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QUASI-SASAKIAN STRUCTURES OF RANK $2p+1$

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Introduction

Quasi-Sasakian structures were defined and studied by D. E. Blair [1]. However, there are some gaps in arguments in § 3 — § 5 of [1]. The first is found in the middle of page 337, namely, for a quasi-Sasakian structure (ϕ, ξ, η, g'), the new (ϕ, ξ, η, g) is not quasi-Sasakian, in general. Moreover, $\mathscr{E}^{2q}, \phi, \theta$ are not uniquely determined.

In this note we give complete statements on quasi-Sasakian structures of rank $2p+1$.

1. Quasi-Sasakian structures

Let ϕ be a $(1,1)$-tensor, ξ a vector field, and η a 1-form on a differentiable manifold M of dimension $2n+1$. Then (ϕ, ξ, η) is an almost contact structure if

$$\eta(\xi) = 1 , \quad \phi\xi = 0 , \quad \eta\phi = 0 , \tag{1.1}$$

$$\phi^2 = -I + \xi \otimes \eta . \tag{1.2}$$

For a (positive definite) Riemannian metric g, (ϕ, ξ, η, g) is an almost contact metric structure if

$$\eta(X) = g(\xi, X) , \tag{1.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1.4}$$

for $X, Y \in \mathscr{E}^{2n+1}$, where $\mathscr{E}^{2n+1}$ denotes the module of vector fields on M. An almost contact metric structure (ϕ, ξ, η, g) is a contact metric structure if

$$(d\eta)(X, Y) = 2g(X, \phi Y) \qquad \text{for } X, Y \in \mathscr{E}^{2n+1} .$$

(ϕ, ξ, η) is said to be normal if

$$\begin{aligned} &-N^1(X, Y) = [\phi, \phi](X, Y) + (d\eta)(X, Y)\xi = 0 . \\ &([\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[X, \phi Y] - \phi[\phi X, Y] .) \end{aligned} \tag{1.5}$$

$N^1 = 0$ implies the followings (cf. [4]):

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$$(1.6) \qquad N^2(X, Y) = (L_{\phi X}\eta)(Y) - (L_{\phi Y}\eta)(X) = 0 ,$$

$$(1.7) \qquad N^3(X) = (L_\xi \phi)X = 0 ,$$

$$(1.8) \qquad N^4(X) = -(L_\xi \eta)X = 0 ,$$

where L_X denotes the Lie derivation with respect to X. Define a 2-form Φ by $\Phi(X, Y) = g(X, \phi Y)$. Then a normal almost contact Riemannian structure (ϕ, ξ, η, g) is said to be quasi-Sasakian, if Φ is closed.

Proposition 1.1. *Let $M(\phi, \xi, \eta, g)$ be a quasi-Sasakian manifold. Then we have*

$$(1.9) \qquad d\eta(\xi, X) = 0 , \qquad X \in \mathscr{E}^{2n+1} ,$$

$$(1.10) \qquad d\eta(\phi X, \phi Y) = d\eta(X, Y) , \qquad X, Y \in \mathscr{E}^{2n+1} ,$$

$$(1.11) \qquad L_\xi \phi = 0 ,$$

$$(1.12) \qquad L_\xi g = 0 .$$

Proof. (1.9) and (1.11) are the same as (1.8) and (1.7). Since $L_{\phi X}\eta = di(\phi X)\eta + i(\phi X)d\eta$, by (1.1) and (1.6) we obtain

$$(1.13) \qquad d\eta(\phi X, Y) - d\eta(\phi Y, X) = 0 .$$

Then replacing Y by ϕY and using (1.9) we have (1.10). (1.12) can be proved by means of $d\Phi = 0$, (1.8) and (1.11) (cf. [1, Lemma 4.1]).

Remark. The condition $d\Phi = 0$ is used only for (1.12).

2. Quasi-Sasakian manifolds of rank $2p + 1$

Let $M(\phi, \xi, \eta, g)$ be a quasi-Sasakian manifold. If $d\eta = 0$ on M, then M is called a cosymplectic manifold (cf. [2]). If $2\Phi = d\eta$, then M is called a Sasakian manifold or a manifold with normal contact metric structure (cf. [4]). In this case, $\eta \wedge (d\eta)^n \neq 0$ holds on M.

