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SOME HOMOLOGICAL PROPERTIES OF SPENCER'S COHOMOLOGY THEORY

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Introduction

It has been observed by Malgrange in [4] that the theory of overdetermined systems of linear partial differential equations can be considered as a subject within the theory of differential modules. Put a little more specifically, it is clear from the results of [4] that the Spencer cohomology groups are equal to $\mathrm{Ext}^p(M, N)$ for appropriately chosen differential modules M and N. This statement is made precise in § 3.

What is given here is an exposition and amplification of this point of view without the restrictive hypotheses of [4]. What is accomplished by this is:

i) Greater clarity in the ideas involved.

ii) A more canonical and natural development of the theory.

These accomplishments are made possible by the introduction of differential module structures on jet bundles which is exploited for the first time in a systematic way here. This is the main innovation of this paper and is the missing link needed to give a fully homological account of the theory.

Concerning the greater generality of the theory presented here, an important qualification needs to be added, namely, for the differentiable case the greater generality is probably an illusion since the objects arising in the non-regular situation are apparently too little understood for us to treat them effectively.

The only contribution of the more general point of view in the differentiable case which one presently expects is the greater clarity it provides. However for the analytic, complex analytic and algebraic cases there is reason to believe that the greater generality will be quite meaningful. In this paper it is primarily the differentiable case which is treated. At appropriate points in the exposition, the minor adaptations of the theory which must be made to handle the other cases of interest are noted.

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0. Notational conventions and preliminary definitions

(ɐ *Miscellaneous notation.* $\boldsymbol{R}$ = field of real numbers. $\boldsymbol{C}$ = field of complex numbers. $\boldsymbol{N}$ = the natural numbers $0, 1, \cdots$. $\boldsymbol{Z}$ = ring of integers.

b) *Notational conventions concerning sheaves.* The notation of [2] will be used throughout when we work with sheaves. All sheaves considered will be sheaves of vector spaces over $\boldsymbol{R}$ or $\boldsymbol{C}$ and in the algebraic case over some fixed field of characteristic zero. This fact will be consistently ignored in the discussion, it being left to the reader to interpret the abelian groups which arise as vector spaces over his favored field when he feels inclined to do so. Every sheaf will be a sheaf of abelian groups, and morphisms of sheaves will always be meant in this sense. If X is a topological space, $\mathbf{Ab}_X$ will denote the category of sheaves of abelian groups on X.

Given X and an object F of $\mathbf{Ab}_X$, a *section* f of F will be any element of $F(U)$ for some (variable) open subset U of X. We will say that f is *defined at* x if $x \in U$. Elements of $F(X)$ will always be called *global sections* (of F). It will be the practice here not to make any mention of the open set U over which a section of a sheaf is defined except when it seems necessary in order to avoid confusion.

If F is a sheaf and f is a section of F defined at x, we will let f_x denote the germ of f at x. If X is a manifold (differentiable, analytic, etc.) and G a vector bundle on X, G will also be the letter used to denote the sheaf of germs of (differentiable, analytic, etc.) sections of G. The stalk of a sheaf F at the point x will be denoted by F_x.

c) *Differential modules.* In what follows X will denote a paracompact differentiable (in the C^∞-sense) manifold and $\mathfrak{O}$ the sheaf of germs of differentiable functions on X.

For each open subset U of X we will let $\mathfrak{D}(U)$ denote the set of all linear differential operators of $\mathfrak{O}(U)$ into itself, and let $\mathfrak{D}_r(U)$ denote the elements of $\mathfrak{D}(U)$ which are of order $\leq r$. Composition of operators defines a multiplication on $\mathfrak{D}(U)$, and $\mathfrak{D}$ is a sheaf of (non-commutative) rings on X.

The natural inclusion $i: \mathfrak{O} \subset \mathfrak{D}$ (whereby every section of $\mathfrak{O}$ acts as a linear differential operator by multiplication) makes $\mathfrak{D}$ into an $\mathfrak{O}$-module in two distinct ways and it is crucial to distinguish between them. The *left* $\mathfrak{O}$-module structure on $\mathfrak{D}$ is the one in which a section a of $\mathfrak{O}$ sends a section D of $\mathfrak{D}$ into the section $i(a)D$, this product being defined by the ring structure on $\mathfrak{D}$. The right $\mathfrak{O}$-module structure is defined similarly.

