad \omega = F_{ij} dx^i \wedge dx^j .$$

By using the multiplication of matrices, from (2.1) we readily see that a necessary condition for the existence of an almost-complex structure on a Riemannian manifold M^n is that the dimension n of the manifold M^n be even. It should also be remarked that an almost-complex manifold is always orientable, and the orientation depends only on the tensor F_i^j .

An almost-Hermitian structure F_i^j defined on a manifold M^n is called an almost-Kählerian structure and the manifold M^n an almost-Kählerian manifold, if the associated form ω is closed, that is,

$$(2.6) \quad d\omega = 0 .$$

From (2.5), (2.6) it follows that an almost-Kählerian structure F_i^j satisfies

$$(2.7) \quad F_{hij} \equiv \nabla_h F_{ij} + \nabla_i F_{jh} + \nabla_j F_{hi} = 0 .$$

The tensor F_{hij} is obviously skew-symmetric in all indices.

An almost-Hermitian structure F_i^j (respectively manifold) satisfying

$$(2.8) \quad F_i \equiv -\nabla_j F_i^j = 0$$

is called an almost-semi-Kählerian structure (respectively manifold). In particular, the structure F_i^j is Kählerian if $\nabla_i F_j^k = 0$. In this case, by means of (2.1) it is easily seen that the torsion tensor

$$t_{ij}^k = F_j^h (\partial F_i^k / \partial x^h - \partial F_h^k / \partial x^i) - F_i^h (\partial F_j^k / \partial x^h - \partial F_h^k / \partial x^j)$$

vanishes, so that the integrability condition of the almost-complex structure F_i^j is satisfied. But in general when $t_{ij}^k = 0$, the almost-Hermitian structure F_i^j is defined to be Hermitian

Multiplying (2.4) by F^{hi} we obtain

$$(2.9) \quad F_{ij} F^{hi} = -\varepsilon_j^h .$$

By taking covariant differentiation of both sides of (2.9), noticing that

$$(2.10) \quad F^{ij} \nabla_h F_{ij} = 0 ,$$

and making use of (2.7), (2.8) it is easily seen that

$$(2.11) \quad F_{hij} F^{ij} = 2F_h^i F_i .$$

Thus an almost-semi-Kählerian structure F_i^j satisfies

$$(2.12) \quad F_{hij} F^{ij} = 0 .$$

Multiplication of (2.11) by F_k^h and use of (2.9) give

$$(2.13) \quad F_k = -\frac{1}{2}F_{hij}F^{ij}F_k^h .$$

From (2.7), (2.8), (2.13) we hence conclude that *an almost-Kählerian structure or manifold is also almost-semi-Kählerian.*

In the proofs of our theorems we shall need the following lemmas.

Lemma 2.1. *An almost-Hermitian structure F satisfying*

$$(2.14) \quad \nabla_i F_{jk} = \nabla_j F_{ik}$$

is Kählerian.

Proof. From the skew-symmetry of F_{ij} we have

$$(2.15) \quad \nabla_i F_{jk} + \nabla_i F_{kj} = 0 .$$

Taking the sum of (2.15) and the two similar equations obtained from it by cyclic permutation of the indices i, j, k , and making use of (2.14) we obtain $\nabla_i F_{jk} + \nabla_i F_{kj} + \nabla_j F_{ki} = 0$, which together with (2.15) implies immediately $\nabla_j F_{ki} = 0$.

Lemma 2.2. *An almost-Hermitian structure F satisfying*

$$(2.16) \quad F^{ij}\nabla^k\nabla_k F_{ij} = 0$$

is Kählerian.

Proof. From (2.9) we have

$$0 = \nabla^k\nabla_k(F_{ij}F^{ij}) = 2(F^{ij}\nabla^k\nabla_k F_{ij} + \nabla_k F_{ij}\nabla^k F^{ij}) ,$$

which together with (2.16) gives $\nabla_k F_{ij}\nabla^k F^{ij} = 0$ and therefore $\nabla_k F_{ij} = 0$.

Lemma 2.3 (S. Kotô [4]). *An almost-Hermitian structure F satisfying*

$$(2.17) \quad \nabla_i F_j^k + \nabla_j F_i^k = 0 ,$$

$$(2.18) \quad R_{hi} = -\frac{1}{2}R_{hjk}l F^{kl} F_i^j$$

is Kählerian.

Proof. (2.17) can be written as

$$(2.19) \quad \nabla_i F_{jk} = \nabla_k F_{ij} .$$

Multiplying (2.19) by F^{ij} , using (2.10) and taking the covariant derivative ∇_l of the resulting equation, we obtain, in consequence of (2.19),

$$(2.20) \quad F^{ij}\nabla_l\nabla_i F_{jk} + \nabla_k F_{ij}\nabla_l F^{ij} = 0 .$$

On the other hand, using (2.19) and the relation $-F^{ij}\nabla_j\nabla_i F_{kl} = F^{ij}\nabla_i\nabla_j F_{kl}$, from the Ricci identity it follows respectively that

$$(2.21) \quad \nabla_l\nabla_i F_{jk} = \nabla_i\nabla_j F_{kl} + R_{ajil}F_k^a - R_{akil}F_j^a,$$

$$(2.22) \quad F^{ij}\nabla_i\nabla_j F_{kl} = -\frac{1}{2}F^{ij}(R^a_{kji}F_{al} + R^a_{lji}F_{ka}).$$

Similarly, the Bianchi identity leads to

$$\begin{aligned} 2R_{hijk}F^{ij} &= R_{hijk}F^{ij} - R_{hjik}F^{ij} \\ &= (R_{hjik} + R_{hikj})F^{ij} - 2R_{hkij}F^{ij}, \end{aligned}$$

and therefore to

$$(2.23) \quad R_{hijk}F^{ij} = -\frac{1}{2}R_{hkij}F^{ij}.$$

Substituting (2.21) in (2.20) and using (2.22), (2.23), (2.1) we can obtain

$$(2.24) \quad \nabla_k F_{ij}\nabla_l F^{ij} = R_{kl} + F^{ij}(\frac{1}{2}R_{akij}F_l^a - R_{alij}F_k^a).$$

Interchanging k, l in (2.24) and subtracting the resulting equation from (2.24) we have

$$(2.25) \quad R_{akij}F_l^a F^{ij} = R_{alij}F_k^a F^{ij},$$

and therefore (2.24) is reduced to

$$(2.26) \quad \nabla_k F_{ij}\nabla_l F^{ij} = R_{kl} - \frac{1}{2}R_{alij}F_k^a F^{ij},$$

which together with (2.18) implies

$$(2.27) \quad \nabla_k F_{ij}\nabla_l F^{ij} = 0.$$

Multiplying (2.27) by g^{kl} we hence obtain $\nabla_k F_{ij} = 0$.

Lemma 2.4. For an almost-Hermitian structure F , condition

$$(2.28) \quad F_i^k R_{jk} = F_k^l R^k_{ijl}$$

implies condition (2.18).

Proof. Since

$$\begin{aligned} F^{kl}R_{kjh l} &= \frac{1}{2}F^{kl}(R_{kjh l} - R_{ljhk}) \\ &= \frac{1}{2}F^{kl}(R_{kjh l} + R_{kh lj}) = \frac{1}{2}F^{kl}R_{kljh} \end{aligned}$$

by the Bianchi identity, from (2.28) we obtain

$$(2.29) \quad F_j^k R_{nk} = \frac{1}{2} R_{jnk\ell} F^{k\ell} .$$

Multiplying (2.29) by F_i^j and using (2.1) lead immediately to (2.18).

