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SELF-LINKING AND THE DIRECTED SECANT SPAN OF A DIFFERENTIABLE MANIFOLD

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1. Introduction

In [2], W. Pohl introduced the concept of the self-linking number of a closed space curve. In [5], the present author generalized the self-linking number to closed oriented n-manifolds in Euclidean $(2n+1)$-space. In this paper we give a new geometric interpretation of this self-linking number by reinterpreting a formula given by Gauss, Pontryagin [3] among others, for the linking number of two manifolds in Euclidean spaces (Proposition 1) and by investigating it in two cases, the first when the two manifolds are the same, the second when one manifold is a submanifold of the other. We will show that, in the case where the two manifolds are the same, the self-linking number is equal to the difference of two intersection numbers. The first is the algebraic number of directed secants through a generic point p of the Euclidean space into which the manifold is imbedded; a directed secant is an infinite half line which begins at one point of the imbedding and passes through another. The second is the algebraic number of infinite half-$(n+1)$-planes which pass through the point p, the half planes being spanned by the full tangent plane and the forward mean curvature vector at each point of the imbedding. This result is then generalized to the second case when one manifold is a submanifold of the other. In both cases we rely heavily on work done in [2], [5], and [6]. The theorem is discussed at some length for curves in three-space and some geometric consequences are noted.

Finally, in the appendix we present a different approach to the main theorem for the case of curves in three-space, an approach more in the spirit of [2]. We prove that under suitable conditions the self-linking number of a closed space curve is equal to one-half the algebraic number of secants through the origin minus the total turning of the projection of the position vector of the curve into the normal plane about the principal normal. In this case a secant is the *full* infinite line through two points of the imbedded curve.

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2. Linking of manifolds

We begin by recalling some well-known facts about the linking of manifolds. Let M^n and K^l be two closed smooth oriented manifolds of dimensions n and l, f and h be C^1 maps of them into oriented Euclidean space E^{n+l+1} such that $f(M^n)$ and $h(K^l)$ do not intersect, and S^{n+l} be the unit $(n+l)$-sphere centered at the origin of E^{n+l+1}. Consider the cartesian product $M^n \times K^l$ given the canonical orientation, and define a map

$$e\colon M^n \times K^l \to S^{n+l}$$

by associating with each $(m, k) \in M^n \times K^l$ the unit vector in E^{n+l+1}:

$$e(m, k) = \frac{h(k) - f(m)}{|h(k) - f(m)|} \,. \tag{1}$$

The degree of this map is the linking number $L(f, h)$, [3]. Let $d0_{n+l}$ be the pull-back of the volume element of the $(n+l)$-sphere under the map e. Then clearly

$$L(f, h) = \frac{1}{0_{n+l}} \int_{M^n \times K^l} d0_{n+l} \,,$$

where 0_{n+l} is the volume of the $(n+l)$-sphere.

We next interpret $L(f, h)$ in a different manner which will be of prime use in what follows. Consider the product $M \times K \times L$, where $L = [0, a]$ is a closed interval of real numbers, and M and K are as above. Then $M \times K \times L$ is a manifold with boundary components $M \times K \times \{0\}$ and $M \times K \times \{a\}$. We define a map $g\colon M \times K \times L \to E^{n+l+1}$ by

$$g(m, k, s) = f(m) + s\left(\frac{h(k) - f(m)}{|h(k) - f(m)|}\right) \,,$$

for $(m, k, s) \in M \times K \times L$. Let p be an arbitrary point of E^{n+l+1} such that p is not a singular value of the map g and is not in the locus $g(\partial(M \times K \times L))$. Let

$$I = \{(m, k, s) \in M \times K \times L \,|\, g(m, k, s) = p\} \,.$$

Because of our assumption on p and the compactness of $M \times K \times L, I$ will be finite in number. Finally, we define a map $e'\colon p \times M \times K \times L - I \to S^{n+l}$ by

$$e'(p, m, k, s) = \frac{g(m, k, s) - p}{|g(m, k, s) - p|} \,.$$

Let $d0_{n+l}$ be the pull-back of the volume element of S^{n+l} under the map e'. In [6], using Stokes' Theorem, we proved