All tensor products will be taken over the sheaf of rings $\mathfrak{O}$ and never over $\mathfrak{D}$. Symbolically $\otimes = \otimes_{\mathfrak{O}}$. When $\mathfrak{D}$ appears in a tensor product, its position will indicate which of the $\mathfrak{O}$-module structures on $\mathfrak{D}$ is being considered. For instance in $F \otimes \mathfrak{D} \otimes E^*$ the first $\otimes$ uses the left structure on $\mathfrak{D}$ and the second $\otimes$ uses the right. If H is an $\mathfrak{O}$-module, we will let $H^* = \mathbf{Hom}_{\mathfrak{O}}(H, \mathfrak{O})$.

A *differential module* (*on* X) will be any sheaf of left $\mathfrak{D}$-modules. By the definition of $\mathfrak{D}$, $\mathfrak{O}$ is a differential module. Also if G is any $\mathfrak{O}$-module, $\mathfrak{D} \otimes G$ has a canonical differential module structure which we will frequently make use of in the sequel. We will let $\mathfrak{M}$ denote the category of differential modules (i.e., of left $\mathfrak{D}$-modules).

Frequently it is helpful to have an alternate definition of differential modules. For this purpose let T be the tangent bundle of X. We have a canonical inclusion $T \subset \mathfrak{D}$. If P is a differential module, the inclusions $\mathfrak{O} \subset \mathfrak{D}$ and $T \subset \mathfrak{D}$ induce an $\mathfrak{O}$-module structure on P and an action of T on P satisfying the following axioms whenever δ and δ' are sections of T, a a section $\mathfrak{O}$ and x a section of P:

1) $\delta(ax) = \delta(a)x + a\delta(x)$,
2) $[\delta, \delta'](x) = \delta(\delta'(x)) - \delta'(\delta(x))$,
3) $(\delta + \delta')(x) = \delta(x) + \delta'(x)$,
4) $(a\delta)(x) = a(\delta(x))$.

It will be left to the reader to satisfy himself that conversely such an action of T on an $\mathfrak{O}$-module P is induced by a unique differential module structure on P.

If E and F are vector bundles on X we will denote by $\mathfrak{D}(E, F)$ the sheaf of linear differential operators from E to F. We have a well-known canonical isomorphism

$$F \otimes \mathfrak{D} \otimes E^* \xrightarrow{\approx} \mathfrak{D}(E, F) \ ;$$

it is defined as follows. Let U be an open subset of X, $e^* \in E^*(U)$, $D' \in \mathfrak{D}(U)$, $f \in F(U)$. To the element $f \otimes D' \otimes e^*$ of $(F \otimes \mathfrak{D} \otimes E^*)(U)$ we associate the linear differential operator of $E(U)$ into $F(U)$ which sends $e \in E(U)$ into the element $fD'(e^*(e))$ of $F(u)$.

In particular, if G is any vector bundle the canonical isomorphism $\mathfrak{D} \otimes G^* \cong \mathfrak{D}(G, \mathfrak{O})$ induces a differential module structure on $\mathfrak{D}(G, \mathfrak{O})$. It is easy to see that if D is a section of $\mathfrak{D}$, this differential module structure is such that $\mathfrak{D}$ acts on $\mathfrak{D}(G, \mathfrak{O})$ by composition, $D' \to D \circ D'$. We will always identify the differential modules $\mathfrak{D} \otimes G^*$ and $\mathfrak{D}(G, \mathfrak{O})$.

1. Homological formulation of the problem of local solvability

Suppose that (differentiable) vector bundles E and F on our differentiable manifold X are given and that $D\colon E \to F$ is a linear differential operator. If f is a global section of F, one would like to know when the equation $D(e) = f$ has a solution e which is a global section of E. Here we shall confine ourselves to the local aspect of this question, namely, when does there exist for each $x \in X$ an $e_x \in E_x$ such that $D(e_x) = f_x$? We limit ourselves to the local situation because this is where the real difficulties seem to lie. We nonetheless keep the sheaf language (rather than focusing our attension on a single $x \in X$) in order that we may at some future moment take advantage of theorems relating local and global phenomena.

There is an obvious necessary condition for the local solvability of the equation $D(e) = f$, namely, if G is another vector bundle on X and $D'\colon F \to G$ a linear differential operator such that $D'D = 0$, we should certainly have $D'f = 0$. Let us call f *compatible* (*for* D) and write $f \in \mathbf{Com}\, D$ if $D'f = 0$ for every such pair (G, D'). This condition is a local one. Let $\mathbf{Com}\, D \subset F$ be the subsheaf of F consisting of all germs of compatible sections of F, and $\mathbf{Im}\, D \subset F$ the image sheaf of D.