We now consider an almost-Hermitian manifold M^n with an almost-Hermitian structure F , and shall follow Spencer (compare [7, Chapter IX]) to introduce complex operators on the manifold M^n . At first we define

$$(2.30) \quad \prod_{1,0} i^j = \frac{1}{2} (\varepsilon_i^j - \sqrt{-1} F_i^j)$$

and its conjugate¹ tensor

$$(2.31) \quad \prod_{0,1} i^j = \overline{\prod_{1,0} i^j} = \frac{1}{2} (\varepsilon_i^j + \sqrt{-1} F_i^j) .$$

A simple calculation gives the following identities:

$$(2.32) \quad \begin{aligned} \prod_{1,0} i^j \prod_{1,0} j^k &= \prod_{1,0} i^k , \\ \prod_{1,0} i^j \prod_{0,1} j^k &= 0 , \\ \prod_{0,1} i^j \prod_{0,1} j^k &= \prod_{0,1} i^k . \end{aligned}$$

Let $\rho + \sigma = p, \rho \geq 0, \sigma \geq 0$, set

$$(2.33) \quad \begin{aligned} \prod_{\rho,\sigma} I_{(p)}^{J(p)} &= \varepsilon_{I(p)}^{M(\rho)N(\sigma)} \prod_{1,0} m_1^{r_1} \cdots \prod_{1,0} m_\rho^{r_\rho} \\ &\quad \cdot \prod_{0,1} n_1^{s_1} \cdots \prod_{0,1} n_\sigma^{s_\sigma} \varepsilon_{R_0(\rho)S_0(\sigma)}^{J(p)} , \end{aligned}$$

and define $\prod_{\rho,\sigma} I_{(p)}^{J(p)}$ to be the identity for $\rho = \sigma = 0$ and to be zero for either $\rho < 0$ or $\sigma < 0$. Then for a form ϕ given by (1.4) we have

$$(2.34) \quad \prod_{\rho,\sigma} \phi = \left(\prod_{\rho,\sigma} \phi \right)_{J_0(p)} dx^{I_0(p)} ,$$

where

$$(2.35) \quad \left(\prod_{\rho,\sigma} \phi \right)_{I(p)} = \prod_{\rho,\sigma} I_{(p)}^{J_0(p)} \phi_{J_0(p)} .$$

We next define a complex covariant differentiator

¹ Throughout this paper a bar over a letter or symbol denotes the conjugate of the complex number or operator defined by the letter or symbol.

$$(2.36) \quad \mathcal{D}_i = \prod_{1,0} i^j \mathcal{V}_j ,$$

and the corresponding contravariant differentiator

$$(2.37) \quad \mathcal{D}^i = g^{ik} \mathcal{D}_k = \prod_{0,1} j^i \mathcal{V}^j = \overline{\prod_{1,0} j^i \mathcal{V}^j} .$$

The conjugate operators of \mathcal{D}_i and \mathcal{D}^i are

$$(2.38) \quad \overline{\mathcal{D}}_i = \prod_{0,1} j^j \mathcal{V}_j ,$$

$$(2.39) \quad \overline{\mathcal{D}}^i = \prod_{1,0} j^i \mathcal{V}^j .$$

Now we define the complex analogues of the real operators d and δ defined by (1.7), (1.15) respectively:

$$(2.40) \quad d_1 = \sum_{\rho+\sigma=p} \prod_{\rho+1,\sigma} d \prod_{\rho,\sigma} ,$$

$$(2.41) \quad d_2 = \sum_{\rho+\sigma=p} \prod_{\rho+2,\sigma-1} d \prod_{\rho,\sigma} ,$$

$$(2.42) \quad \delta_1 = \sum_{\rho+\sigma=p} \prod_{\rho,\sigma-1} \delta \prod_{\rho,\sigma} ,$$

$$(2.43) \quad \delta_2 = \sum_{\rho+\sigma=p} \prod_{\rho+1,\sigma-2} \delta \prod_{\rho,\sigma} .$$

The conjugate operators of d_1, d_2 and δ_1, δ_2 have the forms:

$$(2.44) \quad \overline{d}_1 = \sum_{\rho+\sigma=p} \prod_{\rho,\sigma+1} d \prod_{\rho,\sigma} ,$$

$$(2.45) \quad \overline{d}_2 = \sum_{\rho+\sigma=p} \prod_{\rho-1,\sigma+2} d \prod_{\rho,\sigma} ,$$

$$(2.46) \quad \overline{\delta}_1 = \sum_{\rho+\sigma=p} \prod_{\rho-1,\sigma} \delta \prod_{\rho,\sigma} ,$$

$$(2.47) \quad \overline{\delta}_2 = \sum_{\rho+\sigma=p} \prod_{\rho-2,\sigma+1} \delta \prod_{\rho,\sigma} .$$

Furthermore, for a p -form ϕ given by (1.4) we define

$$(2.48) \quad (\partial_1 \phi)_{I(p+1)} = (2d_2 + d_1 - \overline{d}_2)_{I(p+1)} ,$$

$$(2.49) \quad (\mathcal{D}_1 \phi)_{I(p-1)} = (2\delta_2 + \delta_1 - \overline{\delta}_2)_{I(p-1)} ,$$

$$(2.50) \quad (\partial_2 \phi)_{I(p+1)} = \sum_{\rho+\sigma=p} \prod_{\rho+1,\sigma} \prod_{I(p+1)} {}^j J_0^{(p)} \mathcal{D}_j \phi_{J_0(p)} ,$$

$$(2.51) \quad (\mathcal{D}_2 \phi)_{I(p-1)} = - \sum_{\rho+\sigma=p} \prod_{\rho,\sigma} \prod_{I(p-1)} {}^j J_0^{(p)} \mathcal{D}^j \phi_{J_0(p)} ,$$

together with their conjugate operators:

$$(2.52) \quad (\partial_1\phi)_{I(p+1)} = (2\bar{d}_2 + \bar{d}_1 - d_2)_{I(p+1)},$$

$$(2.53) \quad (\bar{\mathcal{G}}_1\phi)_{I(p-1)} = (2\bar{\delta}_2 + \bar{\delta}_1 - \delta_2)_{I(p-1)},$$

$$(2.54) \quad (\bar{\delta}_2\phi)_{I(p+1)} = \sum_{\rho+\sigma=p} \prod_{\rho, \sigma+1}^{I(p+1)} {}^{jJ_0(p)}\bar{\mathcal{G}}_j\phi_{J_0(p)},$$

$$(2.55) \quad (\bar{\mathcal{G}}_2\phi)_{I(p-1)} = - \sum_{\rho+\sigma=p} \prod_{\rho, \sigma}^{iI(p-1)} {}^{jJ_0(p)}\bar{\mathcal{G}}^i\phi_{J_0(p)}.$$

It is known that (see [3], [5])

$$(2.56) \quad \mathcal{G}_1 = - * \partial_1 * , \quad \mathcal{G}_2 = - * \bar{\partial}_2 * ,$$

and that (see [3]) if the structure F of the manifold M^n is Kählerian, then $\bar{d}_2\phi = d_2\phi = 0$ for any form ϕ , and therefore $\partial_1 = d_1$.

Now we introduce the following complex Laplace-Beltrami operators:

$$(2.57) \quad \square_i = \bar{\mathcal{G}}_i\partial_i + \partial_i\bar{\mathcal{G}}_i, \quad (i = 1, 2),$$

$$(2.58) \quad \square_3 = \bar{\mathcal{G}}_1\partial_2 + \partial_2\bar{\mathcal{G}}_1,$$

$$(5.59) \quad \square_4 = \bar{\mathcal{G}}_2\partial_1 + \partial_1\bar{\mathcal{G}}_2,$$

$$(2.60) \quad \square_5 = \bar{\delta}_1d_1 + d_1\bar{\delta}_1.$$

It should be noted that \square_1 was first defined by Kodaira-Spencer [3], and \square_2 by Hsiung [2].

From [3] we know that $d = \partial_1 + \bar{\partial}_1$. In order to apply $\partial_2 + \bar{\partial}_2$, let ξ be any 0-form. Then we have, in consequence of (2.50), (2.36), (2.32), (2.30),

$$(2.61) \quad (\partial_2\xi)_{i_1} = \prod_{1,0}^{i_1} {}^jV_j\xi = \frac{1}{2}(V_{i_1}\xi - \sqrt{-1}F_{i_1}{}^jV_j\xi),$$

which together with (1.6), (1.7) gives

$$(2.62) \quad d\xi = (\partial_2 + \bar{\partial}_2)\xi.$$

Similarly, for any 1-form η , using (2.50), (2.36), (2.33), (2.34), (2.35), (2.32) we can obtain

$$(2.63) \quad \begin{aligned} (\partial_2\eta)_{i_1i_2} &= \prod_{1,0}^{i_1} {}^j \prod_{1,0}^{i_2} {}^k (V_j\eta_k - V_k\eta_j) + (\prod_{1,0}^{i_1} {}^j \prod_{0,1}^{i_2} {}^k - \prod_{1,0}^{i_2} {}^j \prod_{0,1}^{i_1} {}^k) V_j\eta_k \\ &= \frac{1}{2}[V_{i_1}\eta_{i_2} - V_{i_2}\eta_{i_1} + \sqrt{-1}(F_{i_2}{}^jV_j\eta_{i_1} - F_{i_1}{}^jV_j\eta_{i_2})], \end{aligned}$$

which together with (1.6), (1.7) gives

$$(2.64) \quad d\eta = \frac{1}{2}(\partial_2 + \bar{\partial}_2)\eta.$$

The almost-complex structure F of the manifold M^n is said [3] to be (completely) integrable if and only if $\partial_1^2 = 0$. Now by means of (2.61), (2.50), (2.30), \dots , (2.36) and the relation

$$(2.65) \quad \nabla_i \nabla_j \xi = \nabla_j \nabla_i \xi$$

for any 0-form ξ , an elementary but lengthy calculation gives

$$(2.66) \quad 4(\partial_2^2 \xi)_{i_1 i_2} = (F_{i_2}{}^j \nabla_j F_{i_1}{}^k - F_{i_1}{}^j \nabla_j F_{i_2}{}^k) \nabla_k \xi + \sqrt{-1}(\nabla_{i_2} F_{i_1}{}^k - \nabla_{i_1} F_{i_2}{}^k) \nabla_k \xi .$$

If ∂_2^2 is real for any 0-form ξ , then by taking $\xi = x^i$ for any arbitrary i with respect to any local coordinates x^1, \dots, x^n , from (2.66) we obtain (2.14), and therefore by Lemma 2.1 the structure F is Kählerian.