$$\frac{1}{0_{n+l}} \int_{p\times\partial(M\times K\times L)} d0_{n+l} = I(g, p) ,$$

where $I(g, p)$ is the algebraic number of intersections of $g(M \times K \times L)$ with the point p, or the sum of the indices of the intersections of $g(M \times K \times L)$ with p. In this case we obtain

$$(2) \quad \frac{1}{0_{n+l}} \int_{p\times M\times K\times\{a\}} d0_{n+l} - \frac{1}{0_{n+l}} \int_{p\times M\times K\times\{0\}} d0_{n+l} = (-1)^{n+l} I(g, p) ,$$

the signs coming from the induced orientation from Stokes' Theorem. We first observe that the second integral on the left hand side is zero, for e' restricted to $p \times M \times K \times \{0\}$ gives

$$e'(p, m, k, 0) = \frac{f(m) - p}{|f(m) - p|} ,$$

and hence the image of $p \times M \times K \times \{0\}$ has dimension less than $n + l$. Thus we obtain

$$\frac{1}{0_{n+1}} \int_{p\times M\times K\times\{a\}} d0_{n+l} = (-1)^{n+l} I(g, p) .$$

As with our previous work [6], the main interest lies in the case where a goes to infinity. Let us examine this case more closely. The map e' on $p \times M \times K \times \{a\}$ is

$$e'(p, m, k, a) = \frac{[f(m) - p]\,|h(k) - f(m)| + a\,[h(k) - f(m)]}{|[f(m) - p]\,|h(k) - f(m)| + a\,[h(k) - f(m)]|} .$$

Clearly, as $a \to \infty$, this map becomes

$$e'(p, m, k, \infty) = \frac{h(k) - f(m)}{|h(k) - f(m)|} ,$$

which is the same as the map e on $M \times K$ in equation (1). Thus the integral

$$\frac{1}{0_{n+l}} \int_{p\times M\times K\times\{\infty\}} d0_{n+l}$$

is really just the same as the earlier integral

$$\frac{1}{0_{n+l}} \int_{M\times K} d0_{n+l} .$$

We define the image of $M \times K \times L$ under g, where $L = [0, \infty]$, to be the *directed secant span from* M *to* K (or perhaps more precisely, from $f(M)$ to $h(K)$). We obtain thereby

Proposition 1. *$L(f, h) = (-1)^{n+l} I(g, p)$, where $L(f, h)$ is the linking number of $f(M)$ with $h(K)$, and $I(g,p)$ is the algebraic intersection number of the directed secant span from M to K with p.*

It will be our purpose to use Proposition 1 to give new interpretations of the self-linking number of a differentiable manifold.

3. Extension to secant spaces

Let N^q be a compact orientable differentiable manifold of dimension q possibly with boundary, and M^n a closed submanifold of dimension $n \leq q$ with no boundary. We shall make use of the abstract space $S(M, N)$ of secants of N relative to M canonically defined in [5]. In fact, $S(M, N)$ essentally is a differentiable manifold whose interior is $M \times N - D_M$, where $D_M = \{(m, n) \in M \times N \mid n = m\}$ and whose boundary consists of $M \times \partial N$ and $T(N)_M$ which is the restriction to M of the space of oriented tangent directions of N. We shall understand that if M is a part of the boundary of N, then $T(N)_M$ consists of $T(M)$ and the tangent directions of N pointing to the interior of N from M.

Consider the product manifold $S(M, N) \times L$, where $L = [0, a]$ is a closed interval of real numbers. Then $S(M, N)$ is a manifold with boundary components $M \times \partial N \times L, T(N)_M \times L, S(M, N) \times \{0\}, S(M, N) \times \{a\}$.