It is natural to ask what the quotient $\mathbf{Com}\, D/\mathbf{Im}\, D$ is. In fact it is reasonable to look for conditions under which this quotient is zero; in the real analytic theory, i.e., when X is an analytic manifold, this quotient is always 0 when one poses reasonable "regularity" conditions on D (see [1]). These regularity conditions are not sufficient in the differentiable case as is shown by an example of Lewy [5, (1.6.16)].

The starting point for our investigation of $\mathbf{Com}\, D/\mathbf{Im}\, D$ will be the lemma below which is a global version of a local statement given by Malgrange [4, p. 12]. It allows us to formulate our problem in the category M of differential modules which has the advantage of being an abelian category.

We will define a morphism

$$*\colon \mathfrak{D}(E, F) \to \mathbf{Hom}_{\mathfrak{D}}(\mathfrak{D} \otimes F^*, \mathfrak{D} \otimes E^*)\ .$$

If D_1 is any section of $\mathfrak{D}(E, F)$ over the open subset U of X, let D_1^* be the section of $\mathbf{Hom}_{\mathfrak{D}}(\mathfrak{D}(F, \mathfrak{O}), \mathfrak{D}(E, \mathfrak{O}))$ over U define by $D_1^*(D') = D' \circ D_1$ whenever D' is a section of $\mathfrak{D}(F, \mathfrak{O})$ defined over an open subset V of U. This gives the desired morphism since by part c) of § 0, $\mathfrak{D} \otimes E^* \cong \mathfrak{D}(E, \mathfrak{O})$ and $\mathfrak{D} \otimes F^* \cong \mathfrak{D}(F, \mathfrak{O})$.

Lemma. *Let E and F be vector bundles on X. Then the morphism* $*\colon \mathfrak{D}(E,F) \to \mathbf{Hom}_{\mathfrak{D}}(\mathfrak{D} \otimes F^*, \mathfrak{D} \otimes E^*)$ *is an isomorphism.*

Let D_1 be a section of $\mathfrak{D}(E, F)$, and $\tilde{D}_1$ the section of $\mathbf{Hom}_{\mathfrak{D}}(\mathfrak{D} \otimes F, \mathfrak{D} \otimes E^*)$ obtained from D_1 by applying to D_1 the following sequence of canonical isomorphisms of sheaves:

$$\mathfrak{D}(E, F) \cong F \otimes \mathfrak{D} \otimes E^* \cong \mathbf{Hom}_{\mathfrak{O}}(F^*, \mathfrak{D} \otimes E^*) \cong \mathbf{Hom}_{\mathfrak{D}}(\mathfrak{D} \otimes F^*, \mathfrak{D} \otimes E^*)\ .$$

We will be done if we can show $D_1^* = \tilde{D}_1$.

Let U be the open set over which D_1 is defined. All we say will be valid over U, but U will frequently be left out of the notation for the sake of simplicity. We need to show that the following diagram (defined over U) commutes:

$$\begin{tikzcd}
\mathfrak{D}(F, \mathfrak{O}) \arrow[r, "D_1^*"] & \mathfrak{D}(E, \mathfrak{O}) \\
D \otimes F^* \arrow[u, "i"] \arrow[r, "\tilde{D}_1"] & \mathfrak{D} \otimes E^* \arrow[u, "j"']
\end{tikzcd}$$

where i and j are the canonical identifications mentioned in part c) of § 0.

Since both compositions $D_1^* \circ i$ and $j \circ \tilde{D}_1$ are D-linear, it will suffice to show that they are equal on elements of the form $1 \otimes f^*$ with $f^* \in F^*(U)$. Also one can assume that $D_1 = f \otimes D' \otimes e^*$ with $f \in F(U), D' \in \mathfrak{D}(U)$ and $e^* \in E^*(U)$. Then $\tilde{D}_1(1 \otimes f^*) = f^*(f)D' \otimes e^*$. If $e \in E(U)$, then $(D_1^* \circ i)(1 \otimes f^*)$ sends e into $f^*(fD'(e^*)(e))$, and $(j \circ \tilde{D}_1)(1 \otimes f^*)$ does too. They are hence the same and this completes the proof of the Lemma.