3. Expressions for \square 's

In this section we shall give expressions for $\square_i \xi$ and $\square_i \eta$, where $i = 1, \dots, 4$, and ξ and η are respectively any 0- and 1-forms on an almost-Hermitian manifold M^n with an almost-Hermitian structure F .

3.1. Laplacian \square_2 . In [2, pp. 146-147] we obtained

$$(3.1) \quad 4\square_2 \xi = 2\Delta \xi + \nabla^h F_h{}^j (-F_j{}^k \nabla_k \xi + \sqrt{-1} \nabla_j \xi) ,$$

$$(3.2) \quad 4(\square_2 \eta)_{i_1} = -F_i{}^j \nabla^i F_{i_1}{}^h \nabla_h \eta_j - F_j{}^h \nabla^i F_i{}^j \nabla_h \eta_{i_1} + F_{i_1}{}^i \nabla_i F_h{}^j \nabla^h \eta_j - 2\nabla^i \nabla_i \eta_{i_1} + [\nabla_j, \nabla_{i_1}] \eta^j + F_{i_1}{}^h F^{ij} [\nabla_h, \nabla_i] \eta_j + \sqrt{-1} \{ \nabla^i F_i{}^j \nabla_j \eta_{i_1} - (\nabla_j F_{i_1}{}^k + \nabla_{i_1} F_j{}^k) \nabla_k \eta^j + 2F_j{}^k \nabla^j \nabla_k \eta_{i_1} - F_{i_1}{}^k [\nabla_j, \nabla_k] \eta^j - F^{kj} [\nabla_k, \nabla_{i_1}] \eta_j \} ,$$

where

$$(3.3) \quad [\nabla_h, \nabla_i] = \nabla_h \nabla_i - \nabla_i \nabla_h .$$

3.2. Laplacian \square_1 . At first we notice that as a result of (2.65) we have

$$(3.4) \quad F_i{}^j \nabla^i \nabla_j \xi = 0 .$$

By using (2.57), (2.53), (2.45), \dots , (2.48), (2.33), ((2.40), (2.41), (2.43), (2.1), (2.30), (2.32), (2.34), (2.35), (3.4), (1.17), (1.18) we can obtain

$$(3.5) \quad 2\square_1 \xi = 2\delta \prod_{1,0} d\xi = \Delta \xi + \sqrt{-1} \nabla^i F_i{}^j \nabla_j \xi .$$

In order to compute $\square_1 \eta$, from (2.48), (2.52), (2.40), (2.41), (2.43), (2.45), (2.46), (2.47) we first see that

$$(3.6) \quad \partial_1 = 2 \prod_{2,0} d \prod_{0,1} + \prod_{2,0} d \prod_{1,0} + \prod_{1,1} d \prod_{0,1} - \prod_{0,2} d \prod_{1,0} , \quad \text{for 1-forms ,}$$

$$(3.7) \quad \overline{\mathcal{D}}_1 = 2 \prod_{0,1} \delta \prod_{2,0} + \prod_{1,0} \delta \prod_{2,0} + \prod_{0,1} \delta \prod_{1,1} - \prod_{1,0} \delta \prod_{0,2}, \quad \text{for 2-forms.}$$

Next, by means of (1.6), (1.7), (2.33), (2.34), (2.35), (1.2), (1.3), (2.30), (2.31), we obtain

$$(3.8) \quad \begin{aligned} \prod_{1,1} d \prod_{0,1} \eta &= [\prod_{1,0} i_1^k \prod_{1,0} i_2^l (\nabla_k \prod_{0,1} l^j - \nabla_l \prod_{0,1} k^j) \eta_j + \prod_{1,0} i_1^k \prod_{0,1} i_2^j \nabla_k \eta_j \\ &\quad - \prod_{1,0} i_2^k \prod_{0,1} i_1^j \nabla_k \eta_j + \prod_{1,0} i_2^k \prod_{0,1} i_1^l (\nabla_l \prod_{0,1} k^j - \nabla_k \prod_{0,1} l^j) \eta_j] dx^{I_0(2)} \\ &= \frac{1}{4} \{ \nabla_{i_1} \eta_{i_2} - \nabla_{i_2} \eta_{i_1} + F_{i_1}^k F_{i_2}^j (\nabla_k \eta_j - \nabla_j \eta_k) \\ &\quad + \sqrt{-1} [\eta_j (\nabla_{i_1} F_{i_2}^j - \nabla_{i_2} F_{i_1}^j + F_{i_1}^k F_{i_2}^l (\nabla_k F_l^j - \nabla_l F_k^j)) \\ &\quad + F_{i_2}^j (\nabla_{i_1} \eta_j + \nabla_j \eta_{i_1}) - F_{i_1}^j (\nabla_{i_2} \eta_j + \nabla_j \eta_{i_2}) \} dx^{I_0(2)}, \end{aligned}$$

$$(3.9) \quad \begin{aligned} \prod_{2,0} d \prod_{1,0} \eta &= [\prod_{1,0} i_1^k \prod_{1,0} i_2^l (\nabla_k \prod_{1,0} l^j - \nabla_l \prod_{1,0} k^j) \eta_j \\ &\quad + \prod_{1,0} i_1^j \prod_{1,0} i_2^k (\nabla_j \eta_k - \nabla_k \eta_j)] dx^{I_0(2)} \\ &= \frac{1}{8} \{ \eta_j [F_{i_1}^k (\nabla_{i_2} F_k^j - \nabla_k F_{i_2}^j) + F_{i_2}^k (\nabla_k F_{i_1}^j - \nabla_{i_1} F_k^j)] \\ &\quad + 2 \nabla_{i_1} \eta_{i_2} - 2 \nabla_{i_2} \eta_{i_1} + 2 F_{i_1}^j F_{i_2}^k (\nabla_k \eta_j - \nabla_j \eta_k) \\ &\quad + \sqrt{-1} [\eta_j (\nabla_{i_2} F_{i_1}^j - \nabla_{i_1} F_{i_2}^j + F_{i_1}^k F_{i_2}^l (\nabla_k F_l^j - \nabla_l F_k^j)) \\ &\quad + 2 F_{i_1}^j (\nabla_{i_2} \eta_j - \nabla_j \eta_{i_2}) + 2 F_{i_2}^j (\nabla_j \eta_{i_1} - \nabla_{i_1} \eta_j)] \} dx^{I_0(2)}, \end{aligned}$$

$$(3.10) \quad \begin{aligned} \prod_{2,0} d \prod_{0,1} \eta &= \prod_{1,0} i_1^k \prod_{1,0} i_2^l (\nabla_k \prod_{0,1} l^j - \nabla_l \prod_{0,1} k^j) \eta_j dx^{I_0(2)} \\ &= \frac{1}{8} \{ \eta_j [F_{i_1}^k (\nabla_k F_{i_2}^j - \nabla_{i_2} F_k^j) + F_{i_2}^k (\nabla_{i_1} F_k^j - \nabla_k F_{i_1}^j)] \\ &\quad + \sqrt{-1} \eta_j [\nabla_{i_1} F_{i_2}^j - \nabla_{i_2} F_{i_1}^j \\ &\quad + F_{i_1}^k F_{i_2}^l (\nabla_l F_k^j - \nabla_k F_l^j)] \} dx^{I_0(2)}, \end{aligned}$$