Let F be a C^3 map of N into oriented $(n + q + 1)$-space such that F is a C^3 imbedding in a neighborhood of M, and let f denote the restriction of F to M. We define, as in § 2, a map g: $S(M, N) \times L \to E^{n+q+1}$ by

$$g(m, n, s) = f(m) + s\left(\frac{F(n) - f(m)}{|F(n) - f(m)|}\right)$$

for $(m, n, s) \in (M \times N - D_M) \times L$, and define

$$g(t_m, s) = f(m) + s\left(\frac{F_*(t_m)_{f(m)}}{|F_*(t_m)_{f(m)}|}\right) \tag{3}$$

for $t_m \in T(N)_M$ and $(t_m, s) \in T(N)_M \times L$, where F_* is the induced map on the tangent space. By abuse of notation and where the meaning should be clear, we will write equation (3) as

$$g(t, s) = f(m) + st$$

for $(t, s) \in T(N)_M \times L$, the t on the right-hand side being the "realized" unit tangent vector, as it were. That g is a differentiable map follows from arguments about related maps in [2] and [5].

Let p be an arbitrary point of E^{n+q+1} such that p is not a singular value of the map g and is not in $g(\partial(S(M, N) \times L))$. Let $I = \{(m, n, s) \in S(M, N) \times L \mid g(m, n, s) = p\}$. As before, because of our assumption on p and the compactness of $S(M, N) \times L, I$ will be finite in number. Continuing, we define a map $e\colon p \times S(M, N) \times L - I \to S^{n+q}$ by

$$e(p, m, n, s) = \frac{g(m, n, s) - p}{|g(m, n, s) - p|}$$

for $(m, n, s) \in (M \times N - D_M) \times L$, and

$$e(p, t, s) = \frac{g(t, s) - p}{|g(t, s) - p|} ,$$

for $(t, s) \in T(N)_M \times L$ (where we have made use of the abuse of language mentioned above). Let $d0_{n+q}$ be the pull-back of the volume element of S^{n+q} under the map e. Then we use the main result, the so-called equation (E), of [6] to conclude

$$\frac{1}{0_{n+q}} \int_{p \times \partial(S(M,N) \times L)} d0_{n+q} = I(g, p) ,$$

where $I(g, p)$ is the algebraic number of intersections of $g(S(M, N) \times L)$ with p. This gives

$$(4) \quad \begin{aligned} &-\frac{1}{0_{n+q}} \int_{p \times S(M,N) \times \{0\}} d0_{n+q} + \frac{1}{0_{n+q}} \int_{p \times S(M,N) \times \{a\}} d0_{n+q} \\ &- \frac{1}{0_{n+q}} \int_{p \times M \times \partial N \times L} d0_{n+q} - \frac{1}{0_{n+q}} \int_{p \times T(N)_M \times L} d0_{n+q} = (-1)^{n+q} I(g, p) , \end{aligned}$$

the signs of the above terms coming from the induced orientations from the use of Stokes' Theorem.

The first integral is zero for essentially the same reason as the second integral in equation (2). As before, our main interest in equation (4) occurs when $a \to \infty$. We now investigate the integral

$$\frac{1}{0_{n+q}} \int_{p \times S(M,N) \times \{\infty\}} d0_{n+q} .$$

The map e restricted to $p \times S(M, N) \times \{\infty\}$ is the same as the map e^* defined on $S(M, N)$ as follows:

$$e^*(m, n) = \frac{F(n) - f(m)}{|F(n) - f(m)|}$$

for $(m, n) \in M \times N - D_M$ and

$$e^*(t) = t$$

for $t \in T(N)_M$, the t on the right-hand side once again being the "realized" unit tangent vector. If we use the same notation and denote by $d0_{n+q}$ the pull-back of the volume element of S^{n+q} under e^*, then

$$\frac{1}{0_{n+q}} \int_{p \times S(M,N) \times \{\infty\}} d0_{n+q} = \frac{1}{0_{n+q}} \int_{S(M,N)} d0_{n+q} ,$$

where the integral on the right-hand side is called the Gauss integral for N relative to M. It is defined in § 8 of [5] and is written there as

$$\frac{1}{0_{n+q}} \int_{M \times N} d0_{n+q} .$$

Hereafter, we shall use this latter notation.