It is evident that if E, F and G are vector bundles on $X, D\colon E \to F$ and $D'\colon F \to G$ linear differential operators, then $(D'D)^* = D^*D'^*$. The morphism D^* is called the *adjoint morphism* of D, its kernel K the *adjoint kernel* and its cokernel M the *adjoint cokernel*. We therefore have an exact sequence of differential modules

$$(\dagger) \qquad 0 \longrightarrow K \xrightarrow{\ i\ } \mathfrak{D} \otimes F^* \xrightarrow{\ D^*\ } \mathfrak{D} \otimes E^* \xrightarrow{\ j\ } M \longrightarrow 0\,.$$

It will be shown shortly (parts ii) and iv) of the next proposition) that the adjoint cokernel M contains all the information which is of interest to us.

If $\alpha\colon A \to B$ is a morphism of differential modules on X, let $\alpha^*\colon \mathbf{Hom}_{\mathfrak{D}}(B, \mathfrak{O}) \to \mathbf{Hom}_{\mathfrak{D}}(A, \mathfrak{O})$ be the morphism induced by α. We then have the following proposition. Note that in part i) of it we use implicitly the canonical identification $\mathbf{Hom}_{\mathfrak{D}}(\mathfrak{D} \otimes G^*, \mathfrak{O}) \approx G$ valid for vector bundles G.

Proposition. *Let E and F be vector bundles on X, and $D\colon E \to F$ a linear differential operator. Then*:

i) *$D^{**}\colon E \to F$ is equal to D.*

ii) *j^* identifies $\mathbf{Hom}_{\mathfrak{D}}(M, \mathfrak{O})$ with the kernel of D.*

iii) $\mathbf{Ker}\, i^* = \mathbf{Com}\, D$.

iv) $\mathbf{Com}\, D / \mathbf{Im}\, D = \mathbf{Ext}^1_{\mathfrak{D}}(M, \mathfrak{O})$.

To establish i) we need to show that the following diagram commutes, where α and β are canonical identifications:

$$\begin{tikzcd}
\mathbf{Hom}_{\mathfrak{D}}(\mathfrak{D}(E, \mathfrak{O}), \mathfrak{O}) \arrow[r, "D^{**}"] & \mathbf{Hom}_{\mathfrak{D}}(\mathfrak{D}(F, \mathfrak{O}), \mathfrak{O}) \\
E \arrow[u, "\alpha"] \arrow[r, "D"] & F\,. \arrow[u, "\beta"']
\end{tikzcd}$$

Let e be a section of E, and let $A = D^{**}(\alpha(e)),\ B = \beta(D(e))$. Let D' be a section of $\mathfrak{D}(F, \mathfrak{O})$. We want to show that $A(D') = B(D')$. This is clear because $A = \alpha(e) \circ D^*$ and so $A(D') = \alpha(e)(D'D) = D'(D(e)) = B(D')$. This proves i) and also ii) which is an immediate consequence of i).

To prove iii), we first show that a section f of F which is also a section of $\mathbf{Ker}\, i^*$ must be a section of $\mathbf{Com}\, D$. Indeed let $D'\colon F \to G$ be a linear differential operator such that $D'D = 0$. Then $D^*D'^* = 0$ so there exists $\rho \in \mathrm{Hom}_{\mathfrak{D}}(\mathfrak{D} \otimes G^*, K)$ such that $D'^* = i\rho$. Thus

$$D'f = (D')^{**}f = \rho^*(i^*(f)) = \rho^*(0) = 0 \ ,$$

and f is a section of $\mathbf{Com}\, D$.

Conversely suppose that f is a section of $\mathbf{Com}\, D$. Let $x \in X$ be a point where f is defined and s an element of K_x. We want to show that $f(i(s)) = 0$. There exists a morphism of differential modules $\varphi\colon \mathfrak{D} \to K$ such that if 1 is the identity of $\mathfrak{D}$, $\varphi(1_x) = s$. By the lemma there exists a linear differential operator $D'\colon F \to \mathfrak{O}$ such that $D'^* = i \circ \varphi$. Plainly $D'D = 0$, so $f \circ D'^* = 0$ since by i) this is just $D'f$. Hence

$$f(i(s)) = f(i(\varphi(1_x))) = f(D'^*(1_x)) = 0 \ .$$

This completes the proof of iii).