$$(3.11) \quad \begin{aligned} - \prod_{0,2} d \prod_{1,0} \eta &= \prod_{0,1} i_2^k \prod_{0,1} i_1^l (\nabla_k \prod_{1,0} l^j - \nabla_l \prod_{1,0} k^j) \eta_j dx^{I_0(2)} \\ &= \frac{1}{8} \{ \eta_j [F_{i_1}^k (\nabla_{i_2} F_k^j - \nabla_k F_{i_2}^j) + F_{i_2}^k (\nabla_k F_{i_1}^j - \nabla_{i_1} F_k^j)] \\ &\quad + \sqrt{-1} \eta_j [\nabla_{i_1} F_{i_2}^j - \nabla_{i_2} F_{i_1}^j \\ &\quad + F_{i_1}^k F_{i_2}^l (\nabla_l F_k^j - \nabla_k F_l^j)] \} dx^{I_0(2)}, \end{aligned}$$

$$(3.12) \quad \begin{aligned} 4(\prod_{1,0} d\delta \prod_{1,0} \eta)_{i_1} &= \eta_j F_{i_1}^l \nabla_l \nabla^k F_k^j + F_{i_1}^l \nabla^k F_k^j \nabla_l \eta_j \\ &\quad + F_{i_1}^l \nabla_l F_k^j \nabla^k \eta_j - \nabla_{i_1} \nabla^k \eta_k + F_{i_1}^l F_k^j \nabla_l \nabla^k \eta_j \\ &\quad + \sqrt{-1} (\eta_j \nabla_{i_1} \nabla^k F_k^j + \nabla^k F_k^j \nabla_{i_1} \eta_j + \nabla_{i_1} F_k^j \nabla^k \eta_j \\ &\quad + F_k^j \nabla_{i_1} \nabla^k \eta_j + F_{i_1}^l \nabla_l \nabla^k \eta_k). \end{aligned}$$

Substitution of (3.8), . . . , (3.11) in (3.6) thus gives

$$(3.13) \quad \partial_1 \eta = \frac{1}{2} \{ \nabla_{i_1} \eta_{i_2} - \nabla_{i_2} \eta_{i_1} + \sqrt{-1} [\eta_j (\nabla_{i_1} F_{i_2}^j - \nabla_{i_2} F_{i_1}^j) + F_{i_2}^j \nabla_j \eta_{i_1} - F_{i_1}^j \nabla_j \eta_{i_2}] \} dx^{I_0(2)} .$$

Now put

$$(3.14) \quad \begin{aligned} A_{i_1 i_2} &= \nabla_{i_1} \eta_{i_2} + \sqrt{-1} (\eta_j \nabla_{i_1} F_{i_2}^j + F_{i_2}^j \nabla_j \eta_{i_1}) , \\ B_{s i_1}^{k_1 k_2} &= \varepsilon_s^{k_1} \varepsilon_{i_1}^{k_2} - \varepsilon_{i_1}^{k_1} \varepsilon_s^{k_2} + F_{i_1}^{k_1} F_s^{k_2} - F_s^{k_1} F_{i_1}^{k_2} , \\ C_{s i_1}^{k_1 k_2} &= \varepsilon_{i_1}^{k_1} F_s^{k_2} - \varepsilon_s^{k_1} F_{i_1}^{k_2} - \varepsilon_{i_1}^{k_2} F_s^{k_1} + \varepsilon_s^{k_2} F_{i_1}^{k_1} . \end{aligned}$$

Then

$$(3.15) \quad \partial_1 \eta = \frac{1}{2} (A_{i_1 i_2} - A_{i_2 i_1}) dx^{I_0(2)} .$$

By means of (3.13), (2.33), (2.34), (2.35), (2.30), (2.31), (1.16), (1.17), elementary but rather lengthy calculations give

$$(3.16) \quad \begin{aligned} & -8 \left(\prod_{0,1} \delta \prod_{1,1} \partial_1 \eta \right)_{i_1} \\ &= 2 \prod_{0,1} \nabla^s [\varepsilon_s^{k_1} \varepsilon_{i_1}^{k_2} - \varepsilon_{i_1}^{k_1} \varepsilon_s^{k_2} + F_s^{k_1} F_{i_1}^{k_2} - F_{i_1}^{k_1} F_s^{k_2}] A_{k_1 k_2} \\ &= \eta_j [\nabla^s F_s^k (\nabla_k F_{i_1}^j - \nabla_{i_1} F_k^j) + F_{i_1}^r F_s^k \nabla^s F_r^l (\nabla_l F_k^j - \nabla_k F_l^j) \\ &\quad + F_{i_1}^r (\nabla^s \nabla_r F_s^j - \nabla^s \nabla_s F_r^j) + F_s^k (\nabla^s \nabla_k F_{i_1}^j - \nabla^s \nabla_{i_1} F_k^j)] \\ &\quad + (2 F_{i_1}^l \nabla^s F_s^k - F_s^l \nabla^s F_{i_1}^k) \nabla_k \eta_l + F_{i_1}^r \nabla^s \eta_j (\nabla_r F_s^j - 2 \nabla_s F_r^j) \\ &\quad - F_s^k \nabla_{i_1} F_k^j \nabla^s \eta_j - F_{i_1}^r \nabla^s F_r^j \nabla_j \eta_s + 2 \nabla^s \nabla_s \eta_{i_1} \\ &\quad + 2 F_{i_1}^k F_s^l \nabla^s \nabla_l \eta_k + \sqrt{-1} \{ \eta_j [F_s^k \nabla^s F_{i_1}^l (\nabla_k F_l^j - \nabla_l F_k^j) \\ &\quad + F_{i_1}^k \nabla^s F_s^l (\nabla_l F_k^j - \nabla_k F_l^j) + \nabla^s \nabla_s F_{i_1}^j - \nabla^s \nabla_{i_1} F_s^j \\ &\quad + F_{i_1}^k F_s^l (\nabla^s \nabla_l F_k^j - \nabla^s \nabla_k F_l^j)] + (2 \nabla_s F_{i_1}^j - \nabla_{i_1} F_s^j) \nabla^s \eta_j \\ &\quad - 2 \nabla^s F_s^j \nabla_j \eta_{i_1} + \nabla^s F_{i_1}^j \nabla_j \eta_s - F_{i_1}^k F_s^l (\nabla_k F_l^j \nabla^s \eta_j + \nabla^s F_k^j \nabla_j \eta_l) \\ &\quad + 2 F_{i_1}^j \nabla^s \nabla_s \eta_j - 2 F_s^j \nabla^s \nabla_j \eta_{i_1} \} , \end{aligned}$$

$$\begin{aligned} & -8 \left(\prod_{1,0} \delta \prod_{2,0} \partial_1 \eta \right)_{i_1} \\ &= \prod_{1,0} \nabla^s [(B_{s i_1}^{k_1 k_2} + \sqrt{-1} C_{s i_1}^{k_1 k_2}) A_{k_1 k_2}] \\ &= \eta_j [\frac{1}{2} \nabla^s F_{i_1}^k (\nabla_s F_k^j - \nabla_k F_s^j) + \frac{1}{2} F_{i_1}^r F_s^k \nabla^s F_r^l (\nabla_l F_k^j - \nabla_k F_l^j) \\ &\quad + \nabla^s F_s^k (\nabla_k F_{i_1}^j - \nabla_{i_1} F_k^j) + F_{i_1}^k (\nabla^s \nabla_s F_k^j - \nabla^s \nabla_k F_s^j) \\ &\quad + F_s^k (\nabla^s \nabla_k F_{i_1}^j - \nabla^s \nabla_{i_1} F_k^j)] + 2 F_{i_1}^k \nabla^s F_s^l (\nabla_k \eta_l - \nabla_l \eta_k) \end{aligned}$$