Finally, equation (4) then becomes

$$(5) \quad \begin{aligned} &\frac{1}{0_{n+q}} \int_{M \times N} d0_{n+q} - \frac{1}{0_{n+q}} \int_{p \times M \times \partial N \times L} d0_{n+q} \\ &\qquad - \frac{1}{0_{n+q}} \int_{p \times T(N)_M \times L} d0_{n+q} = (-1)^{n+q} I(g, p) . \end{aligned}$$

In what follows we shall consider two special cases of this equation, the first being when $N = M$, the second being when $M \subset N$ and $\partial N = \emptyset$.

4. Self-linking

In this section we give a new interpretation of the self-linking number of a differentiable manifold. If we set $N = M$, equation (5) becomes

$$\frac{1}{0_{2n}} \int_{M \times M} d0_{2n} - \frac{1}{0_{2n}} \int_{p \times T(M) \times L} d0_{2n} = I(g, p) ,$$

where $T(M)$ is the space of oriented tangent directions of M, and $I(g, p)$ is the intersection number of $g(S(M) \times L)$ with the point p. Define the image of $S(M) \times L$ under g to be the *directed secant span* of the differentiable manifold M, and call $I(g, p)$ the *algebraic number of directed secants through the point p*.

Theorem 2. *Let $I(g, p)$ be the algebraic number of directed secants through the point p. Then*

$$\frac{1}{0_{2n}} \int_{M \times M} d0_{2n} - \frac{1}{0_{2n}} \int_{p \times T(M) \times L} d0_{2n} = I(g, p) .$$

We now use Theorem 2 to tie together work done in [2], [5], and [6]. For the convenience of the reader we recall the pertinent theorems from these works. (These will be listed with capital Latin letters.)

Theorem A. *Let $f\colon M^n \to E^{2n+1}$ be a C^3 imbedding of a closed oriented differentiable manifold into Euclidean $(2n+1)$-space, and v a non-vanishing unit normal vector field on M^n. If n is odd, then*

$$\frac{1}{0_{2n}}\int_{M\times M} d0_{2n} + \frac{1}{0_n}\int_M \tau_v dV = L(f, f_{\varepsilon_v}) ,$$

where $L(f, f_{\varepsilon_v})$ is the linking number of the imbedded manifold with the same manifold deformed a distance ε along the vector field v, and $\tau_v dV$ is the torsion form of the imbedded manifold with respect to the vector field v. If n is even, then

$$-\tfrac{1}{2}\chi(v^c) = L(f, f_{\varepsilon_v}) ,$$

where $\chi(v^c)$ is the Euler characteristic of the complementary (to v) oriented subbundle of the normal bundle.

Theorem B. *If n is even, then*

$$\frac{1}{0_{2n}}\int_{M\times M} d0_{2n} = 0 ,$$

that is, the Gauss integral for M is zero.

It is also shown in [5] that $L(f, f_{\varepsilon_v})$ is the algebraic number of forward cross normals, which is the algebraic number of times forward lines along the vector field v intersect the imbedding $f(M)$.

Theorem C. *$L(f, f_{\varepsilon_v})$ is the algebraic number of forward cross-normals.*

If v is chosen to be along the mean curvature vector field, then $L(f, f_{\varepsilon_v})$ is called the self-linking number of the manifold M.

Combining Theorems 2, A and B, we have

Lemma 3. *If n is odd, then*

$$-\frac{1}{0_{2n}}\int_{p\times T(M)\times L} d0_{2n} - \frac{1}{0_n}\int_M \tau_v dV = I(g, p) - L(f, f_{\varepsilon_v}) .$$

Lemma 4. *If n is even, then*

$$-\frac{1}{0_{2n}}\int_{p\times T(M)\times L} d0_{2n} = I(g, p) ,$$

and

$$-\frac{1}{0_{2n}}\int_{p\times T(M)\times L} d0_{2n} + \tfrac{1}{2}\chi(v^c) = I(g, p) - L(f, f_{\varepsilon_v}) .$$