To prove iv) break-up the exact sequence (†) into the exact sequences

$$0 \longrightarrow K \longrightarrow \mathfrak{D} \otimes F^* \longrightarrow P \longrightarrow 0$$
$$0 \longrightarrow P \longrightarrow \mathfrak{D} \otimes E^* \longrightarrow M \longrightarrow 0 \ .$$

We obtain exact sequences

$$0 \longrightarrow \mathbf{Hom}_{\mathfrak{D}}(P, \mathfrak{O}) \xrightarrow{a} \mathbf{Hom}_{\mathfrak{D}}(\mathfrak{D} \otimes F^*, \mathfrak{O}) = F \longrightarrow \mathbf{Hom}_{\mathfrak{D}}(K, \mathfrak{O}) \ ,$$
$$\mathbf{Hom}_{\mathfrak{D}}(\mathfrak{D} \otimes E^*, \mathfrak{O}) = E \xrightarrow{b} \mathbf{Hom}_{\mathfrak{D}}(P, \mathfrak{O}) \longrightarrow \mathbf{Ext}^1_{\mathfrak{D}}(M, \mathfrak{O}) \longrightarrow 0 \ ,$$

and $a \circ b = D$. As a identifies $\mathbf{Hom}_{\mathfrak{D}}(P,\mathfrak{O})$ with $\mathbf{Com}\, D$, iv) results immediately.

2. A spectral sequence for the Ext of differential modules

By the proposition of the preceding paragraph determining $\mathbf{Com}\, D/\mathbf{Im}\, D$ for a linear differential operator D is a special case of determining the functors $\mathbf{Ext}^p_{\mathfrak{D}}(M, N)$ for differential modules M and N on X. The next lemma motivates the approach to this problem which is followed here.

Let P be a differential module on X and U an open subset of X. An element f of $P(U)$ will be called a *constant* if for every $\delta \in T(U)$, $\delta f = 0$. Let $(C((P))(U)$ denote the set of constants of $P(U)$. Then $C(P)$ is a subsheaf of abelian groups of P called the *constant sheaf* (*of* P).

Lemma. *Let M and N be differential modules on X. Then* $\mathbf{Hom}_{\mathfrak{O}}(M, N)$ *has a canonical structure of differential module and* $C(\mathbf{Hom}_{\mathfrak{O}}(M, N)) = \mathbf{Hom}_{\mathfrak{D}}(M,N)$.

Let U be an open subset of X and $\delta \in T(U)$. Then δ induces sections of $\mathbf{Hom}_{\mathbf{Z}}(M, M)$ and $\mathbf{Hom}_{\mathbf{Z}}(N, N)$ over U both of which will again be denoted by the same letter δ. If f is a section of $\mathbf{Hom}_{\mathfrak{O}}(M, N)$ over U, let

$$\delta(f) = \delta \circ f - f \circ \delta \ .$$

It is immediate that $\delta(f)$ is again a section of $\mathbf{Hom}_{\mathfrak{O}}(M,N)$ over U. This defines an action of T on $\mathbf{Hom}_{\mathfrak{O}}(M,N)$ which satisfies 1)–4) of part c) of § 0. Thus $\mathbf{Hom}_{\mathfrak{O}}(M,N)$ is a differential module. The equality of the lemma is immediate.

Fix the differential module M, and consider the functor $h^M\colon N \to \mathbf{Hom}_{\mathfrak{D}}(M,N)$ of $\mathfrak{M}$ into $\mathbf{Ab}_X$. We are looking for the derived functors $R^p h^M$ of h^M, and the lemma tells us that h^M is the composite of the functor $\mathbf{Hom}_{\mathfrak{O}}(M,\ )$ of $\mathfrak{M}$ into itself with $C\colon \mathfrak{M} \to \mathbf{Ab}_X$. It is therefore natural to apply the theorem [3, p. 148] which relates the derived functors $R^m H$ of a composite functor $H = G \circ F$ with the composites $(R^p G) \circ (R^q F)$. We first take note of a corollary to the lemma just proven.

Corollary. *If P is a differential module on X, then*

$$C(P) = \mathbf{Hom}_{\mathfrak{D}}(\mathfrak{O}, P)\ .$$

Consequently C is left exact, and its right derived functors are given by the formula

$$(R^p C)(P) = \mathbf{Ext}^p_{\mathfrak{D}}(\mathfrak{O}, P)\ .$$

Indeed

$$C(P) = C(\mathbf{Hom}_{\mathfrak{O}}(\mathfrak{O}, P)) = \mathbf{Hom}_{\mathfrak{D}}(\mathfrak{O}, P)\ .$$

We will prove the following theorem.

Theorem. *Let M and N be differential modules on X. Then there is a spectral sequence converging to $\mathbf{Ext}^{p+q}_{\mathfrak{D}}(M,N)$ with initial term*

$$E_2^{pq} = (R^p C)(\mathbf{Ext}^q_{\mathfrak{O}}(M,N))\ .$$

To prove this theorem one needs to show that if the differential module N is injective and p is an integer > 0, then $R^P C(\mathbf{Hom}_{\mathfrak{O}}(M,N)) = 0$. The proof of this will be postponed so that applications of this theorem can be presented at the earliest possible moment.

