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METRICS AND ISOMETRIC EMBEDDINGS OF THE 2-SPHERE

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Since any compact C^2 two-dimensional submanifold of euclidean 3-space R^3 must have positive Gaussian curvature at some point, it follows that the 2-torus with flat metric and the compact orientable 2-manifolds of genus greater than 1 with metrics of everywhere negative curvature have no C^2 isometric embeddings in R^3. Of course, compact non-orientable 2-manifolds cannot be embedded in R^3 for topological reasons. A manifold of dimension $d > 2$ always admits a metric for which there is no isometric embedding in R^{d+1}: There certainly exists a metric such that all sectional curvatures are negative at some point of the manifold, by a standard extension argument for Riemannian metrics defined in a neighborhood of a point. There exists no C^2 isometric embedding in R^{d+1} for such a metric and in fact no C^2 isometric embedding in R^{d+1} of any neighborhood of a point of a d-dimensional manifold, $d > 2$, where all sectional curvatures are negative, as the expression for the sectional curvature of hypersurfaces in terms of the eigenvalues of the second fundamental form shows immediately.

The reasoning used in the cases already discussed fails to apply to any metric on the 2-sphere S^2, since $d = 2$ and the Gauss-Bonnet theorem guarantees at least one point of positive curvature for any given C^2 metric on the sphere. The purpose of this article is to exhibit a C^∞ metric on S^2 for which there is no C^2 isometric embedding in R^3. The proof of the non-existence of a C^2 embedding of S^2 in R^3 isometric for this metric is based on the analysis of the structure of flat submanifolds of R^3 given in Hartman and Nirenberg [1] (see also Massey [2]).

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1. The following results on flat submanifolds of R^3 will be used to show that the metric on S^2 constructed in § 2 has no C^2 isometric embedding in R^3.

Lemma 1 (*Hartman-Nirenberg*). *Let X be a C^2 surface with zero Gaussian curvature in R^3 with simple, nonsingular projection $P|X\colon X \to D_1$ onto a connected open set D_1 in the xy-plane, where $P : R^3 \to R^2$ is the canonical orthogonal*

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projection of R^3 *onto the xy-plane. Let* $x_0 \in D$ *and let* $C(x_0)$ *be the arc-component containing* x_0 *of the set of all points* x *of* D *such that the normal to* X *at* $X \cap P^{-1}(x)$ *is equal to the normal to* X *at* $X \cap P^{-1}(x_0)$. *Then the boundary of* $C(x_0)$ *in the xy-plane is the union of a subset of the boundary of* D_1 *and straight line segments whose and points lie on the boundary of* D_1, *and these straight line segments are disjoint except for (possibly) having common end points.*

Proof. This result is essentially a restatement of Part a), Theorem A, § 9, Hartman-Nirenberg [1]. The fact that the straight line segments do not intersect except perhaps for end points in common follows from Corollary 2 of Lemma 2, § 3, Hartman-Nirenberg [1].

In the notation of Lemma 1, let L be any line segment in $C(x_0)$ (not necessarily a boundary segment). Then $P^{-1}(L)$ is a plane, and, moreover, the normal to the nonsingular plane curve $X \cap P^{-1}(L)$ in the plane $P^{-1}(L)$ is the (normalized) orthogonal projection of the surface normal to X onto the plane $P^{-1}(L)$. Since the surface normal along $X \cap P^{-1}(L)$ is constant by assumption, it follows that $X \cap P^{-1}(L)$ is a nonsingular plane curve with constant planar normal and hence that $X \cap P^{-1}(L)$ is a straight line in $P^{-1}(L)$ and so in R^3. It is now clear that the assumption of the existence of a global simple nonsingular projection on the xy-plane can be dropped in Lemma 1, and we obtain the following version, independent of coordinate projections.