A quasi-Sasakian manifold M (or more generally, an almost contact manifold M) is said to be of rank $2p$ if $(d\eta)^p \neq 0$ and $\eta \wedge (d\eta)^p = 0$ on M, and to be of rank $2p + 1$ if $\eta \wedge (d\eta)^p \neq 0$ and $(d\eta)^{p+1} = 0$ on M. It is known that there are no quasi-Sasakian structures of even rank (cf. [1]).

Let M be a quasi-Sasakian manifold of rank $2p + 1$, and define a submodule $\mathscr{E}^{2q}$ of $\mathscr{E}^{2n+1}$ $(2q = 2n - 2p)$ by

$$\mathscr{E}^{2q} = \{X \in \mathscr{E}^{2n+1};\ i(X)d\eta = 0 \text{ and } \eta(X) = 0\} .$$

$\mathscr{E}^{2q}$ is well defined and $\mathscr{E}^{2q}_x$ is of dimension $2q$ at each point x of M. We denote by $\mathscr{E}^1$ a submodule of $\mathscr{E}^{2n+1}$ composed of $\{f\xi\}$ for C^∞-functions f on M, and by $\mathscr{E}^{2p}$ the orthogonal complement of $\mathscr{E}^1 \oplus \mathscr{E}^{2q}$ in $\mathscr{E}^{2n+1}$. Put $\mathscr{E}^{2p+1} = \mathscr{E}^{2p} \oplus \mathscr{E}^1$, and let $X \in \mathscr{E}^{2q}$. Then by $\eta(\phi X) = 0$ and (1.13) or (1.10) we have $\phi X \in \mathscr{E}^{2q}$. Since $X = \phi(-\phi X)$ for $X \in \mathscr{E}^{2q}$, we get

$$(2.1) \qquad \phi\mathscr{E}^{2q} = \mathscr{E}^{2q}, \qquad \phi\mathscr{E}^{2p} = \mathscr{E}^{2p}.$$

Define (1,1)-tensors ψ and θ by

$$\begin{aligned} \psi(X) &= \phi X & &\text{if } X \in \mathscr{E}^{2p}, \\ &= 0 & &\text{if } X \in \mathscr{E}^{2q} \oplus \mathscr{E}^{1}, \\ \theta(X) &= \phi X & &\text{if } X \in \mathscr{E}^{2q}, \\ &= 0 & &\text{if } X \in \mathscr{E}^{2p+1}. \end{aligned}$$

Then $-\psi^2$, $-\psi^2 + \xi \otimes \eta$ and $-\theta^2$ are projection tensors to $\mathscr{E}^{2p}$, $\mathscr{E}^{2p+1}$ and $\mathscr{E}^{2q}$ respectively, and we have $\phi = \psi + \theta$ and

$$(2.2) \qquad \phi\psi = \psi\phi = \psi^2, \qquad \phi\theta = \theta\phi = \theta^2$$

by the definitions of ψ and θ and by (2.1) respectively. We define a (0,2)-tensor $g^\#$ by

$$(2.3) \qquad 2g^\#(X, Y) = -d\eta(X, \phi Y), \qquad X, Y \in \mathscr{E}^{2n+1}.$$

By (1.13), $g^\#$ is symmetric. Assume that $g^\#$ is positive definite on $\mathscr{E}^{2p}$, and define a new metric $\bar{g}$ by

$$(2.4) \qquad \bar{g}(X, Y) = \eta(X)\eta(Y) + g^\#(\psi^2X, \psi^2Y) + g(\theta^2X, \theta^2Y).$$

Then we have

$$\bar{g}(\xi, X) = \eta(X), \qquad \bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \eta(X)\eta(Y)$$

by (1.10) and (2.2), etc. $(\phi, \xi, \eta, \bar{g})$ is a normal almost contact metric structure.

Proposition 2.1. *Let $M(\phi, \xi, \eta, g)$ be a quasi-Sasakian manifold of rank $2p + 1$, and assume that*

(i) $[\theta, \theta] = 0$,

(ii) *$g^\#$ defined by* (2.3) *is positive definite on $\mathscr{E}^{2p}$. Then M has a normal almost contact metric structure $(\phi, \xi, \eta, \bar{g})$ such that for each point x of M we have two submanifolds U^{2p+1} and V^{2q} of M containing x, where U^{2p+1} is a Sasakian manifold and V^{2q} is a Kählerian manifold.*