3. Derivation of Spencer's theory

We first note the following corollary to the theorem of § 2.

Corollary. *Let M and N be differential modules on X. If $\mathbf{Ext}^q_{\mathfrak{O}}(M,N) = 0$ when $q > 0$, then for all non-negative integers P we have*

$$\mathbf{Ext}^p_{\mathfrak{D}}(M,N) = R^p C\,(\mathbf{Hom}_{\mathfrak{O}}(M,N))\ .$$

The manner in which the infinite Spencer theory results from the this corollary will now be explained. Let E and F be vector bundles on $X, D\colon E \to F$ a linear differential operator, and M the adjoint cokernel of D. We will suppose that D is *quasi-regular* which will mean by definition that M is locally free as an $\mathfrak{O}$-module.

Let $R_\infty = \mathbf{Hom}_{\mathfrak{O}}(M, \mathfrak{O})$. By applying $\mathbf{Hom}_{\mathfrak{O}}(\ , \mathfrak{O})$ to the exact sequence (†) of § 1 we get an exact sequence (using the notation of [5])

$$(\dagger)^* \qquad 0 \to R_\infty \to J_\infty(E) \to J_\infty(E) \to \mathbf{Hom}_{\mathfrak{O}}(K, \mathfrak{O}) \to 0\ .$$

Indeed it is evident for instance that $J_\infty(E) = \mathbf{Hom}_{\mathfrak{O}}(\mathfrak{D} \otimes E^*, \mathfrak{O})$ since when r is finite we have $J_r(E) = \mathfrak{D}_r^* \otimes E$ (their duals are the same) and

$$\begin{aligned} J_\infty(E) &= \varprojlim_r J_r(E) = \varprojlim_r \mathfrak{D}_r^* \otimes E = \varprojlim_r \mathbf{Hom}_{\mathfrak{O}}(\mathfrak{D}_r \otimes E^*, \mathfrak{O}) \\ &= \mathbf{Hom}_{\mathfrak{O}}(\mathfrak{D} \otimes E^*, \mathfrak{O})\ . \end{aligned}$$

From this it follows that if we choose a linear differential operator D which is formally integrable (a stronger hypothesis than quasi-regularity) our R_∞ is canonically isomorphic to the R_∞ of [5]. We have in addition placed a differential module structure on R_∞. We have[1] $R^pC(R_\infty) = \mathbf{Ext}^p_{\mathfrak{D}}(M, \mathfrak{O})$. It will be shown that $R^pC(R_\infty)$ is the *p-th* Spencer cohomology of D. For this we will use the following proposition concerning an arbitrary differential module P; in it the sections of $\Lambda^pT^* \otimes P$ are identified in the usual way with alternating p-forms on T having values in P.

Proposition. *Let P be a differential module on X. Then for $p \geq 0$ there exists a morphism (of sheaves of abelian groups) $d^p\colon \Lambda^pT^* \otimes P \to \Lambda^{p+1}T^* \otimes P$ satisfying the formula*

$$\begin{aligned} d^p\omega(\xi_1, \cdots, \xi_{p+1}) = &\sum_{j=1}^{p+1} (-1)^{j-1}\xi_j(\omega(\xi_1, \cdots, \hat{\xi}_j, \cdots, \xi_{p+1})) \\ &+ \sum_{i<j} (-1)^{i+j}\omega([\xi_i, \xi_j], \xi_1, \cdots, \hat{\xi}_i, \cdots, \hat{\xi}_j, \cdots, \xi_{p+1})\ . \end{aligned}$$

whenever U is an open subset of $X, \xi_1, \cdots, \xi_p \in T(U)$ and $\omega \in (\Lambda^pT^* \otimes P)(U)$. With these morphisms $\Lambda^{\cdot}T^* \otimes P$ is a complex of sheaves of abelian groups and

$$H^p(\Lambda^{\cdot}T^* \otimes P) = (H^pC)(P)\ .$$

Let $D_p\colon \mathfrak{D} \otimes \Lambda^{p+1}T \to \mathfrak{D} \otimes \Lambda^pT$ be the unique morphism of differential modules such that if $\xi_1, \cdots, \xi_{p+1}$ are sections of T, then

$$\begin{aligned} D_p(1 \otimes \xi_1 \wedge \cdots \wedge \xi_{p+1}) = &\sum_{j=1}^{p+1} (-1)^{j-1}\xi_j \otimes \xi_1 \wedge \cdots \wedge \hat{\xi}_j \wedge \cdots \wedge \xi_{p+1} \\ &+ \sum_{i<j} (-1)^{i+j} 1 \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \\ &\qquad \wedge \hat{\xi}_i \wedge \cdots \wedge \hat{\xi}_j \wedge \cdots \wedge \xi_{p+1}\ . \end{aligned}$$

[1] It is not too difficult to show that $\mathbf{Ext}^q_{\mathfrak{O}}(M, \mathfrak{O}) = 0$ when $q > 0$.

