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ANALYTIC COMPLEX STRUCTURES ON HILBERT MANIFOLDS

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Always by a Hilbert manifold we mean a paracompact separable infinite dimensional C^∞-manifold whose local model is the infinite dimensional separable Hilbert space and by differentiable, C^∞-differentiable. In this note we construct, for any such Hilbert manifold M, many nonequivalent complex analytic structures (for the definition of complex analytic structures we refer to [4]), namely, an infinite family of different analytic structures, all of whose holomorphic functions are constant (Theorem 4.1), and infinitely many different analytic structures which have sufficient holomorphic functions, i.e., for any two different points x, y there exists a holomorphic function with different values at x and y (Corollary 5.2). We invite comparison of these results with the following ones: Any two homotopy equivalent Hilbert manifolds are diffeomorphic, and any two homotopic diffeomorphisms are isotopic, [2], [1]. To prove the stated results we need some differential topology of Hilbert manifolds which will be developed in § 1, the Calabi-Eckmann equivalent in Hilbert space (§ 2), and Hartogs' theorem in Hilbert space (§ 3). §§ 4 and 5 deal with the construction of the stated complex structures.

1.

Theorem 1.1 (*Eells and Elworthy* [5]). *Any Hilbert manifold is diffeomorphic to an open set of the real Hilbert space H.*

Since all infinite dimensional separable Hilbert spaces are isomorphic, we will denote them by H and sometimes by H^R, H^C, when we indicate the field over real R and complex C respectively; of course, H^R and H^C are R (real) isomorphic.

Theorem 1.2 ([2], *see also* [1]). *Any Hilbert manifold M is Palais stable* (*stable, for short*), *i.e., M is diffeomorphic to $M \times H$.*

Theorem 1.3 [2]. *Two homotopy equivalent Hilbert manifolds are diffeomorphic.*

Proposition 1.3′ (*Bessaga* [2]). *The unit sphere $S^\infty = \{v \in H \mid \| v \| = 1\}$ is diffeomorphic to H.*

Proposition 1.4 [2]. *Any Hilbert manifold can be closed and bounded differentiably imbedded in H.*

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Proposition 1.5. *For any given open set $U \subset H$ and any Hilbert manifold M, there exists a closed and bounded imbedded manifold with boundary L such that*[1]

1) $L \backslash \partial L$ *is an open set in* H,
2) $L \subset U$,
3) $H \backslash L$ *is diffeomorphic to* M.

Proof. Let $f: M \to H$ be a closed imbedding which exists according to Proposition 1.4. Choose $\bar{f}: B(\nu) \to H$ to be a closed tubular neighborhood of f, where $B(\nu)$ denotes the total space of the fibre bundle with discs associated with the normal bundle of f. Let $T = \bar{f}(B(\nu))$(Hilbert manifold with boundary) and $P = H \backslash \text{Int}\, T$. Take points $q_1 \in \text{Int}\, T$ and $q_2 \in U$ and the closed discs $D_1^\infty \subset \text{Int}\, T$, $D_2^\infty \subset U$ centered at q_1, q_2 respectively. Because H is diffeomorphic to S^∞ (Proposition 1.3′) there exists a diffeomorphism $l: H \to H$ such that $l(H \backslash D_1^\infty) \subset \text{Int}\, D_2^\infty$. The theorem follows taking $L = l(P)$ because of Theorem 1.3 and the remark that $\text{Int}\, B(\nu)$ has the same homotopy type as M.

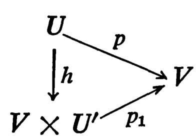
Theorem 1.6. *Given a Hilbert manifold M, $p \in H$, and an open neighborhood U of $p \times R^2 \subset H \times R^2$, there exists a closed imbedded manifold with boundary $(\mathscr{L}, \partial \mathscr{L}) \subset H \times R^2$ such that*

1) $\mathscr{L} \backslash \partial \mathscr{L}$ *is an open set in* $H \times R^2$,
2) $\mathscr{L} \subset U$,
3) $H \times \{t\} \backslash \mathscr{L}$ *diffeomorphic to* M, $t \in R^2$.

Proof. Choose a C^∞-function $\rho: R^2 \to R_+$ such that for any t, the disc $D^\infty(p, \rho(t))$ centered at p with radius $\rho(t)$ is contained in $U \cap H \times t$. Now let us consider $U = \text{Int}\, D^\infty(1)$ in Proposition 1.5, define $m: H \times R^2 \to H \times R^2$ by $m(v, t) = (\rho(t)v + p, t)$, and take $\mathscr{L} = m(\mathscr{L} \times R^2)$.