We now quote a theorem from [6] which gives a different result for the left-hand sides of the equation in Lemma 3 and the second equation in Lemma 4. First, we observe that the image of $T(M) \times L$ under the map g is really just the same as the span of all the full tangent planes of the imbedding $f(M)$. The theorem from [6] concerns precisely the same image. For each point $m \in M$, consider the half $(n+1)$-plane spanned by the tangent plane of M at $f(m)$ and the forward unit normal vector v of M at $f(m)$. We count the algebraic intersection number of these half $(n+1)$-planes with the point p and call it $I(v, p)$. Theorem 5 of [6] states the following

Theorem D. *If n is odd, then*

$$-\frac{1}{0_{2n}}\int_{p\times T(M)\times L} d0_{2n} - \frac{1}{0_n}\int_M \tau_v dV = I(v,p) .$$

If n is even, then

$$-\frac{1}{0_{2n}}\int_{p\times T(M)\times L} d0_{2n} + \tfrac{1}{2}\chi(v^c) = I(v,p) .$$

(We note for the sake of the reader that the notation in [6] is substantially different.)

Combining Lemmas 3 and 4 with Theorem D, we obtain

Theorem 5. $L(f, f_{\epsilon_v}) = I(g, p) - I(v, p)$.

From Theorem C, we therefore obtain

Corollary 6. *The algebraic number of forward cross normals* $= I(g,p) - I(v,p)$.

Corollary 7. *If v is along the mean curvature vector, then*

$$SL = I(g,p) - I(v,p) ,$$

where SL is the self-linking number of the differentiable manifold M.

To get a further understanding of Corollary 7, let us examine it more closely for the case of curves in three-space. Let $f \colon M \to E^3$ be a C^3 imbedding of a curve in three-space for which the curvature never vanishes. In this case the vector field v will be the principal normal vector field. What the corollary states is that the self-linking number of the curve is equal to the algebraic number of directed secants through a generic point p minus the algebraic number of infinite half osculating planes passing through p. Hence, for example, if no osculating planes pass through a point p, then the self-linking number is equal to the algebraic number of directed secants through p.

Suppose, further, that we let the point p go to infinity along a certain fixed direction. Then, the corollary implies that if no infinite half osculating planes pass through this point at infinity (which will occur, for example, if no osculating plane is parallel to the given fixed direction), then the self-linking number of the curve is the algebraic number of directed secants along the given direction. One may state, for example,

Corollary 8. *If all the binormals of a curve make an angle less than $\pi/2$ with a fixed direction, then the self-linking number of the curve is the algebraic number of directed secants along the given direction.*

The condition of Corollary 8 is satisfied, for example, when the binormal indicatrix is contained in a hemisphere.

As an interesting sidelight we mention that such curves have been studied before by B. Segre [4] and W. Fenchel [1], the former giving an estimate for the total absolute torsion of a curve whose binormal indicatrix is contained in a hemisphere. In fact, if B is the binormal indicatrix and δ is the spherical diameter of the circumscribed circle of B, then

$$\int_M |\tau|\, ds > 2\delta \; .$$

The self-linking number, however, is related to the total torsion

$$\int_M \tau ds \; .$$

Continuing our line of thought, suppose that we have a curve immersed in the xy-plane such as that in Fig. 1A with the binormal pointing along the positive z-axis. By a slight deformation lift branch 1 away from branch 2 out of the plane in such a way that the curvature never vanishes (Fig. 1B). For the direction in Corollary 8 choose the z-axis in its positive sense. If the deformation is small enough, then it is clear that the binormals will make an

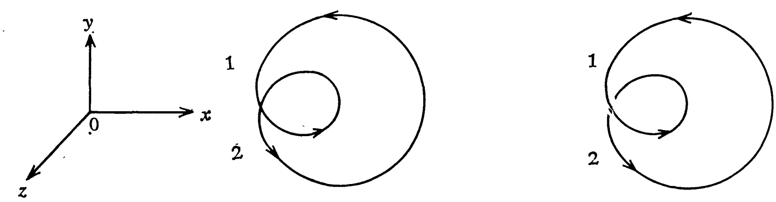

Fig. 1A Fig. 1B

angle less than $\pi/2$ with the z-axis. An easy observation shows that there is only one directed secant along the positive z-axis, and hence by Corollary 8 the self-linking number is ± 1. This method can be used for constructing a curve of arbitrary self-linking number. There is a similar method described by W. Pohl in [2] for computing the self-linking number of a closed space curve.