$$\begin{aligned}
& + F_{i_1}{}^r \nabla_j \eta^s (\nabla_r F_s^j + \nabla_s F_r^j) + F_s^k \nabla^s F_{i_1}{}^l (\nabla_l \eta_k - 2\nabla_k \eta_l) \\
& - F_s^k \nabla_{i_1} F_k^j \nabla^s \eta_j + 2\nabla^s \nabla_s \eta_{i_1} - 2\nabla^s \nabla_{i_1} \eta_s \\
(3.17) \quad & + 2F_{i_1}{}^k F_s^l (\nabla^s \nabla_k \eta_l - \nabla^s \nabla_l \eta_k) \\
& + \sqrt{-1} \{ \eta_j [\frac{1}{2} F_{i_1}{}^r \nabla^s F_r^k (\nabla_k F_s^j - \nabla_s F_k^j) + F_{i_1}{}^k \nabla^s F_s^l (\nabla_k F_l^j \\
& - \nabla_l F_k^j) + \frac{1}{2} F_s^k \nabla^s F_{i_1}{}^l (\nabla_l F_k^j - \nabla_k F_l^j) + \nabla^s \nabla_s F_{i_1}{}^j \\
& - \nabla^s \nabla_{i_1} F_s^j + F_{i_1}{}^k F_s^l (\nabla^s \nabla_k F_l^j - \nabla^s \nabla_l F_k^j)] + \nabla^s F_{i_1}{}^j \nabla_j \eta_s \\
& - \nabla_{i_1} F_s^j \nabla^s \eta_j + F_{i_1}{}^k F_s^l \nabla_k F_l^j \nabla^s \eta_j - F_{i_1}{}^r F_s^l \nabla^s F_r^k \nabla_k \eta_l \\
& + 2\nabla^s F_s^j (\nabla_{i_1} \eta_j - \nabla_j \eta_{i_1}) - 2F_s^k F_{i_1}{}^l \nabla_k F_l^j \nabla^s \eta_j \\
& + 2F_s^k (\nabla^s \nabla_{i_1} \eta_k - \nabla^s \nabla_k \eta_{i_1}) + 2F_{i_1}{}^j (\nabla^s \nabla_j \eta_s - \nabla^s \nabla_s \eta_j) \} ,
\end{aligned}$$

$$\begin{aligned}
& - 8(\prod_{0,1} \delta \prod_{2,0} \partial_i \eta)_{i_1} \\
& = \eta_j [\frac{1}{2} \nabla^s F_{i_1}{}^k (\nabla_s F_k^j - \nabla_k F_s^j) + \frac{1}{2} F_{i_1}{}^r F_s^k \nabla^s F_r^l (\nabla_k F_l^j - \nabla_l F_k^j)] \\
(3.18) \quad & + F_s^k \nabla^s F_{i_1}{}^l (\nabla_l \eta_k - \nabla_k \eta_l) + F_{i_1}{}^r \nabla^s F_r^k (\nabla_s \eta_k - \nabla_k \eta_s) \\
& + \sqrt{-1} \{ \eta_j [\frac{1}{2} F_s^k \nabla^s F_{i_1}{}^l (\nabla_l F_k^j - \nabla_k F_l^j) \\
& \quad + \frac{1}{2} F_{i_1}{}^r \nabla^s F_r^k (\nabla_s F_k^j - \nabla_k F_s^j)] \\
& \quad + F_{i_1}{}^r F_s^k \nabla^s F_r^l (\nabla_l \eta_k - \nabla_k \eta_l) + \nabla^s F_{i_1}{}^j (\nabla_j \eta_s - \nabla_s \eta_j) \} ,
\end{aligned}$$

$$\begin{aligned}
& - 16(\prod_{1,0} \delta \prod_{0,2} \partial_i \eta)_{i_1} = 2 \prod_{1,0} \nabla^s [(B_s^{k_1 k_2} - \sqrt{-1} C_s^{k_1 k_2}) A_{k_1 k_2}] \\
& = \eta_j [\nabla^s F_{i_1}{}^k (\nabla_k F_s^j - \nabla_s F_k^j) + F_{i_1}{}^r F_s^k \nabla^s F_r^l (\nabla_l F_k^j - \nabla_k F_l^j)] \\
(3.19) \quad & + \sqrt{-1} \eta_j [F_s^k \nabla^s F_{i_1}{}^l (\nabla_l F_k^j - \nabla_k F_l^j) \\
& \quad + F_{i_1}{}^r \nabla^s F_r^k (\nabla_s F_k^j - \nabla_k F_s^j)] .
\end{aligned}$$

Substituting (3.6), (3.7), (3.12), (3.16), \dots , (3.19) in (2.57) and using (2.32) and

$$(3.20) \quad 2F_j^k \nabla^j \nabla_k \eta_{i_1} = F^{jk} [\nabla_j, \nabla_k] \eta_{i_1} ,$$

we can obtain, after some elementary simplification,

$$\begin{aligned}
4(\square_i \eta)_{i_1} & = 4[(2 \prod_{0,1} \delta \prod_{2,0} + \prod_{1,0} \delta \prod_{2,0} + \prod_{0,1} \delta \prod_{1,1} - \prod_{1,0} \delta \prod_{0,2}) \partial_i \eta + \prod_{1,0} d\delta \prod_{1,0} \eta]_{i_1} , \\
& = \eta_j [\nabla^s F_s^k (\nabla_{i_1} F_k^j - \nabla_k F_{i_1}{}^j) + \nabla^s F_{i_1}{}^k (\nabla_k F_s^j - \nabla_s F_k^j) \\
& \quad + F_s^k (\nabla^s \nabla_{i_1} F_k^j - \nabla^s \nabla_k F_{i_1}{}^j) + F_{i_1}{}^l \nabla_l \nabla^k F_k^j] \\
& \quad + F_s^k \nabla^s \eta_j (\nabla_{i_1} F_k^j - 2\nabla_k F_{i_1}{}^j) + F_s^k \nabla^s F_{i_1}{}^l (\nabla_k \eta_l - \nabla_l \eta_k) \\
(3.21) \quad & + F_{i_1}{}^l \nabla_l F_k^j \nabla^k \eta_j + F_{i_1}{}^l \nabla^j F_l^k \nabla_k \eta_j - 2\nabla^s \nabla_s \eta_{i_1} \\
& \quad + [\nabla_k, \nabla_{i_1}] \eta^k + F_{i_1}{}^l F^{kj} [\nabla_l, \nabla_k] \eta_j \\
& \quad + \sqrt{-1} \{ \eta_j (\nabla^s \nabla_{i_1} F_s^j - \nabla^s \nabla_s F_{i_1}{}^j + \nabla_{i_1} \nabla^s F_s^j) + 2\nabla^s F_s^j \nabla_j \eta_{i_1} \}
\end{aligned}$$

$$\begin{aligned}
 & - 2(\nabla_s F_{i_1}{}^j + \nabla_{i_1} F_s{}^j) \nabla_j \eta^s + F^{sj} [\nabla_s, \nabla_j] \eta_{i_1} \\
 & - F^{sj} [\nabla_s, \nabla_{i_1}] \eta_j - F_{i_1}{}^j [\nabla_s, \nabla_j] \eta^s \}.
 \end{aligned}$$

3.3. Laplacian \square_3 . In the same way as above we can compute $\square_{i_1} \xi$ and $\square_{i_1} \eta$, $i = 3, 4, 5$, but we shall omit the details in this section and §§ 3.4, 3.5. We find that

$$(3.22) \quad \square_3 \xi = \square_1 \xi ,$$

$$(3.23) \quad 2\bar{\mathcal{D}}_1 \eta = -\nabla^k \eta_k + \sqrt{-1} (F_k{}^j \nabla^k \eta_j + \eta_j \nabla^k F_k{}^j) ,$$

$$(3.24) \quad \begin{aligned} 4(\partial_2 \bar{\mathcal{D}}_1 \eta)_{i_1} = & \eta_j F_{i_1}{}^l \nabla_l \nabla^k F_k{}^j + F_{i_1}{}^l \nabla_k{}^j \nabla^k \eta_j + F_{i_1}{}^l \nabla^k F_k{}^j \nabla_l \eta_j \\ & - \nabla_i \nabla^k \eta_k + F_{i_1}{}^l F_k{}^j \nabla_l \nabla^k \eta_j + \sqrt{-1} [\eta_j \nabla_{i_1} \nabla^k F_k{}^j \\ & + \nabla_{i_1} F_k{}^j \nabla^k \eta_j + \nabla^k F_k{}^j \nabla_{i_1} \eta_j + F_k{}^j \nabla_{i_1} \nabla^k \eta_j + F_{i_1}{}^l \nabla_l \nabla^k \eta_k] , \end{aligned}$$