According to [8] an analytic family of complex structures on a differentiable manifold M with parameter $t \in N$ (complex analytic manifold) is a complex analytic manifold $\mathscr{P}$ and a holomorphic map $p: \mathscr{P} \to N$ such that

1) p is holomorphic locally-locally trivial, i,e., for any $x \in \mathscr{P}$ there exist open neighborhoods $U \ni x$ and $V \ni p(x)$ and an analytic isomorphism $h: U \to V \times U'$ such that the diagram



is commutative.

[1] $A \backslash B$ denotes the subset consisting of those points of A, which do not belong to B.

2.

Consider the (complex) Hilbert space H^C denoted for short by H. Let $S^\infty = \{v \mid \|v\| = 1\}$, and let $p\colon S^\infty \to P(H)$ be the canonical map where $P(H)$ denotes the projective space of the complex Hilbert space which is an analytic manifold, [4]. $p \times p\colon S^\infty \times S^\infty \to P(H) \times P(H)$ is a differentiable bundle (neglecting the complex structure of $P(H) \times P(H)$) whose fibre is $S^1 \times S^1 (S^1 = \{\lambda \in C \mid |\lambda| = 1\})$.

Let us consider $C \backslash R = \{\alpha \in C \mid \mathrm{Imag}\,\alpha \neq 0\}$. Following Calabi-Eckmann [3] we will define a complex family of complex analytic structures on $S^\infty \times S^\infty$ with parameters in $C\backslash R$ such that:

For any $\tau \in C\backslash R$ the corresponding structure on $S^\infty \times S^\infty, S^\infty \times S^\infty_\tau$ makes $p \times p$ an analytic fibre bundle whose fibre is $S^1 \times S^1_{(1,\tau)} = T_\tau$, the complex tori obtained as the quotient-space of the $Z \oplus Z$-free action τ^* on C and τ^* defined by $\tau^*((m, n), z) = z + m + n\tau$. To distinguish between the first and the second components of $H \times H$ we will denote the first H by H_1 and the second by H_2. Choose the orthonormal basis $e_1 e_2, \cdots, e_n, \cdots$ in H_1 and $f_1 f_2, \cdots, f_n, \cdots$ in H_2. Consider the map $l_{kj}\colon H_1 \times H_2 \to C$ defined by $l_{kj}(v, w) = \langle v, e_k\rangle\langle w, f_j\rangle$ and look at the restriction l'_{kj} of l_{kj} to $S^\infty_1 \times S^\infty_2$. Let $V_{kj} = l'^{-1}_{kj}(C\backslash\{0\})$, and for any $\tau \in C\backslash R$ define the homeomorphism $h^\tau_{kj}\colon V_{kj} \to H^\perp_{1(e_k)} \times H^\perp_{2(f_j)} \times T_\tau$ given by

$$h^\tau_{kj}(v, w) = \left(\frac{v - \langle v, e_k\rangle e_k}{\langle v, e_k\rangle}, \frac{w - \langle w, f_j\rangle f_j}{\langle w, f_j\rangle}, \frac{1}{2\pi_i}[\log\langle v, e_k\rangle + \tau \log\langle w, f_j\rangle]\right) .$$

($H^\perp_{1(e_k)}$ and $H^\perp_{2(f_j)}$ are respectively the orthogonal complements of e_k and f_j.) h^τ_{kj} is a C^∞-diffeomorphism, and $h^\tau_{kj} \cdot (h^\tau_{k'j'})^{-1}$ is an analytic homeomorphism [3]. Moreover

$$\begin{array}{ccc} V_{kj} \times C\backslash R & \xrightarrow{h_{kj}(x,\tau) = (h^\tau_{kj}(x), \tau)} & (H^\perp_{1(e_k)} \times H^\perp_{2(f_j)}) \times \mathscr{T} \\ & \searrow \quad \swarrow {\scriptstyle p, p_2} & \\ & C\backslash R & \end{array}$$

where $\mathscr{T} \xrightarrow{p} C\backslash R$ is a complex family of complex tori obtained as the complex $Z \oplus Z$-free action in $C \times (C\backslash R)$ defined by $((m, n), z, \tau)) = (z + m + n\tau, \tau)$; in fact, $\mathscr{T} = \bigcup_\tau T_\tau$. One remarks that $h_{kj} \cdot (h_{k'j'})^{-1}$ is holomorphic, hence h_{kj} defines on $S^\infty \times S^\infty \times C\backslash R$ a complex analytic structure. One verifies easily that $S^\infty \times S^\infty \times C\backslash R \xrightarrow{p \times p \times id} P(H) \times P(H) \times C\backslash R$ is holomorphic, and

$$\begin{array}{ccc} S^\infty \times S^\infty \times C\backslash R & \longrightarrow & P(H) \times P(H) \times C\backslash R \\ & \searrow \quad \swarrow & \\ & C\backslash R & \end{array}$$

is a complex family of analytic fibrebundles (whose definition is obvious, see, for instance, [8]).