5. Linking of submanifolds

In this section we assume that N is an oriented closed differentiable manifold without boundary and that M is a closed submanifold of N also without boundary. Then equation (5) becomes

$$(6) \qquad \frac{1}{0_{n+q}}\int_{M\times N} d0_{n+q} - \frac{1}{0_{n+q}}\int_{p\times T(N)_M\times L} d0_{n+q} = (-1)^{n+q}I(g,p) .$$

Our analysis in this section is quite similar to that of § 4 in that we wish to use equation (6) to tie together results in [5] and [6].

Suppose the map F is a C^3 imbedding of N into Euclidean $(n+q+1)$-space, and there exists on N a non-vanishing unit normal vector field v. In [5, Theorem 9], we proved

Theorem E. *If n is odd, then*

$$\frac{1}{0_{n+q}}\int_{M\times N} d0_{n+q} + \frac{1}{0_n}\int_M (\tau_M)_v dV = L(f, F_{\varepsilon_v}) ,$$

where f is the restriction of F to M, $(\tau_M)_v dV$ is the torsion form of the imbedded manifold $F(N)$ with respect to the imbedded submanifold $f(M)$, and $L(f, F_{\varepsilon_v})$ is the linking number of the imbedded submanifold $f(M)$ with $F(N)$ deformed a small distance ε along v.

If n is even, then

$$\frac{1}{0_{n+q}}\int_{M\times N} d0_{n+q} - \tfrac{1}{2}\chi(v^c)[M] = L(f, F_{\varepsilon_v}) ,$$

where $\chi(v^c)[M]$ is the Euler class of the complementary (to v) subbundle of the normal bundle of N evaluated on the fundamental class of M.

Combining equation (6) with Theorem E, we have

Lemma 9. *If n is odd, then*

$$-\frac{1}{0_{n+q}}\int_{p\times T(N)_M\times L} d0_{n+q} - \frac{1}{0_n}\int_M (\tau_M)_v dV = (-1)^{n+q}I(g,p) - L(f, F_{\varepsilon_v}) .$$

Lemma 10. *If n is even, then*

$$-\frac{1}{0_{n+q}}\int_{p\times T(N)_M\times L} d0_{n+q} + \tfrac{1}{2}\chi(v^c)[M] = (-1)^{n+q}I(g,p) - L(f, F_{\varepsilon_v}) .$$

The result we quote from [6] concerns the left-hand sides of these two above equations. We observe, again, that the image of $T(N)_M \times L$ under the map g is the same as the span of all the full tangent planes of $F(N)$ restricted to

$f(M)$. For each point $m \in M$, consider the half $(q+1)$-plane spanned by the tangent plane to N at $f(m)$ and the forward unit normal vector v of N at $f(m)$. We count the algebraic intersection number of these $(q+1)$-half planes with the point p and call it $I_M(v,p)$. Theorem 5 of [6] states

Theorem F. *If n is odd, then*

$$-\frac{1}{0_{n+q}}\int_{p\times T(N)_M\times L} d0_{n+q} - \frac{1}{0_n}\int_M (\tau_M)_v dV = I_M(v,p)\ .$$

If n is even, then

$$-\frac{1}{0_{n+q}}\int_{p\times T(N)_M\times L} d0_{n+q} + \tfrac{1}{2}\chi(v^c)[M] = I_M(v,p)\ .$$

Finally, combining Lemmas 9 and 10 with Theorem F we obtain

Theorem 11. $L(f, F_{\varepsilon_v}) = (-1)^{n+q} I(g,p) - I_M(v,p)$.