$$(3.25) \quad \begin{aligned} 2(\bar{\mathcal{D}}_1 \partial_2 \eta)_{i_1} = & 2[(2 \prod_{0,1} \delta \prod_{2,0} + \prod_{1,0} \delta \prod_{2,0} + \prod_{0,1} \delta \prod_{1,1} - \prod_{1,0} \delta \prod_{0,2}) \partial_2 \eta]_{i_1} \\ & = F_s{}^j \nabla^s F_{i_1}{}^k (\nabla_j \eta_k - \nabla_k \eta_j) + F_{i_1}{}^r (\nabla^s F_r{}^j \nabla_j \eta_s - \nabla^s F_s{}^j \nabla_r \eta_j) \\ & - 2\nabla^s \nabla_s \eta_{i_1} + \nabla^s \nabla_{i_1} \eta_s - F_{i_1}{}^k F_s{}^l \nabla^s \nabla_k \eta_l \\ & + \sqrt{-1} [\nabla^s F_{i_1}{}^j (\nabla_s \eta_j - 2\nabla_j \eta_s) + \nabla^s F_s{}^j (2\nabla_j \eta_{i_1} - \nabla_{i_1} \eta_j) \\ & - F_{i_1}{}^j \nabla^s \nabla_j \eta_s + F_s{}^j (2\nabla^s \nabla_j \eta_{i_1} - \nabla^s \nabla_{i_1} \eta_j)] , \end{aligned}$$

$$(3.26) \quad \begin{aligned} 4(\square_3 \eta)_{i_1} = & \eta_j F_{i_1}{}^l \nabla_l \nabla^k F_k{}^j + 2F_s{}^j \nabla^s F_{i_1}{}^k (\nabla_j \eta_k - \nabla_k \eta_j) \\ & + F_{i_1}{}^r (\nabla_r F_k{}^j \nabla^k \eta_j - \nabla^k F_k{}^j \nabla_r \eta_j) + 2F_{i_1}{}^r \nabla^s F_r{}^j \nabla_j \eta_s \\ & - 4\nabla^s \nabla_s \eta_{i_1} + 2\nabla^s \nabla_{i_1} \eta_s - \nabla_{i_1} \nabla^k \eta_k \\ & + F_{i_1}{}^l F_k{}^j (\nabla_l \nabla^k \eta_j - 2\nabla^k \nabla_l \eta_j) \\ & + \sqrt{-1} [\eta_j \nabla_{i_1} \nabla^k F_k{}^j + \nabla^s F_s{}^j (4\nabla_j \eta_{i_1} - \nabla_{i_1} \eta_j) \\ & + 2\nabla^s F_{i_1}{}^j (\nabla_s \eta_j - 2\nabla_j \eta_s) \\ & + \nabla_{i_1} F_k{}^j \nabla^k \eta_j + 2F_s{}^j (2\nabla^s \nabla_j \eta_{i_1} - \nabla^s \nabla_{i_1} \eta_j) \\ & + F_{i_1}{}^j (\nabla_j \nabla^k \eta_k - 2\nabla^k \nabla_j \eta_k) + F_k{}^j \nabla_{i_1} \nabla^k \eta_j] . \end{aligned}$$

3.4. Laplacian \square_4 . For $\square_4 \xi$, $\square_4 \eta$ we obtain the following equations:

$$(3.27) \quad 2(\partial_1 \xi)_{i_1} = \nabla_{i_1} \xi - \sqrt{-1} F_{i_1}{}^j \nabla_j \xi ,$$

$$(3.28) \quad \square_4 \xi = \square_2 \xi ,$$

$$(3.29) \quad 2\bar{\mathcal{D}}_2 \eta = -\nabla^k \eta_k + \sqrt{-1} F_k{}^j \nabla^k \eta_j ,$$

$$(3.30) \quad \begin{aligned} 4(\partial_1 \bar{\mathcal{D}}_2 \eta)_{i_1} = & F_{i_1}{}^l \nabla_l F_k{}^j \nabla^k \eta_j - \nabla_{i_1} \nabla^k \eta_k + F_{i_1}{}^l F_k{}^j \nabla_l \nabla^k \eta_j \\ & + \sqrt{-1} (\nabla_{i_1} F_k{}^j \nabla^k \eta_j + F_k{}^j \nabla_{i_1} \nabla^k \eta_j + F_{i_1}{}^j \nabla_j \nabla^k \eta_k) , \end{aligned}$$

$$\begin{aligned}
 -4(\bar{\mathcal{G}}_2\partial_1\eta)_{i_1} &= \eta_j F_k^l (\nabla^k \nabla_l F_{i_1}^j - \nabla^k \nabla_{i_1} F_l^j) + (\nabla_l F_{i_1}^j - \nabla_{i_1} F_l^j) F_k^l \nabla^k \eta_j \\
 &\quad + F_l^j (\nabla^l F_{i_1}^k \nabla_k \eta_j - \nabla^l F_j^k \nabla_k \eta_{i_1}) + 2\nabla^j \nabla_j \eta_{i_1} - \nabla^j \nabla_{i_1} \eta_j \\
 &\quad + F_l^j F_{i_1}^k \nabla^l \nabla_k \eta_j + \sqrt{-1} [(\nabla^j \nabla_j F_{i_1}^k - \nabla^j \nabla_{i_1} F_j^k) \eta_k \\
 &\quad + (\nabla_k F_{i_1}^j - \nabla_{i_1} F_k^j) \nabla^k \eta_j + \nabla^j F_{i_1}^k \nabla_k \eta_j - \nabla^j F_j^k \nabla_k \eta_{i_1} \\
 &\quad + F_{i_1}^j \nabla^k \nabla_j \eta_k - 2F_j^k \nabla^j \nabla_k \eta_{i_1} + F_j^k \nabla^j \nabla_{i_1} \eta_k] ,
 \end{aligned}
 \tag{3.31}$$

$$\begin{aligned}
 -4(\square_1\eta)_{i_1} &= \eta_j F_k^l (\nabla^k \nabla_l F_{i_1}^j - \nabla^k \nabla_{i_1} F_l^j) + (\nabla_l F_{i_1}^j - \nabla_{i_1} F_l^j) F_k^l \nabla^k \eta_j \\
 &\quad + F_l^j (\nabla^l F_{i_1}^k \nabla_k \eta_j - \nabla^l F_j^k \nabla_k \eta_{i_1}) - F_{i_1}^l \nabla_l F_k^j \nabla^k \eta_j \\
 &\quad + [\nabla_{i_1}, \nabla_j] \eta^j + 2\nabla^j \nabla_j \eta_{i_1} + F_l^j F_{i_1}^k \nabla^l \nabla_k \eta_j \\
 &\quad - F_{i_1}^l F_k^j \nabla_l \nabla^k \eta_j \\
 &\quad + \sqrt{-1} \{ (\nabla^j \nabla_j F_{i_1}^k - \nabla^j \nabla_{i_1} F_j^k) \eta_k \\
 &\quad \quad + (\nabla_k F_{i_1}^j - 2\nabla_{i_1} F_k^j) \nabla^k \eta_j \\
 &\quad \quad + \nabla^j F_{i_1}^k \nabla_k \eta_j - \nabla^j F_j^k \nabla_k \eta_{i_1} + F^{kj} [\nabla_k, \nabla_{i_1}] \eta_j \\
 &\quad \quad + F^{kj} [\nabla_j, \nabla_k] \eta_{i_1} + F_{i_1}^j [\nabla_k, \nabla_j] \eta^k \} .
 \end{aligned}
 \tag{3.32}$$

3.5. Laplacian \square_5 . Finally, for the remaining Laplacian \square_5 we first have

$$\square_5 \xi = \square_1 \xi ,
 \tag{3.33}$$

$$\bar{\delta}_1 \eta = -\eta_j \nabla^i \prod_{1,0} i^j - \prod_{1,0} i^j \nabla^i \eta_j .
 \tag{3.34}$$

Adding (3.8) to (3.9) gives

$$\begin{aligned}
 8(d_1\eta)_{i_1 i_2} &= 4\nabla_{i_1} \eta_{i_2} - 4\nabla_{i_2} \eta_{i_1} + \eta_j [F_{i_1}^k (\nabla_{i_2} F_k^j - \nabla_k F_{i_2}^j) \\
 &\quad + F_{i_2}^k (\nabla_k F_{i_1}^j - \nabla_{i_1} F_k^j)] \\
 &\quad + \sqrt{-1} \{ \eta_j [3F_{i_1}^k F_{i_2}^l (\nabla_k F_l^j - \nabla_l F_k^j) \\
 &\quad \quad + \nabla_{i_1} F_{i_2}^j - \nabla_{i_2} F_{i_1}^j] + 4F_{i_2}^k \nabla_k \eta_{i_1} \\
 &\quad \quad - 4F_{i_1}^k \nabla_k \eta_{i_2} \} .
 \end{aligned}
 \tag{3.35}$$