Remark 2.1. The complex family, namely, the analytic structure induced on $S^\infty \times S^\infty$ for any τ, is independent of the chosen basis of H.

Remark 2.2. Suppose C_n is the subspaces generated by the first n vectors $e_1, \cdots, e_n$ of the chosen basis. Then in the commutative diagram

$$\begin{array}{ccc} S^{2n-1} \times S^{2n-1} \times C\backslash R & \xrightarrow{\ t\ } & S^\infty \times \ S^\infty \times \ C\backslash R \\ \big\downarrow & & \big\downarrow p \quad \big\downarrow p \quad \big\downarrow i \\ P(C^n) \times P(C^n) \times C\backslash R & \longrightarrow & P(H) \times P(H) \times C\backslash R \end{array}$$

all maps are analytic, t being an analytic imbedding. It will be convenient for us to consider some canonical charts. Let us denote by $\mathscr{T}^{\mathrm{I}}, \mathscr{T}^{\mathrm{II}}, \mathscr{T}^{\mathrm{III}}, \mathscr{T}^{\mathrm{IV}}$ the following open sets of $\mathscr{T}$,

$$\begin{aligned} \mathscr{T}^{\mathrm{I}} &= \bigcup_{\tau \in C\backslash R} \{(z, t) \mid z = \lambda + \mu\tau, 0 < \lambda, \mu < 1\} , \\ \mathscr{T}^{\mathrm{II}} &= \bigcup_{\tau \in C\backslash R} \{(z, \tau) \mid z = \lambda + \mu\tau, 0 < \mu < 1, 1/2 < \lambda < 3/2\} , \\ \mathscr{T}^{\mathrm{III}} &= \bigcup_{\tau \in C\backslash R} \{(z, \tau) \mid z = \lambda + \mu\tau, 0 < \lambda < 1, 1/2 < \mu < 3/2\} , \\ \mathscr{T}^{\mathrm{IV}} &= \bigcup_{\tau \in C\backslash R} \{(z, \tau) \mid z = \lambda + \mu t, 1/2 < \lambda, \mu < 3/2\} . \end{aligned}$$

Let us define

$$h^{(?)}_{k,j}\colon \gamma^{(?)}_{k,j} = h^{-1}_{kj}(H^{\perp}_{1(e_k)} \times H^{\perp}_{2(fj)} \times \mathscr{T}^{(?)}) \to H^{\perp}_{1(e_k)} \times H^{\perp}_{2(fj)} \times \mathscr{T}^{(?)} ,$$

where $(?) = {}^{\mathrm{I}}, {}^{\mathrm{II}}, {}^{\mathrm{III}}, {}^{\mathrm{IV}}$, and $h^{(?)}_{kj}$ is a complex family of *R-convex* charts on $S^\infty \times S^\infty$. Then any complex structure $S^\infty \times S^\infty$ of our family can be covered by the canonical charts $\{{}^{\{\tau\}}V^{\mathrm{I}}_{k,j}, {}^{\{\tau\}}V^{\mathrm{II}}_{k,j}, {}^{\{\tau\}}V^{\mathrm{III}}_{k,j}, {}^{\{\tau\}}V^{\mathrm{I}}_{k,j}\}$, where k, j are integers, and ${}^{\{\tau\}}V^{(?)}_{k,j}$ is the fibre over τ of the family $\gamma^{(?)}_{k,j}$.

Remark 2.3. The Calabi-Eckmann construction is functorial on the category of complex Hilbert spaces and closed linear imbeddings, transforming the imbedding of Hilbert spaces in an analytic imbedding of complex family.