Appendix. In this appendix we briefly outline a different approach to the above investigations for curves in three-space. This is done more in the spirit of W. Pohl in [2] and gives a slightly different result for the self-linking number. For the details of the kind of analysis which follows, the reader is referred to [2].

Let $f\colon M \to E^3$ be a C^3 imbedding of a closed curve in three-space with non-vanishing curvature, and $S(M)$ be the abstract space of secants of M, i.e., $S(M) = M \times M - D \cup T(M)$, where D is the diagonal and $T(M)$ is the space of oriented tangent directions of M. We define a map $e_1\colon S(M) \to S^2$, S^2 being the unit 2-sphere in E^3, by

$$e_1(x,y) = \frac{f(y)-f(x)}{|f(y)-f(x)|}$$

for $(x,y) \in M \times M - D$, and

$$e_1(t) = t$$

for $t \in T(M)$, where the right-hand side is the "realized" unit tangent vector.

Let p be an arbitrary point of E^3 such that no tangent line of $f(M)$ passes through p. With each $(x,y) \in M \times M - D$ such that the secant line $f(x)f(y)$ does not pass through p, we associate $e_2(x,y)$, the unit vector in the plane spanned by secant line $f(x)f(y)$ and the line $pf(x)$, perpendicular to e_1 and so oriented that e_1e_2 agrees with the orientation e_1a where a is a unit vector directed from p to $f(x)$. The vector function e_2 extends smoothly to the boundary $T(M)$ and gives there a unit vector along the projection of the line $pf(x)$ into the normal plane at $f(x)$. We observe that e_2 is not defined when the secant line passes through p, and call such a secant a cross-secant. We note that p may be chosen in such a way that the number of cross-secants are finite. Finally, set $e_3 = e_1 \times e_2$.

Let $d0_2$ be the pull-back of the volume element of S^2 under the map e_1 If we set $de_i \cdot e_j = \omega_{ij}$, then $d0_2$ may be written as $\omega_{12} \wedge \omega_{13} = -d\omega_{23}$, where d denotes the exterior derivative. An analysis similar to that of [2] (the reader is referred there for details) gives, by a use of Stokes' Theorem,

$$(7) \qquad \frac{1}{4\pi}\int_{M\times M} d0_2 + \frac{1}{2\pi}\int_M \omega_{23} = \tfrac{1}{2}I ,$$

where the first integral is the Gauss integral for M (cf. § 4) and I is the algebraic number of cross-secants (or secants through p). We observe without proof that $\frac{1}{2}I$ is an integer. This is essentially due to the fact that in the analysis here each cross-secant is counted twice, once for (x, y), and once for (y, x).

We examine now the ω_{23} of the second integral. Recall that on M, e_2 is a unit vector along the projection of the line $pf(x)$ onto the normal plane at $f(x)$. Hence we may write, e_1 being along the unit tangent vector,

$$e_2 = \cos\theta N + \sin\theta B ,$$
$$e_3 = -\sin\theta N + \cos\theta B ,$$

where N and B are the principal normal and binormal of the curve at $f(x)$ and θ is the angle e_2 makes with N. A direct computation shows

$$\omega_{23} = \tau ds + d\theta ,$$

where τ is the torsion and ds the arc-element. Hence

$$\frac{1}{2\pi}\int_M \omega_{23} = \frac{1}{2\pi}\int_M \tau ds + \frac{1}{2\pi}\int_M d\theta .$$

Combining this equation with equation (7) we obtain

$$\frac{1}{4\pi}\int_{M\times M} d0_2 + \frac{1}{2\pi}\int_M \tau ds = \tfrac{1}{2}I - \frac{1}{2\pi}\int_M d\theta .$$

But the left-hand side is just the self-linking number of the curve, so that

$$SL = \tfrac{1}{2}I - \frac{1}{2\pi}\int_M d\theta .$$

We may state this in the following way: Suppose without loss of generality that p is the origin of coordinates. Then we have shown

Theorem 12. *The self-linking number of a closed space curve is equal to* *one-half the algebraic number of secants through the origin minus the total turning of the projection of the position vector into the normal plane about the principal normal.*

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