Now put

$$\begin{aligned}
 8G_{i_1 i_2} &= 4\nabla_{i_1} \eta_{i_2} + \eta_j (F_{i_1}^k \nabla_{i_2} F_k^j + F_{i_2}^k \nabla_k F_{i_1}^j) \\
 &\quad + \sqrt{-1} (3\eta_j F_{i_1}^k F_{i_2}^l \nabla_k F_l^j + \eta_j \nabla_{i_1} F_{i_2}^j + 4F_{i_2}^j \nabla_j \eta_{i_1}) .
 \end{aligned}
 \tag{3.36}$$

Then

$$8d_1\eta = (G_{i_1 i_2} - G_{i_2 i_1}) dx^{I_0(2)} .
 \tag{3.37}$$

As in the derivation of (3.16), (3.17) we can obtain

$$-2(\prod_{0,1} \delta \prod_{1,1} d_1\eta)_{i_1} = \prod_{0,1} \nabla^s [(\varepsilon_s^{k_1 \varepsilon_{i_1}^{k_2}} - \varepsilon_{i_1}^{k_1 \varepsilon_s^{k_2}} + F_s^{k_1} F_{i_1}^{k_2} - F_{i_1}^{k_1} F_s^{k_2}) G_{i_1 i_2}] ,
 \tag{3.38}$$

$$(3.39) \quad -4\left(\prod_{1,0} \delta \prod_{2,0} d_1\eta\right)_{i_1} = \prod_{1,0} \mathcal{V}^s[(B_{s i_1}^{k_1 k_2} + \sqrt{-1}C_{s i_1}^{k_1 k_2})G_{k_1 k_2}] ,$$

where $B_{s i_1}^{k_1 k_2}$ and $C_{s i_1}^{k_1 k_2}$ are defined in (3.14). After some calculations we can thus have

$$(3.40) \quad \begin{aligned} 16(\square_5\eta)_{i_1} &= 16\left[\left(\prod_{1,0} \delta \prod_{2,0} + \prod_{0,1} \delta \prod_{1,1}\right)d_1\eta + \prod_{1,0} d\bar{\delta}_1\eta\right]_{i_1} \\ &= \eta_j[\mathcal{V}^s F_{i_1}^k(\mathcal{V}_s F_k^j - \mathcal{V}_k F_s^j) + F_{i_1}{}^r F_s^k \mathcal{V}^s F_r{}^l (\mathcal{V}_k F_l^j \\ &\quad - \mathcal{V}_l F_k^j) + 4F_{i_1}{}^k (\mathcal{V}^s \mathcal{V}_s F_k^j - \mathcal{V}^s \mathcal{V}_k F_s^j + \mathcal{V}_k \mathcal{V}^l F_l^j)] \\ &\quad + 8F_{i_1}{}^k \mathcal{V}_s F_k^j \mathcal{V}^s \eta_j - 8\mathcal{V}^s \mathcal{V}_s \eta_{i_1} + 4[\mathcal{V}_k, \mathcal{V}_{i_1}] \eta^k \\ &\quad + 4F_{i_1}{}^l F^{kj} [\mathcal{V}_l, \mathcal{V}_k] \eta_j + \sqrt{-1} \{\eta_j [4F_{i_1}{}^k \mathcal{V}^s F_s{}^l (\mathcal{V}_k F_l^j \\ &\quad - \mathcal{V}_l F_k^j) + F_{i_1}{}^r \mathcal{V}^s F_r{}^k (\mathcal{V}_k F_s^j - \mathcal{V}_s F_k^j) \\ &\quad + 3F_s^k \mathcal{V}^s F_{i_1}{}^l (\mathcal{V}_l F_k^j - \mathcal{V}_k F_l^j) + 4\mathcal{V}_{i_1} \mathcal{V}^k F_k^j \\ &\quad + 4F_{i_1}{}^k F_s{}^l \mathcal{V}^s (\mathcal{V}_k F_l^j - \mathcal{V}_l F_k^j)] + 8\mathcal{V}^s F_s^k \mathcal{V}_k \eta_{i_1} \\ &\quad + 4F_{i_1}{}^r F^{sk} (\mathcal{V}_r F_s{}^l + \mathcal{V}_s F_r{}^l) \mathcal{V}_l \eta_k - 4(\mathcal{V}_{i_1} F_j^k + \mathcal{V}_j F_{i_1}^k) \mathcal{V}_k \eta^j \\ &\quad + 4F^{kj} [\mathcal{V}_{i_1}, \mathcal{V}_k] \eta_j + 4F_{i_1}{}^l [\mathcal{V}_l, \mathcal{V}_k] \eta^k + 4F^{sk} [\mathcal{V}_s, \mathcal{V}_k] \eta_{i_1} \} . \end{aligned}$$

4. Realization of \square 's

Theorem 4.1. *The complex Laplacian \square_i , $i = 1, \dots, 5$, for an almost-Hermitian structure is real with respect to every 0-form if and only if the structure is almost-semi-Kählerian. Moreover, with respect to every 0-form, if \square_i , $i = 1, \dots, 5$, for an almost-Hermitian structure is real, then $\square_i = \Delta/2$ for $i = 1, \dots, 5$.*

Proof. The theorem follows immediately from (3.1), (3.5), (3.22), (3.28), (3.33) and (2.8) by choosing the 0-form ξ to be x^k for an arbitrary k with respect to any local coordinates x^1, \dots, x^n .

Theorem 4.2. *For an almost-Hermitian structure, if the Laplacian \square_i , $i = 1, 2$ or 4 , is real with respect to all 0- and 1-forms, then the structure is Kählerian.*

Kodaira and Spencer [3] have shown that if the relation

$$(4.1) \quad \square_1 = \Delta/2$$

holds for an almost-Hermitian structure, then the structure is integrable. The particular case of Theorem 4.2 in which

$$(4.2) \quad \square_i = \Delta/2 \quad (i = 1, 2 \text{ or } 4)$$

holds was a conjecture for some time; it was proved by Hsiung [2] for $i = 2$ and by A. W. Adler [1] for $i = 1$ by a different method under a stronger as-

sumption that (4.1) holds for a Hermitian structure and all 0-, 1- and 2-forms. Theorem 4.2 was proved by Hsiung [2] and Ogawa [5] for $i = 2$, and by Ogawa [5] for $i = 1$ by a somewhat different method.

Proof. (i) $i = 2$. In [2, p. 148] Hsiung proved that under the assumption of the theorem the structure F satisfies² (2.17) and (2.28). Then the theorem follows immediately from Lemmas 2.4 and 2.3; this was pointed out to one of the authors by H. Wakakuwa.

(ii) $i = 1$. Using the Ricci and Bianchi identities and (2.23) we can easily obtain

$$(4.3) \quad F_{i_1}{}^k[\nabla_j, \nabla_k]\eta^j = F_{i_1}{}^k R_k{}^j \eta_j,$$

$$(4.4) \quad F^{kj}[\nabla_k, \nabla_{i_1}]\eta_j - F^{jk}[\nabla_j, \nabla_k]\eta_{i_1} = -\frac{1}{2}F^{kl}R^j{}_{i_1kl}\eta_j,$$

$$(4.5) \quad \nabla_s \nabla_{i_1} F^{sj} - \nabla_{i_1} \nabla_s F^{sj} = F_a{}^j R_{i_1}{}^a - \frac{1}{2}F^{sa}R^j{}_{i_1sa}.$$

By assumption, for any 1-form η , $\text{Im } \square_i \eta = 0$ which is reduced to, in consequence of Theorem 4.1, (2.8), (3.21), (4.3), (4.4), (4.5),

$$(4.6) \quad 2(\nabla_s F_{i_1}{}^j + \nabla_{i_1} F_s{}^j)\eta_j + (\nabla^s \nabla_s F_{i_1}{}^j + F_{i_1}{}^k R_k{}^j - R_{i_1}{}^k F_k{}^j)\eta_j = 0.$$

By choosing

$$(4.7) \quad \eta = dx^h, \quad \text{for an arbitrary } h$$

with respect to any local coordinates x^1, \dots, x^n , from (4.6) it thus follows that

$$(4.8) \quad \nabla^s \nabla_s F_{i_1}{}^h + F_{i_1}{}^k R_k{}^h - R_{i_1}{}^k F_k{}^h = 0.$$

Multiplying (4.8) by $F_h{}^{i_1}$ and using (2.1) we obtain (2.16), and therefore the structure F is Kählerian by Lemma 2.2.