Remark 2.4. If H has a basis $e_1, \cdots, e_k, \cdots$, and C^k denotes the subspace generated by $e_1, \cdots, e_k$, according to 2) we have an imbedding of families, namely:

$$\begin{array}{ccc} (S^{2k-1} \times S^{2k-1}) \times C\backslash R & \longrightarrow & (S^\infty \times S^\infty) \times C\backslash R \\ & {\scriptstyle p_2} \searrow \quad \swarrow {\scriptstyle p_2} & \\ & C\backslash R & \end{array}$$

Moreover, the family of charts $\gamma_{i,j}^{C\,k(?)} = \gamma_{i,j}^{H(?)} \cap (S^{2k-1} \times S^{2k-1} \times C/R)$ for $i, j \leq k$.

Remark 2.5. For any $\tau \in C\backslash R$ the complex analytic manifold $S^\infty \times S^\infty_\tau$ has no holomorphic functions[2], because we have a sequence of compact analytic submanifolds

$$S^{2k-1} \times S_\tau^{2k-1} \subset S^{2k+1} \times S_\tau^{2k+1} \subset \cdots \subset S^{2(k+r)-1} \times S_\tau^{2(k+r)-1} \subset \cdots,$$

whose union is everywhere dense in $S^\infty \times S^\infty_\tau$.

Remark 2.6. If h denotes a diffeomorphism of $S^\infty \times S^\infty \to H$ which exists because of Proposition 1.3′, then $h \times \mathrm{id}\colon S^\infty \times S^\infty \times C\backslash R \to H \times C\backslash R$ will be a diffeomorphism commuting with the projection on the component $C\backslash R$, and one can consider this family of complex structures on $S^\infty \times S^\infty$ as a family of complex stuctures on H.

Remark 2.7. $S^\infty \times S^\infty_\tau$ is isomorphic to $S^\infty \times S^\infty_{\tau'}$ iff τ and τ' are related by the following equation:

$$\tau = \frac{a_{11}\tau' + a_{12}}{a_{21}\tau' + a_{22}}, \tag{1}$$

where a_{ij} are integers and $\det |a_{ij}| = \pm 1$. To prove this, notice that τ and τ' related by (1) imply that the identity map is holomorphic with respect to the analytic structures τ and τ' (as one can easily see

$$\left(\frac{1}{2\pi i}[\log\langle x, e_k\rangle + \tau \log\langle x, f_j\rangle] \to \frac{1}{2\pi i}[\log\langle v, e_k\rangle + \tau' \log\langle v, f_j\rangle]\right)$$

is an analytic isomorphism of torus $T_{(1,\tau)} \to T_{(1,\tau')}$ as soon as (1) is satisfied).

Conversely, one uses the same argument as in the proof of Theorem 4.1 below.

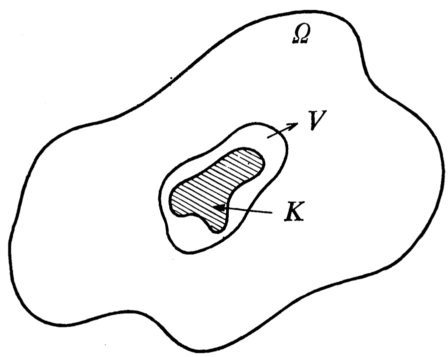
3.

Hartogs' theorem. *Let Ω be a bounded open set in H^C, and K a closed bounded set in H^C such that $K \subset \Omega$ and $\Omega\backslash K$ is connected. For every holomorphic function u on $\Omega\backslash K$ one can find a holomorphic function U on Ω so that $U = u$ in $\Omega\backslash K$.*

This theorem is the well-known Hartogs theorem in the case $H = C^n$, $n \geq 2$. The proof is exactly the same as in the finite dimensional case, namely, as the proof of [7, p. 30], but we still give it here for the convenience of the reader.

First, choose a C^∞-function $\varphi\colon \Omega \to R_+$ such that $\operatorname{Sup} \varphi \subset \Omega$ and $\varphi = 1$ on a neighborhood V of K. This is always possible because of the partition of the unity.

[2] The index τ for $S^\infty \times S^\infty_\tau$ refers to $S^\infty \times S^\infty$ and not to the second S^∞.



Now let us consider the function $u_0 = (1 - \varphi)u \in C^\infty$ defined on all Ω. Then u_0 is 0 on V. To seek U of the form $U = u_0 - v$, v has to satisfy the following differential equation $\bar\partial(v) = \bar\partial u_0 = -u\bar\partial\varphi = f$. (We can consider $H^C = H^R \otimes_R C \approx H^R \oplus H^R$, so $z \in H^C$ corresponds to $x + y, x \in H^R, y \in H^R$. Then $f(z) = f(x, y)$ and $\bar\partial f(z_0) = \frac{1}{2}\, df_x(z_0) + \frac{i}{2}\, df_y(z_0), \bar\partial f(z_0) \in \operatorname{Hom}_R (H^R \oplus H^R, C)$ $= \operatorname{Hom}_R (H^C, C)$.) Notice that function $f \in C^\infty$ can be extended on H with zero outside Ω; $f = 0$ on V (f has bounded support because Ω is bounded).