Lemma 2. *Let* V *be an open set in* R^2 *and* $f: V \to R^3$ *be a* C^2 *isometric embedding. Let* $x_0 \in V$ *and* $C(x_0)$ *be the arc-component containing* x_0 *of the set of all* x *in* V *such that the surface normal in* R^3 *to* $f(V)$ *at* $f(x)$ *is equal to the surface normal to* $f(V)$ *at* $f(x_0)$. *Then the boundary of* $C(x_0)$ *in* R^2 *is the union of a subset of the boundary of* V *and straight line segments whose end points lie on the boundary of* V, *and these straight line segments are disjoint except for (possibly) having common end points.*

Proof. This Lemma is an immediate consequence of the previous discussion, together with the facts that the inverse under f of a straight line segment of R^3 contained in $f(V)$ is a straight line segment in V and that $f(V)$ has locally a nonsingular projection on some coordinate plane.

Proposition. *Let* D *denote the closed unit disc in* R^2 *with the usual metric, and* $f: V \to R^3$ *be a* C^2 *isometric embedding of an open set* V *in* R^2 *with* $D \subset V$. *Then there exist points* p_1, p_2 *of* ∂D, *the boundary of* D *in* V, *such that the distance in* R^3 *from* $f(p_1)$ *to* $f(p_2)$ *is at least* $\sqrt{3}$.

Proof. We shall apply Lemma 2 with $x_0 =$ the center of the unit disc D. We consider two cases:

1) Suppose x_0 is a boundary point of $C(x_0)$ in V. Then, by Lemma 2, there exists a straight line segment $L \subset V$ with end points on the boundary of V in R^2 such that the surface normal to $f(V)$ in R^3 along $f(L)$ is constant. We show that $f(L)$ is a straight line segment in R^3: Since L is a geodesic in V, the curvature of $f(L)$ as a space curve is equal to the normal curvature of the surface $f(V)$ along the direction of $f(L)$. But, since the (surface) normal of $f(V)$ is con-

stant along $f(L)$, this curvature is zero. Thus $f(L)$ is a space curve of zero curvature and hence is a straight line segment in R^3. It follows that the images under f of the two points of $\partial D \cap L$ are separated by distance 2 in R^3.

2) Now suppose that x_0 is in the interior of $C(x_0)$. Again by Lemma 2, the boundary of $C(x_0) \cap D$ consists of disjoint line segments extending to the boundary of D together with arcs of the unit circle. Note that, since D is compact in the open set V, these line segments cannot have even end points in common. Then (as in the argument for Theorem B, § 9, Hartman and Nirenberg [1]), one can show that $C(x_0)$ contains three rays from x_0 to the boundary ∂D of D, two of which rays make a smallest angle greater than $2\pi/3$.

Let $\mathscr{L}$ be the set of $u \in \partial D$ such that the line segment from x_0 to u does not lie entirely in $C(x_0)$. From the structure of the boundary of $C(x_0) \cap D$, it is clear that $\mathscr{L}$ is a union of open arcs of length less than π. Let A be a component arc of $\mathscr{L}$ of maximal length and let u_1, u_2 be the end points of A. Then $-A$, the arc diametrically opposite to A, must contain in its closure (in ∂D) a point, say u_3, not in $\mathscr{L}$; for otherwise A would not be of maximal length. Two of the line segment $l_1 = \overrightarrow{x_0u_1}$, $l_2 = \overrightarrow{x_0u_2}$, $l_3 = \overrightarrow{x_0u_3}$ must make a (smallest) angle of more than $2\pi/3$.

Say l_1 and l_2 make a smallest angle of more than $2\pi/3$. $f(l_1)$ and $f(l_2)$ are line segments in R^3 by the argument used in case 1); both lie in the plane perpendicular to the surface normal to $f(V)$ at $f(x_0)$, and the angle between them is equal to the angle between l_1 and l_2 since f is an isometry. Thus the end points $f(u_1)$ and $f(u_2)$ are at least $\sqrt{3}$ apart in R^3.

2. Let S denote the 2-sphere, p_1 the south pole, p_2 the north pole, H_1 and H_2 the open southern and northern hemispheres, respectively, and E the equator of S. Let Δ be a stereographic projection diffeomorphism of $S - \{p_2\}$ onto R^2, which takes the equator E of S onto the unit circle of R^2 and H_1 onto the open unit disc. Δ induces a flat C^∞ metric on $S - \{p_2\}$ from the eulidean metric on R^2; let G_1 denote this metric on $S - \{p_2\}$.