(iii) $i = 4$. At a general point P of the manifold M^n we choose orthogonal geodesic local coordinates x^1, \dots, x^n so that

$$(4.9) \quad g_{ij}(P) = \delta_{ij}, \quad \Gamma_{ij}^k(P) = 0,$$

where δ_{ij} are Kronecker deltas. By using Theorem 4.1, and choosing η to satisfy (4.7) first and then

$$(4.10) \quad \eta = x^h dx^l, \quad \text{for any fixed distinct } h \text{ and } l$$

with respect to the geodesic local coordinates x^1, \dots, x^n , from (3.32) the condition $\text{Im } (\square_i \eta) = 0$ for any 1-form η is reduced to

$$(4.11) \quad \nabla^j \nabla_j F_{i_1}{}^h - \nabla^j \nabla_{i_1} F_j{}^h = 0,$$

² By mistake, (2.28) was printed as $F_i{}^h R^i{}_{jkl} = F_j{}^i R^h{}_{ikl}$ in [2, p. 148].0

$$(4.12) \quad \nabla^h F_{i_1 l} + 2\nabla_{i_1} F_l^h + \nabla_l F_{i_1}^h = 0 .$$

Interchanging l, i_1 in (4.12) and adding the resulting equation to (4.12) we obtain

$$(4.13) \quad \nabla_{i_1} F_l^h + \nabla_l F_{i_1}^h = 0 .$$

From (4.11), (4.13) it thus follows that

$$(4.14) \quad \nabla^j \nabla_j F_{i_1}^h = 0 ,$$

and hence by Lemma 2.2 the structure is Kählerian.

5. Relationships among \square 's

Theorem 5.1. *If for an almost-Hermitian structure the relation*

$$(5.1) \quad \text{Im } \square_1 = \text{Im } \square_i \quad (i = 2 \text{ or } 4)$$

holds for all 0- and 1-forms, then the structure is Kählerian.

Proof. (i) $i = 2$. From (3.5), (3.1) and condition (5.1) for any 0-form ξ , we have

$$(5.2) \quad \nabla^h F_{i_1}^j \nabla_j \xi = 0 .$$

By choosing $\xi = x^i$ for an arbitrary i with respect to any local coordinates x^1, \dots, x^n , from (5.2) follows immediately (2.8), which together with (3.2), (3.21), (3.20) reduces condition (5.1) for any 1-form η to

$$(5.3) \quad (\nabla_s F_{i_1}^j + \nabla_{i_1} F_s^j) \nabla_j \eta^s - (\nabla^s \nabla_{i_1} F_s^j - \nabla^s \nabla_s F_{i_1}^j) \eta_j = 0 .$$

Choosing η to satisfy (4.7) first and then (4.10) with respect to the local coordinates x^1, \dots, x^n defined by (4.9) we therefore obtain (4.11), (4.13), and hence the structure is Kählerian for the same reasoning given in the proof (iii) of Theorem 4.2.

(ii) $i = 4$. As in part (i), from (3.5), (3.28), (3.1) and condition (5.1) for any 0-form ξ , we obtain (2.8), which together with (3.21), (3.32) reduces condition (5.1) for any 1-form η to

$$(5.4) \quad (\nabla_j F_{i_1}^k - \nabla^k F_{i_1 j}) \nabla_k \eta^j = 0 .$$

By choosing η to satisfy (4.10) with respect to the local coordinates x^1, \dots, x^n defined by (4.9), we have

$$(5.5) \quad \nabla_i F_{h i_1} - \nabla_h F_{l i_1} = 0 .$$

Thus by Lemma 2.1 the structure is Kählerian.

Theorem 5.2. *If for an almost-Hermitian structure either the relation*

$$(5.6) \quad \text{Im } \square_2 = \text{Im } \square_4$$

or

$$(5.7) \quad \text{Re } \square_2 = \text{Re } \square_4$$

holds for all 1-forms, where Re denotes the real part, then the structure is Kählerian.

Proof. From (3.1), (3.32), by the same argument as in the proof of Theorem 5.1 for $i = 4$ it is easily seen that conditions (5.6), (5.7) imply

$$(5.8) \quad \nabla_h F_{i_1}{}^l = \nabla_{i_1} F_h{}^l = 0,$$

$$(5.9) \quad F_h{}^j \nabla_{i_1} F_j{}^l - F_h{}^j \nabla_j F_{i_1}{}^l = 0,$$

respectively. By multiplying (5.9) by $F_k{}^h$, we can reduce (5.9) to (5.8). Hence by Lemma 2.1, the structure is Kählerian under either (5.6) or (5.7).

Theorem 5.3. *If for an almost-Hermitian structure the relation*

$$(5.10) \quad \text{Im } \square_2 = \text{Im } \square_5$$

holds for all 0- and 1-forms, then the structure is Kählerian.

Proof. From (3.33), (3.5), (3.1) and condition (5.10) for any 0-form ξ we obtain (2.8). Then by the same argument as in the proof of Theorem 5.1 for $i = 2$, (2.8), (3.2), (3.40) reduce condition (5.10) for any 1-form η to

$$(5.11) \quad F_{i_1}{}^\tau \nabla^\tau F_\tau{}^k (\nabla_k F_s{}^h - \nabla_s F_k{}^h) + 3F_s{}^k \nabla^s F_{i_1}{}^l (\nabla_l F_k{}^h - \nabla_k F_l{}^h) \\ + 4F_{i_1}{}^k F_s{}^l \nabla^s (\nabla_k F_l{}^h - \nabla_l F_k{}^h) = 0,$$

$$(5.12) \quad F_{i_1}{}^\tau F^{s l} (\nabla_\tau F_s{}^h + \nabla_s F_\tau{}^h) = 0.$$

Multiplying (5.12) by $F_j{}^{i_1} F_l{}^k$ and use of (2.1) give

$$(5.13) \quad \nabla_j F_k{}^h + \nabla_k F_j{}^h = 0.$$

Substituting (5.13) in (5.11) we can easily obtain

$$(5.14) \quad 2F_{i_1}{}^k F_s{}^l \nabla^s \nabla_k F_l{}^h - F_s{}^k \nabla^s F_{i_1}{}^l \nabla_k F_l{}^h = 0.$$

Multiplying (5.14) by $F_h{}^{i_1}$ and using (2.1), (2.8), (5.13) we therefore have

$$(5.15) \quad \nabla_s F_{i_1 l} \nabla^s F^{i_1 l} = 0,$$

which implies that $\nabla_s F_{i_1 l} = 0$. Hence the structure is Kählerian. q.e.d.

Finally, it should be remarked that there are no theorems involving the Laplacian \square_3 similar to Theorems 4.2, 5.1, 5.2, 5.3. However, we have the following two theorems, the proofs of which are omitted.

Theorem 5.4. *If for an almost-Hermitian structure the relation*

$$(5.16) \quad \text{Im } \square_3 = \text{Im } \square_1 + \frac{1}{2} \text{Im } (\bar{\mathcal{J}}_1 \partial_2)$$

holds for all 1-forms, then the structure is Kählerian.

Theorem 5.5. *If for an almost-semi-Kählerian structure the relation*

$$(5.17) \quad \text{Im } \square_3 = \text{Im } \square_i + \frac{1}{2} \text{Im } (\bar{\mathcal{J}}_1 \partial_2) \quad (i = 2 \text{ or } 4)$$

holds for all 1-forms, then the structure is Kählerian.

References

- [1] A. W. Adler, *Classifying spaces for Kähler metrics IV: The relation $\Delta = 2 \cdot \square$* , Math. Ann. **160** (1965) 41–58.
- [2] C. C. Hsiung, *Structures and operators on almost-Hermitian manifolds*, Trans. Amer. Math. Soc. **122** (1966) 136–152.
- [3] K. Kodaira & D. C. Spencer, *On the variation of almost-complex structure*, Algebraic Geometry and Topology, A Symposium in Honor of S. Lefschetz, Princeton University Press, Princeton, 1957, 139–150.
- [4] S. Kotô, *Some theorems on almost Kählerian spaces*, J. Math. Soc. Japan **12** (1960) 422–433.
- [5] Y. Ogawa, *Operators on almost Hermitian manifolds*, J. Differential Geometry **4** (1970) 105–119.
- [6] G. de Rham & K. Kodaira, *Harmonic integrals*, Mimeographed notes, Institute for Advanced Study, Princeton, 1950.
- [7] M. Schiffer & D. C. Spencer, *Functionals of finite Riemann surfaces*, Princeton University Press, Princeton, 1954.

LEHIGH UNIVERSITY