Proof. An almost product structure (defined by $-\theta^2$ and $-\psi^2 + \xi \otimes \eta$) is integrable (see [5, p. 240]), since $[\theta, \theta] = 0$ implies $[\theta^2, \theta^2] = 0$. For a point x of M, let V^{2q} and U^{2p+1} be integral submanifolds of $-\theta^2$ and $-\psi^2 + \xi \otimes \eta$ passing through x. Consider the imbeddings $r\colon V^{2q} \to M$ and $s\colon U^{2p+1} \to M$, and let u, v be vector fields on U^{2p+1}. Define $\phi_0, \xi_0, \eta_0, \bar{g}_0$ by

$$\begin{gathered} \phi_0 u = s^{-1}\psi s u = s^{-1}\phi s u, \qquad \xi_0 = s^{-1}\xi, \\ \eta_0(u) = \eta(su), \qquad \eta_0 = s^*\eta, \qquad \bar{g}_0(u, v) = \bar{g}(su, sv), \end{gathered}$$

where by s we also mean the differential of s; these are well defined. $(\phi_0, \xi_0, \eta_0, \bar{g}_0)$ is an almost contact metric structure, and is normal since

$$s\{[\phi_0, \phi_0](u, v) + (d\eta_0)(u, v)\xi_0\} = [\phi, \phi](su, sv) + (d\eta)(su, sv)\xi = 0 .$$

Further, we have

$$\begin{aligned} 2\bar{g}_0(u, \phi_0 v) &= 2\bar{g}(su, \phi sv) = 2g^{\#}(su, \phi sv) = -(d\eta)(su, \phi\phi sv) \\ &= (d\eta)(su, sv) = (s^*d\eta)(u, v) = (d\eta_0)(u, v) . \end{aligned}$$

Hence U^{2p+1} is a Sasakian manifold.

Let w, z be vector fields on V^{2p}, and define J_0 and G_0 by

$$J_0 w = r^{-1}\theta rw = r^{-1}\phi rw , \qquad G_0(w, z) = \bar{g}(rw, rz) .$$

Then J_0 and G_0 are well defined and define an almost Hermitian structure. Moreover, J_0 is integrable since

$$r\{[J_0, J_0](w, z)\} = [\theta, \theta](rw, rz) = 0 .$$

Define $\Omega_0(w, z) = G_0(w, J_0 z)$. Then

$$\begin{aligned} \Omega_0(w, z) &= \bar{g}(rw, rJ_0 z) = \bar{g}(rw, \phi rz) \\ &= g(\theta^2 rw, \theta^2\phi rz) \qquad \text{by (2.4)} \\ &= \Phi(rw, rz) = (r^*\Phi)(w, z) , \end{aligned}$$

and therefore $d\Omega_0 = dr^*\Omega = r^*d\Phi = 0$. Hence V^{2q} is Kählerian.

Remark. $d\Phi = 0$ is used only for $d\Omega_0 = 0$. Thus, if $d\bar{\Theta} = 0$, then $d\Phi = 0$ is unnecessary, where $\bar{\Theta}$ is defined below.

We define 2-forms $\Psi, \bar{\Psi}, \Theta, \bar{\Theta}$ by

$$\Psi(X, Y) = g(X, \phi Y) , \qquad \bar{\Psi}(X, Y) = \bar{g}(X, \phi Y) ,$$
$$\Theta(X, Y) = g(X, \theta Y) , \qquad \bar{\Theta}(X, Y) = \bar{g}(X, \theta Y) .$$

Lemma 2.2. *$\mathscr{E}^{2p}$ and $\mathscr{E}^{2q}$ are invariant under* $\exp t\xi$, *and we have*

(2.5) $$L_\xi\phi = 0 , \qquad L_\xi\Psi = L_\xi\bar{\Psi} = 0 ,$$

(2.6) $$L_\xi\theta = 0 , \qquad L_\xi\Theta = L_\xi\bar{\Theta} = 0 ,$$

(2.7) $$L_\xi g^{\#} = 0 , \qquad L_\xi\bar{g} = 0 .$$

Proof. Let $X \in \mathscr{E}^{2q}$ and put $\alpha = \exp t\xi$, t being a real number (sufficiently small, if necessary). If ξ is complete, α is a diffeomorphism of M. If ξ is not complete, we understand that α is a map: $W \to \alpha W$ for some open set W, and also that $X \in \mathscr{E}^{2q}$ implies $X | W \in \mathscr{E}^{2q} | W$. Since α leaves η invariant, we have $\eta(\alpha X) = 0$. For $Z \in \mathscr{E}^{2n+1}$,