Given a base $e_1, e_2, \cdots$ in H and $\alpha \in \operatorname{Hom}_R (H, C)$, the 1-component of α will be α restricted to $\{Ce_1\}$, and $z = z_1 \oplus z_1^\perp, z_1 \in \{Ce_1\}, z_1^\perp \in H_{e_1}^\perp$. Now define

$$v(z) = \frac{1}{2\pi i} \iint_{C = R \oplus R} (\tau - z_1)^{-1} f_1(\tau, z_1^\perp) \,, \qquad d\tau \wedge d\bar\tau \,,$$

where f_1 is the 1-component of f. Notice v is continuous in Ω because for any point $w \in H, w = w_1 \oplus w_1^\perp \in Ce_1 \oplus H_{e_1}^\perp = H$, there exists a neighborhood $D(w_1, \varepsilon) \times D(w_1^\perp, \eta) \subset Ce_1 \oplus H_{e_1}^\perp = H$, where $D(w_1, \varepsilon)$ and $D(w_1^\perp, \eta)$ are discs in Ce_1 and $H_{e_1}^\perp$ centered at $w_1 \in C\{e_1\}$ and $w_1^\perp \in H_{e_1}^\perp$ with radii ε and η, respectively, such that v is continuous on this neighborhood which can be found in the following way:

(1) If $w \in V$, take (ε, η) such that $D(w_1, \varepsilon) \times D(w_1^\perp, \eta) \subset V$. Then for any $z \in D(w_1, \varepsilon) \times D(w_1^\perp, \eta)$,

$$\begin{aligned} v(z) &= \frac{1}{2\pi i} \iint_{\mathring{D}(w_1, \varepsilon)} (\tau - z_1)^{-1} f_1(\tau, z_1^\perp)\, d\tau \wedge d\bar\tau \\ &\quad + \frac{1}{2\pi i} \iint_{C \setminus \mathring{D}(w_1, \varepsilon)} (\tau - z_1)^{-1} f_1(\tau, z_1^\perp) d\tau \wedge d\bar\tau \\ &= \frac{1}{2\pi i} \iint_{C \setminus \mathring{D}(w_1, \varepsilon)} (\tau - z_1)^{-1} f_1(\tau, z_1^\perp) d\tau \wedge d\bar\tau \,, \end{aligned}$$

since f restricts on V to zero. The last term becomes equal to $\frac{1}{2\pi i}\iint_{D(0,N)\setminus \overset{\circ}{D}(w_1,\varepsilon)} (\tau - z_1)^{-1} f_1(\tau, z_1^\perp) d\tau \wedge d\bar{\tau}$, where $D(0,N)$ denotes the disc in C centered 0 with radius N since f has bounded support. The last integral is obviously continuous on $D(w_1,\varepsilon) \times D(w_1^\perp, \eta)$, because $g(\tau, z_1, z_1^\perp) = \frac{1}{(\tau - z_1)^{-1}} f_1(\tau, z_1^\perp)$ restricts on $(D(0,N)\setminus D^0(w_1;\varepsilon) \times D(w_1,\varepsilon) \times D(w_1^\perp, \eta)$ to a continuous function, ane $D(0,N)\setminus D^0(w_1;\varepsilon)$ is compact.

(2) If $w \in \Omega\setminus V$, it is easy to verify that $v(w)$ is well defined and equal to $u_0(w) - v(w)$ by applying, for instance, Theorem 121 of [7]. Hence v is locally continuous on $\Omega\setminus V$.

4.

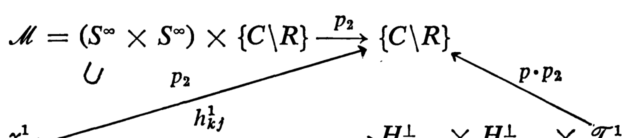
Theorem 4.1. *Given a Hilbert manifold M, there exists a complex family of complex analytic structures M_τ, $\tau \in C\setminus R$, such that:*

1) *M_τ has no nonconstant holomorphic functions,*

2) *if M_τ and $M_{\tau'}$ are analytic isomorphic, then $\tau = \frac{a_{11}\tau' + a_{12}}{a_{21}\tau' + a_{22}}$ with integers a_{ij} and* $\det|a_{ij}| = \pm 1$.