(This is actually a special case of the formula in [4, p. 23].) We get in this way a resolution

$$(\dagger\dagger) \qquad 0 \longrightarrow \mathfrak{D} \otimes \Lambda^n T \xrightarrow{D_{n-1}} \cdots \xrightarrow{D_0} \mathfrak{D} \longrightarrow \mathfrak{O} \longrightarrow 0$$

of $\mathfrak{O}$ by locally free $\mathfrak{D}$-modules. The desired complex is

$$\mathbf{Hom}_{\mathfrak{D}}(\mathfrak{D} \otimes \Lambda^{\cdot} T, P) = \Lambda^{\cdot} T^* \otimes P .$$

By Corollary 1 in [3, p. 189],

$$\mathbf{Ext}^{p}_{\mathfrak{D}}(\mathfrak{O}, P) = H^p(\Lambda^{\cdot} T^* \otimes P) .$$

and this is $R^pC(P)$ by the corollary to the lemma of § 2. The proof of the proposition is now complete since the formula for d^p is a trivial consequence of the formula for D_p.

To show that $R^pC(R_\infty)$ is the *p-th* Spencer cohomology for D choose an arbitrary point $x \in X$ and an open neighborhood U of x. Suppose that U has a coordinate system $u_1, \cdots, u_m$ and that the $\mathfrak{O}(U)$ module $E(U)$ has a free basis $e_1, \cdots, e_m$. Now $J_\infty(E)$ is the differential module $(\mathfrak{D} \otimes E^*)^* = \mathfrak{D}^* \otimes E$. Let D^α for each multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in N^n$ be the section $D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ of $\mathfrak{D}$ where $D_1, \cdots, D_n$ denote the partial derivations with respect to $u_1, \cdots, u_n$.

A section s of $(D \otimes E^*)^*$ over U is determined by giving its values on the $D^\alpha \otimes e_i^*$, say $s(D^\alpha \otimes e_i^*) = s_{\alpha,i}$. Now

$$(D_i s)(D^\alpha \otimes e_i^*) = D_i(s(D^\alpha \otimes e_i^*)) - s(D_i D^\alpha \otimes e_i^*) = D_i(s_{\alpha,i}) - s_{\alpha+1_i,i} ,$$

i.e., the difference between "honest" differentiation and "formal" differentiation.

For any section ω of $\Lambda^p T^* \otimes J_\infty(E)$ over U let $\omega_{i_1 \cdots i_p} = \omega(D_{i_1}, \cdots, D_{i_p})$ where as before we consider ω as an alternating form on $T(U)$ with values in $J_\infty(E)(U)$. The formula of the proposition tells us that

$$(d^p \omega)_{i_1 \cdots i_{p+1}} = \sum_{j=1}^{p+1} (-1)^{j-1} D_{i_j}(\omega_{i_1 \cdots \hat{i}_j \cdots i_{p+1}}) .$$

By substituting the above formula for D_i's we get Spencer's formula for differentiation in the complex $\Lambda^{\cdot} T^* \otimes J_\infty(E)$.

4. Proof of the theorem 1

As has been observed we only need to show that if M and N are differential modules on X and N is injective, then $R^pC(\mathbf{Hom}_{\mathfrak{O}}(M, N)) = 0$ for all $p > 0$. For this the corollary to the following lemma is used. In the lemma $P \otimes M$ is considered as a differential module by letting $\delta(x \otimes y) = \delta(x) \otimes y + x \otimes \delta(y)$ when δ is a section of T.

Lemma. *Let M, N and P be differential modules on X. Then the canonical isomorphism of $\mathfrak{O}$-modules*

$$\mathbf{Hom}_{\mathfrak{O}}(P \otimes M, N) \cong \mathbf{Hom}_{\mathfrak{O}}(P, \mathbf{Hom}_{\mathfrak{O}}(M, N))$$

is an isomorphism of differential $\mathfrak{O}$-modules.