$$(d\eta)(\alpha X, Z) = (d\eta)(\alpha X, \alpha(\alpha^{-1}Z)) = \alpha^*(d\eta)(X, \alpha^{-1}Z) = d\eta(X, \alpha^{-1}Z) = 0 \ ,$$

which implies $i(\alpha X)d\eta = 0$. Therefore $\mathscr{E}^{2q}$ and also $\mathscr{E}^{2p}$ are invariant under α. Next, we show (2.5). Let $X \in \mathscr{E}^{2p}$. Then we get

$$(2.8) \qquad (L_\xi\psi)X = L_\xi(\psi X) - \psi L_\xi X \ .$$

By the definition of ψ we have $\psi X = \phi X$. Since $\mathscr{E}^{2p}$ is invariant under $\exp t\xi$, $L_\xi X \in \mathscr{E}^{2p}$ and therefore $\psi L_\xi X = \phi L_\xi X$. Thus

$$(L_\xi\psi)X = L_\xi(\phi X) - \phi L_\xi X = (L_\xi\phi)X \ ,$$

and $(L_\xi\psi)X = 0$ by (1.11). If $X \in \mathscr{E}^{2q} \oplus \mathscr{E}^1$, then $(L_\xi\psi)X = 0$ follows from (2.8). Hence we have $L_\xi\psi = 0$. Further, $L_\xi\Psi = 0$ follows from $\Psi(X, Y) = g(X, \psi Y)$ and (1.12), $L_\xi\theta = 0$ from $L_\xi\phi = 0, L_\xi\psi = 0$ and $\phi = \psi + \theta$, and $L_\xi g^\# = 0$ from (2.3) and $L_\xi d\eta = dL_\xi\eta = 0$. Finally, by (2.4) we have $L_\xi\bar{g} = 0$.

Remark. $d\Phi = 0$ is used only for $L_\xi\bar{g} = 0$.

Lemma 2.3. *For $X \in \mathscr{E}^{2n+1}$, we have*

$$(2.9) \qquad \bar{\nabla}_X\xi = -\psi X \ .$$

Proof. Since $L_\xi\bar{g} = 0$ by Lemma 2.2, we have $(\bar{\nabla}_X\eta)Y + (\bar{\nabla}_Y\eta)X = 0$, which implies

$$(2.10) \qquad d\eta(X, Y) = (\bar{\nabla}_X\eta)Y - (\bar{\nabla}_Y\eta)X = -2(\bar{\nabla}_Y\eta)X = -2\bar{g}(\bar{\nabla}_Y\xi, X) \ .$$

Next, we show that

$$(2.11) \qquad d\eta(X, Y) = 2\bar{g}(X, \psi Y)$$

for X, $Y \in \mathscr{E}^{2n+1}$. If X, $Y \in \mathscr{E}^{2p}$, then (2.11) is (2.3). If $X \in \mathscr{E}^{2q} \oplus \mathscr{E}^1$ or $Y \in \mathscr{E}^{2q} \oplus \mathscr{E}^1$, then both sides of (2.11) vanish. Thus we have (2.11), and finally (2.10) and (2.11) give (2.9).

Remark. $d\Phi = 0$ is used to apply $L_\xi\bar{g} = 0$. Thus, if $L_\xi\bar{g} = 0$, then Lemma 2.3 holds for a normal almost contact Riemannian manifold of rank $2p + 1$.

By $K(X_x, Y_x)$ we denote the sectional curvature with respect to $\bar{g}$ for a 2-plane determined by X_x and Y_x at x of M.

Theorem 2.4. *Let $M(\phi, \xi, \eta, g)$ be a quasi-Sasakian manifold of rank $2p + 1$, and assume that $g^\#$ defined by* (2.3) *is positive definite on $\mathscr{E}^{2p}$. Then, with respect to $\bar{g}$, we have*

$$\begin{aligned} \bar{K}(\xi_x, X_x) &= 1 \qquad \text{if } X_x \in \mathscr{E}^{2p}_x - 0 \\ &= 0 \qquad \text{if } X_x \in \mathscr{E}^{2q}_x - 0 \ . \end{aligned}$$

Proof. Let $X \in \mathscr{E}^{2p} \oplus \mathscr{E}^{2q}$ and assume that X is a unit vector field (locally). Then, by (2.5) and (2.9),

$$\bar{g}(\bar{R}(\xi, X)\xi, X) = \bar{g}((\bar{\nabla}_{[\xi,X]} + \bar{\nabla}_X\bar{\nabla}_\xi - \bar{\nabla}_\xi\bar{\nabla}_X)\xi, X) = -\bar{g}(\psi^2 X, X) .$$

Thus, if $X_x \in \mathscr{E}_x^{2p}$, then $K(\xi_x, X_x) = 1$; if $X_x \in \mathscr{E}_x^{2q}$, then $K(\xi_x, X_x) = 0$.