Proof. Consider the complex family of complex (Calabi-Eckmann)-structures defined in § 2. Take the complex family of charts γ_{kj}^1.

As we have seen, the diagram



is commutative and the fibre of p_2 in τ is a convex chart of the complex structure $S^\infty \times S_\tau^\infty$. Consider the unit discs $(D_1(H_{ek}^\perp)$ and $D_1(H_{fj}^\perp)$ in $H_{ek}^\perp$ ane $H_{fj}^\perp$ respectively. Applying Theorem 1.6 we can construct a closed manifold with boundary $\mathscr{L}$ such that $\mathscr{L} \subset h_{kj}^{-1}(D_1(H_{ek}^k) \times D_1(H_{fj}^j) \times \mathscr{T}')$ and $p_2^{-1}(\tau)\setminus\mathscr{L}$ is diffeomorphic to M because $S^\infty \times S^\infty \times \{C\setminus R\} \xrightarrow{p_2} \{C\setminus R\}$ is the trivial differential fibre bundle with fibre diffeomorphic to $S^\infty \times S^\infty = H^R$ (according to Proposition 1.3'). The map $(S^\infty \times S^\infty \times \{C\setminus R\})\setminus\mathscr{L} \xrightarrow{p_2} \{C\setminus R\}$ is surjective; moreover, it is a complex family of complex analytic structures $(S^\infty \times S_\tau^\infty\setminus\mathscr{L})$, and, by Hartogs' theorem, $S^\infty \times S_\tau^\infty\setminus\mathscr{L}$ has no nonconstant holomorphic functions. (If f is a holomorphic function on $S^\infty \times S_\tau^\infty\setminus\mathscr{L}$, consider the restriction of f to $V_{kj}\setminus\mathscr{L}$, and by applying Hartogs' theorem we get an extension of f to

V_{kj} and hence an extension of f on $S^\infty \times S^\infty_\tau$ which, according to Remark 3.2, must be constant.

Now let us consider M_τ and $M_{\tau'}$, both of which are open manifolds of the complex manifolds $S^\infty \times S^\infty_\tau$, complex fibre bundles over $P(H) \times P(H)$. This is represented by the following diagram:

$$\begin{array}{ccccc} S^\infty \times S^\infty_\tau & \xrightarrow{p} & P(H) \times P(H) & \xleftarrow{p} & S^\infty \times S^\infty_{\tau'} \\ \uparrow & & & & \uparrow \\ M_\tau & & & & M_{\tau'} \end{array}$$

Suppose there exists an analytic isomorphism $l\colon M_\tau \to M_{\tau'}$. By construction we can find a point $x \in P(H) \times P(H)$ such that $p^{-1}(x) \subset M_{\tau'}$. Looking at the diagram

$$\begin{array}{ccc} S^\infty \times S^\infty_{\tau'} & \xrightarrow{p} & P(H) \times P(H) \\ \cup & & \\ M_{\tau'} & & p^{-1}(x) \\ {\scriptstyle l} \nearrow & & \parallel \\ M_\tau & \longleftarrow & S^1 \times S^1_\tau = T_\tau \end{array}$$

we notice that if $p \cdot l(T_\tau)$ is just one point denoted by y, then l maps T_τ in $T_{\tau'} = p^{-1}_{(y)}$ holomorphically and injectively, and it follows $l\colon T_\tau \to T_{\tau'}$ is an analytic isomorphism since T_τ and $T_{\tau'}$ are 1-(complex)-dimensional analytic manifolds. Thus $\tau = \dfrac{a_{11}\tau' + a_{12}}{a_{21}\tau' + a_{22}}$ with integers a_{ij} and $\det |a_{ij}| = \pm 1$.

It remains to prove $p \cdot l(T_\tau) =$ one point. Notice that $(p \cdot l)_*\colon H_2(T_\tau) \to H_2(PH \times PH)$, where $H_2(\ )$ denotes the second group of homology with integral coefficients, is the zero homomorphism, since $p \cdot l$ factorizes by $S^\infty \times S^\infty$ (contractible).