Let Φ be the canonical map of $\mathbf{Hom}_{\mathfrak{O}}(P \otimes M, N)$ into $\mathbf{Hom}_{\mathfrak{O}}(P, \mathbf{Hom}_{\mathfrak{O}}(M, N))$ defined by $g = \Phi(f)$ where $g(x)y = f(x \otimes y)$ when x and y are sections of P and M respectively. If δ is a section of T, then

$$\begin{aligned}((\delta g)x)y &= (\delta(gx) - g(\delta x))y = \delta(g(x)y) - g(x)(\delta y) - g(\delta x)y \\ &= \delta(f(x \otimes y)) - f(\delta(x \otimes y)) = (\delta f)(x \otimes y) = (\Phi(\delta f))(x)y ,\end{aligned}$$

so $\delta g = \Phi(\delta f)$ which was to be proven.

Corollary. *Under the assumptions of the preceding lemma,*

$$\mathbf{Hom}_{\mathfrak{D}}(P \otimes M, N) \cong \mathbf{Hom}_{\mathfrak{D}}(P, \mathbf{Hom}_{\mathfrak{O}}(M, N)) .$$

The proof follows immediately by applying C to both sides of the isomorphism of the lemma.

Let us now prove that if $p > 0$, then $R^p C(\mathbf{Hom}_{\mathfrak{O}}(M, N)) = 0$ when N is injective. The corollary shows that if in addition we assume that M is flat as an $\mathfrak{O}$-module, $\mathbf{Hom}_{\mathfrak{D}}(\ , \mathbf{Hom}_{\mathfrak{O}}(M, N))$ is an exact functor, i.e., that $\mathbf{Hom}_{\mathfrak{O}}(M, N)$ is an injective differential module.

In any case it can easily be shown that M has a flat resolution[2]

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0 .$$

(For instance take the F_i to be locally free, or even direct sums of copies of $\mathfrak{D}$.) Since $\mathbf{Hom}_{\mathfrak{O}}(\ , N) = \mathbf{Hom}_{\mathfrak{D}}(\mathfrak{D} \otimes\ , N)$, N is injective as an $\mathfrak{O}$-module. It follows that

$$0 \longrightarrow \mathbf{Hom}_{\mathfrak{O}}(M, N) \longrightarrow \mathbf{Hom}_{\mathfrak{O}}(F., N)$$

is an injective resolution of the differential module $\mathbf{Hom}_{\mathfrak{O}}(M, N)$. Applying C to this resolution, we get

$$R^p C(\mathbf{Hom}_{\mathfrak{O}}(M, N)) = H^p(C(\mathbf{Hom}_{\mathfrak{O}}(F., N))) = H^p(\mathbf{Hom}_{\mathfrak{D}}(F., N)) = 0 ,$$

if $p > 0$. This proves the theorem.

[2] Our proof for the analytic and algebraic cases proceedes identically once this fact is established in those cases.

5. The analytic and algebraic cases of the theorem

The following is the fact needed to extend the theorem to these other cases.

Lemma. *Let X be an analytic manifold, a complex analytic manifold or an algebraic variety without singularities. If M is any differential module on X (analytic, complex analytic or algebraic respectively) there is a surjective morphism of differential modules $F \to M$ where F is flat as an $\mathfrak{O}$-module.*

Here of course $\mathfrak{O}$ and $\mathfrak{D}$ are to be defined in the appropriate way in analogy to the definitions previously employed.

If U is an open subset of X, define $\mathfrak{D}^U(V)$ for every connected open subset V of X by

$$\mathfrak{D}^U(V) = \begin{cases} \mathfrak{D}(V), & \text{if } V \subset U\,, \\ 0, & \text{if } V \not\subset U\,. \end{cases}$$

This defines a sheaf $\mathfrak{D}^U$ on the family of connected open subsets of X. Denote also by $\mathfrak{D}^U$ its canonical extension to a sheaf on X. Then $\mathfrak{D}^U$ is canonically a sheaf of $\mathfrak{D}$-modules and clearly flat as an $\mathfrak{O}$-module.

If $f \in M(U)$ with U a connected open subset of X, there exists a unique morphism of $\mathfrak{D}$-modules $\varphi_{(U,f)}\colon \mathfrak{D}^U \to M$ such that the element 1 of $\mathfrak{D}^U(U)$ is mapped onto f. Let $F = \sum_{(U,f)} \mathfrak{D}^U$ where the sum is taken over all the pairs (U, f) just mentioned. There is a unique morphism of F to M which on the summand $\mathfrak{D}^U$ corresponding to (U, f) is $\varphi_{(U,f)}$. It is clearly surjective and the proof is complete.

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