Proposition 2.5. *In a quasi-Sasakian manifold, we have*

$$(2.12)\qquad \begin{aligned}(\nabla_X\Phi)(Y, Z) &= \eta(Z)(\nabla_X\eta)(\phi Y) - \eta(Y)(\nabla_X\eta)(\phi Z)\\ &(= \eta(Z)g(\nabla_X\xi, \phi Y) - \eta(Y)g(\nabla_X\xi, \phi Z)) .\end{aligned}$$

If M is of rank $2p + 1$ and $\nabla_X\xi = -\psi X$, then

$$(2.13)\qquad \begin{aligned}(\nabla_X\Phi)(Y, Z) &= \eta(Y)g(X, Z) - \eta(Z)g(X, Y)\\ &+ \eta(Y)g(\theta^2 X, Z) - \eta(Z)g(\theta^2 X, Y) .\end{aligned}$$

If M is of rank $2p + 1$ and $\bar{\Phi}$ is also closed for the metric $\bar{g}$ defined by (2.4), *then* (2.13) *holds for $\bar{\nabla}, \bar{\Phi}, \bar{g}$.*

Proof. In [4] under the assumptions $N^1 = 0, d\Phi = 0$ and $L_\xi g = 0$, it was proved that

$$\nabla_i\Phi_{lk} = -\eta_l\nabla_i\eta_h\phi_k^h - \eta_k\nabla_j\eta_i\phi_l^j ,$$

which is nothing but (2.12) since $\nabla_j\eta_i = -\nabla_i\eta_j$. If M is of rank $2p + 1$ and $\nabla_X\xi = -\psi X$, then we obtain (2.13) from (2.12) on account of (1.4), $\phi\psi = \psi^2$, and $\psi^2 = -I + \xi \otimes \eta - \theta^2$. If $\bar{\Phi}$ is closed, we have (2.12) for $\bar{\nabla}, \bar{\Phi}, \bar{g}$, and hence the last statement of Proposition 2.5 follows from (2.9).

Next we have (cf. [1, Theorem 5.2])

Corollary 2.6. *A quasi-Sasakian manifold is cosymplectic if and only if $\nabla\Phi = 0$ (or equivalently $\nabla\phi = 0$).*

In fact, if a quasi-Sasakian manifold is cosymplectic, then $d\eta = 0$ and $L_\xi g = 0$, which imply $\nabla\eta = 0$. Thus by (2.12) we have $\nabla\Phi = 0$. The converse follows from $[\phi, \phi] = 0$ and (1.5).

3. Locally product quasi-Sasakian manifolds

Let $M_1^{2p+1}(\phi_1, \xi_1, \eta_1, g_1)$ be a Sasakian manifold, and $M_2^{2q}(J_2, G_2)$ a Kählerian manifold. Then $M_1 \times M_2$ has a quasi-Sasakian structure (ϕ, ξ, η, g) of rank $2p + 1$ scuh that

$$(3.1)\qquad \phi X = (\phi_1 X_1, J_2 X_2) ,$$

$$(3.2)\qquad \xi = (\xi_1, 0) ,$$

$$(3.3)\qquad \eta(X) = \eta_1(X_1) ,$$

$$(3.4)\qquad g(X, Y) = g_1(X_1, Y_1) + G_2(X_2, Y_2)$$

for the canonical decompodition $X = (X_1, X_2)$ of a vector field X on $M_1 \times M_2$ (cf. [1, Theorem 3.2]).