Since $p \cdot l(T_\tau)$ is a compact set, if we denote by $\{U_k\}$ the canonical charts on $P(H)$ with respect to a given basis $e_1, e_2 \cdots$, i.e., if $U_k = P(H) \backslash P(H^\perp_{e_k})$, then there exists N such that $p \cdot l(T_\tau) \subset \bigcup_{i,j \le N} \{U_i \times U_j\}$; but as one can easily see $\bigcup_{i,j \le N} \{U_i \times U_j\} = E_N \times E_N$ where E_N is the total space of an analytic fibre bundle over $P(C^{N-1})$ with fibre $H^\perp_{\{e_1, \cdots, e_N\}}$ to be orthogonal complement of the space generated by $e_1 \cdots e_N$. Hence $p \cdot l$ can be factorized by $E_N \times E_N$ as is indicated by the following diagram:

$$\begin{array}{ccc} & P(H) \times P(H) & \\ {}^{p\cdot l}\nearrow & \cup \quad \cup & \\ T_\tau \xrightarrow{p\cdot l} & E_N \times E_N & \xrightarrow{p'} P(C^{N-1}) \times P(C^{N-1}) \end{array}$$

Since $E_N \times E_N \subset P(H) \times P(H)$ induces an isomorphism for homology up to dimension $(N-1)$, $p' \cdot (p \cdot l)$ is an analytic map such that $(p' \cdot (p \cdot l))_*: H_2(T_\tau) \to H_2(P(C^{N-1}) \times P(C^{N-1}))$ is zero, and hence $(p_i \cdot p' \cdot (p \cdot l))_*: H_2(T_\tau) \to H_2(P(C^{N-1}))$ is zero where p_i, $i = 1, 2$, are the projections of $P(C^{N-1}) \times P(C^{N-1})$ onto its factors.

Because of the "intersection theory", $p_i p' pl(T_\tau)$ is a discrete set; otherwise $p_i p'(p \cdot l)_*(H_2(T_\tau)) \neq 0$. Thus $p'(p \cdot l)(T_\tau)$ is a discrete set and is just one point $y \in P(C^{N-1}) \times P(C^{N-1})$ because of the connectivity of T_τ. It therefore follows that $p \cdot l(T_\tau)$ lies in the fibre of p' over y. Since T_τ is compact, all holomorphic functions are constant, and hence $p \cdot l(T_\tau)$ is a point.

Corollary 4.2. *There exist infinitely many complex analytic structures on any Hilbert manifold, which have no nonconstant holomorphic functions.*

Remark 4.3. $M_\tau \times H^C$ analytically isomorphic to $M_{\tau'} \times H^C$ implies M_τ analytically isomorphic to $M_{\tau'}$.

Proof. Suppose $l: M_\tau \times H \to M_{\tau'} \times H$ is an analytic isomorphism, and p_2 is the projection of $M_{\tau'} \times H \to H$. Since M_τ has no holomorphic functions, $M_\tau \times H \xrightarrow{p_2 \cdot l} H$ is factorized by p_2, and therefore $p_2 \cdot l = r \cdot p_2$. Hence we get the commutative diagram

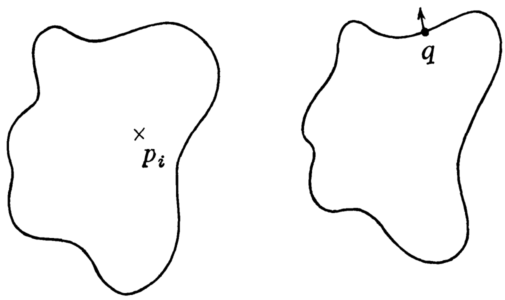
$$\begin{array}{ccc} M_\tau \times H & \xrightarrow{l} & M_{\tau'} \times H \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_2} \\ H & \xrightarrow{r} & H \end{array}$$

which implies that M_τ is analytically isomorphic to $M_{\tau'}$.

5.

Proposition 5.1. *Let $(M, \partial M)$ be a closed differentiable manifold in H^C such that $M \backslash \partial M$ is an open set in H^C (∂M is a closed differentiable submanifold of H^C), and let $p_1, \cdots, p_n$ be different points in $M \backslash \partial M$. Then $M \backslash \{\partial M \cup p_1 \cup \cdots \cup p_{n-1}\}$ is analytically nonequivalent to $M \backslash \{\partial M \cup p_1 \cup \cdots \cup p_n\}$.*

Proof. Suppose there exists an analytic isomorphism $l: M \backslash \{\partial M \cup p_1 \cup \cdots \cup p_n\} \to M \backslash \{\partial M \cup p_1 \cup \cdots \cup p_{n-1}\} \subset H$. Then applying Hartogs' theorem we can extend l to $\tilde{l} M \to H$. We first notice that either $l(p_i)$ becomes p_j or $l(p_i) \in \partial M$ since $\tilde{l}$ is continuous, and then show the latter case not to be possible.