We now wish to define a C^∞ metric G on S with the following properties:

a) There is an open set $V \subset S - \{p_2\}$ with $\mathrm{Cl}\, H_1 \subset V \ni G|V = G_1|V$.

b) $\sup_{q_1, q_2 \in \mathrm{Cl}\, H_2} \mathrm{DIS}_G(q_1, q_2) \leq 1/2$,

where DIS_G denotes the distance function on S induced by the Riemannian metric G.

For $\varepsilon > 0$, define U_ε by

$$U_\varepsilon = \{p \in S - \{p_2\} \mid \operatorname*{Inf}_{q \in E} \mathrm{DIS}_{G_1}(q, p) < \varepsilon\} ,$$

where DIS_{G_1} denotes the distance function on $S - \{p_2\}$ induced by the Riemannian metric G_1. Then there is a C^∞ function $f_\varepsilon \colon S \to R$ such that

$$f_\varepsilon(p) = 1, p \in H_1 \cup U_{\varepsilon/2} , \qquad 0 < f_\varepsilon(p) \leq 1, p \in S ,$$
$$f_\varepsilon(p) < \varepsilon, p \in S - (H_1 \cup U_\varepsilon) .$$

Let G_2 be any extension of $G_1 | H_1 \cup U_{1/2}$ to all of S. Then for $\varepsilon > 0$ sufficiently small, $G = f_\varepsilon G_2$ satisfies properties a) and b). Property a) is immediate since $\mathrm{Cl}\, H_1 \cup U_{\varepsilon/2}$ is open and $f_\varepsilon(p) = 1$ for $p \in \mathrm{Cl}\, H_1 \cup U_{\varepsilon/2} = H_1 \cup U_{\varepsilon/2}$. To show that property b) holds for $f_\varepsilon G_2$ with $\varepsilon > 0$ sufficiently small, observe that every point in $H_2 - U_\varepsilon$ is within G-distance

$$\varepsilon \sup_{q \in H_2 - U_\varepsilon} \mathrm{DIS}_{G_2}(p_2, q)$$

of p_2 while every point in $U_\varepsilon \cap H_2$ is, for $\varepsilon < 1/2$, within G_2-distance ε of a point in $H_2 - U_\varepsilon$ and hence within G-distance ε of a point in $H_2 - U_\varepsilon$. Thus, for $\varepsilon < 1/2$, every point in H_2 is within G-distance

$$\varepsilon(1 + \sup_{q \in H_2 - U_\varepsilon} \mathrm{DIS}_{G_2}(p_2, q))$$

of p_2. Since

$$\sup_{q \in H_2 - U_\varepsilon} \mathrm{DIS}_{G_2}(p_2, q) < \infty ,$$

there are ε such that

$$0 < \varepsilon < [4(1 + \sup_{q \in H_2 - U_\varepsilon} \mathrm{DIS}_{G_2}(p_2, q))]^{-1}$$

and $\varepsilon < 1/2$, and for such an ε, $DIS_G(q, p_2) < 1/4$ for all $q \in H_2$ and hence

$$\sup_{q_1, q_2 \in \mathrm{Cl}\, H_2} \mathrm{DIS}_G(q_1, q_2) \leq 1/2 .$$

It remains to show that S with the metric G has no C^2 isometric embedding in R^3. Suppose on the contrary that $f\colon S \to R^3$ is a C^2 embedding isometric for G on S. By property a) of the metric G and the construction of G_1, there is an open set V with $H_1 \cup E = \mathrm{Cl}\, H_1 \subset V$ such that V is isometric to an open set in R^2 and $H_1 \cup E$ corresponds to the closed unit disc D under this isometry. Thus, the Proposition of § 1 is applicable to the isometry $f | V$ of V into R^3. It follows that there are points $q_1, q_2 \in E$ such that the distance in R^3 of $f(q_1)$ from $f(q_2)$ is at least $\sqrt{3}$. But by property b) of the metric G, the G-distance of q_1 from q_2 on S is less than or equal to $1/2 < \sqrt{3}$. Thus f cannot be an isometric embedding for S with the metric G.

References

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[2] W. S. Massey, *Surfaces of Gaussian curvature zero in Euclidean 3-space,* Tôhoku Math. J. **14** (1962) 73–79.

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