Suppose $q = \tilde{l}(p_i) \in \partial M$. Let us take the normal vector $\vec{v}$ at q (outside the manifold) and consider $\varepsilon\vec{v}$ such that $|\varepsilon\vec{v}| \cap M = q$, where $|\varepsilon\vec{v}|$ denotes all points $t\vec{v}, t \in R_+$, $0 < t < \varepsilon$. Consider the origin q of the complex Hilbert space and the complex line $\{\vec{v}\}$ generated by v. There exists at least one complex line $\{t\}$ passing through p_i such that $s = pr_v \cdot \tilde{l} \mid \{t\}$ is a nonconstant holomorphic function. Consider a small open disc D in $\{v\}$ and $s^{-1}(D) \subset \{t\}$. Since s is nonconstant and holomorphic, it has to be open, i.e., $s(s^{-1}(D))$ has to be an open set. But as we can see, $s(s^{-1}(D))$ is not open. The origin q of $\{\vec{v}\}$ belongs to $s(s^{-1}(D))$, but neighborhoods of q contained in $s(s^{-1}(D))$ do *not* exist. Thus $\tilde{l}(p_i) \notin \partial M$, and hence $\tilde{l}(p_i) = p_j$, so that there exist at least two points p_{i_1} and p_{i_2} such that $\tilde{l}(p_{i_1}) = \tilde{l}(p_{i_2})$. Choose again a complex line $\{t\}$ passing through p_j. Since $\tilde{l}$ is nonconstant for any points p_{i_1} and p_{i_2}, we can get a line $\{t_1\}$ and $\{t_2\}$ such that its projection on $\{t\}$ composed by $\tilde{l}$ is holomorphic and nonconstant and is therefore open. This implies that there exist $x_1 \in \{t_1\} \cap (M \backslash \partial M \cup p_1 \cup \cdots \cup p_{n-1})$ and $x_2 \in \{t_2\} \cap (M \backslash \partial M \cup p_1 \cup \cdots \cup p_{n-1})$ such that $l(x_1) = \tilde{l}(x_1) = \tilde{l}(x_2) = l(x_2)$. But this is impossible because l is injective.

Corollary 5.2. *For any given Hilbert manifold M, there exist infinitely many different complex analytic structures with many holomorphic functions (i.e., given two points $x, y \in M$, there exists a holomorphic function f such that $f(x) \neq f(y)$).*

Moreover, we can construct infinitely many different structures with many holomorphic functions, which have nonconstant bounded holomorphic functions, and infinitely many different analytic structures which have no nonconstant bounded holomorphic functions.

Proof. (a) Start with M and imbed M closely in H. Take a closed tubular neighborhood of M, denote it by T, and notice that $(T, \partial T) \subset H^C$ is a closed differentiable submanifold and that $T \backslash \partial T = \mathring{T}$ is open set homotopy equivalent to M and therefore is diffeomorphic to M according to Theorem 1.3. If H is the complex Hilbert space $\mathring{T}$, then $\mathring{T} \backslash p_1, \mathring{T} \backslash \{p_1 \cup p_2\}, \mathring{T} \backslash \{p_1, p_2, p_3\}, \cdots$ are complex analytic manifolds with the induced structure which has sufficient holomorphic functions, all of which are diffeomorphic to M by Theorem 1.3 and analytically nonequivalent by Proposition 5.1.

(b) If we apply Proposition 1.5 to $U = \operatorname{Int} D^\infty(1)$, we will get an $\mathscr{L}$ and $((H\backslash \operatorname{Int} L), (\partial(H\backslash \operatorname{Int} L))$ is a closed differentiable manifold with $M = H\backslash L$. With the induced complex structure, the manifolds $M, M\backslash\{p_1\}, \cdots, M\backslash\{p_1, \cdots, p_k\}, \cdots$ are different complex manifolds according to Proposition 5.1, but all are diffeomorphic. Moreover, any holomorphic function $f\colon M\backslash\{p_1, \cdots, p_n\}$ can be extended to H because of Hartogs' theorem, and if f is nonconstant, then it cannot be bounded. In fact, if f is bounded, Hartogs' extension is also bounded and then is constant according to Liouville's theorem.

We can construct M as an open and bounded set in H, whose boundary is a differentiable manifold by Proposition 1.4, and then we get a structure which has of course, sufficiently many, bounded nonconstant holomorphic functions.

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