Conversely, we have

Theorem 3.1′. *Let $M(\phi, \xi, \eta, g)$ be a quasi-Sasakian manifold (more generally, a normal almost contact Riemannian manifold) of rank $2p + 1$. If $g^\#$ defined by (2.3) is positive definite on $\mathscr{E}^{2p}$, and $\bar{\nabla}\theta = 0$ with respect to the Riemannian metric $\bar{g}$ defined by (2.4), then $(\phi, \xi, \eta, \bar{g})$ is also a quasi-Sasakian structure of rank $2p + 1$, and $M(\phi, \xi, \eta, \bar{g})$ is locally the product of a Sasakian manifold and a Kählerian manifold.*

Proof. Clearly, $\bar{\nabla}_X\theta = 0$ implies $\bar{\nabla}_X\theta^2 = 0$ and $[\phi, \phi] = 0$. Then the almost product Riemannian structure (defined by $-\phi^2 + \xi \otimes \eta$ and $-\theta^2$) is integrable. Let x be an arbitrary point of M. Then we have some open set W containing x such that $W = U^{2p+1} \times V^{2q}$, which is a Riemannian product. From (2.11) and $\bar{\nabla}\theta = 0$, it follows that $2\bar{\Psi} = d\eta$ is closed, $\bar{\nabla}\bar{\Theta} = 0$ and, in particular, $d\bar{\Theta} = 0$, so that $\bar{\Phi} = \bar{\Psi} + \bar{\Theta}$ is closed. Hence the structure $(\phi, \xi, \eta, \bar{g})$ is quasi-Sasakian, and $L_\xi\bar{g} = 0$ by (1.12). In order that $U^{2p+1} \times V^{2q}$ be the product of a Sasakian manifold U^{2p+1} and a Kählerian manifold V^{2q}, it must be shown that

$$\bar{\nabla}_X\xi = 0 \qquad \text{for } X \in \mathscr{E}^{2q}\,, \tag{3.5}$$

$$\bar{\nabla}_X\phi = 0 \qquad \text{for } X \in \mathscr{E}^{2q}\,. \tag{3.6}$$

(3.5) follows from Lemma 2.3 (cf. remark to Lemma 2.3), and (3.6) is equivalent to $\bar{\nabla}_X\bar{\Psi} = 0$ for $X \in \mathscr{E}^{2q}$. Since $\bar{\Phi} = \bar{\Psi} + \bar{\Theta}$ and $\bar{\nabla}\bar{\Theta} = 0$, we have $(\bar{\nabla}_X\bar{\Phi})(Y, Z) = 0$. On the other hand, an application of Proposition 2.5 to the quasi-Sasakian structure $(\phi, \xi, \eta, \bar{g})$ yields

$$(\bar{\nabla}_X\bar{\Phi})(Y, Z) = \eta(Z)(\bar{\nabla}_X\eta)(\phi Y) - \eta(Y)(\bar{\nabla}_X\eta)(\phi Z)\,. \tag{3.7}$$

Since $\bar{\nabla}_X\xi = 0$ implies $\bar{\nabla}_X\eta = 0$ for $X \in \mathscr{E}^{2q}$, we have $\bar{\nabla}_X\bar{\Phi} = 0$.

Now the Sasakian structure on U^{2p+1} and the Kählerian structure on V^{2q} defined in Proposition 2.1 (cf. remark to Proposition 2.1) give the product quasi-Sasakian structure on $U^{2p+1} \times V^{2q}$, which and the quasi-Sasakian structure on W, restriction of $(\phi, \xi, \eta, \bar{g})$ to W, are isomorphic by (3.5), (3.6) and $\bar{\nabla}\theta = 0$.

Theorem 3.1. *Let $M(\phi, \xi, \eta)$ be a normal almost contact manifold such that*

(i) $\qquad \eta \wedge (d\eta)^p \neq 0$ *and* $(d\eta)^{p+1} = 0$ *on* M,

(ii) $\qquad -(d\eta)(X, \phi X) \geq 0$ *for any* $X \in \mathscr{E}^{2n+1}$.

Then we have a normal almost contact Riemannian structure (ϕ, ξ, η, g) which admits the canonical almost product structure $(-\phi^2 + \xi \otimes \eta, -\theta^2)$. If $\nabla\theta = 0$, then $M(\phi, \xi, \eta, g)$ is locally the product of a Sasakian manifold of dimension $2p + 1$ and a Kählerian manifold of dimension $2n - 2p$.

In fact, let g' be any Riemannian metric associated with (ϕ, ξ, η). Then (ϕ, ξ, η, g') is a normal almost contact Riemannian structure, and therefore we obtain Theorem 3.1 by using Theorem 3.1′ for (ϕ, ξ, η, g').

4. A simple example

Let E^3 be a 3-dimensional Euclidean space with coordinates (x, y, z), and define ϕ, ξ, η, g by

$$\phi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & y & 0 \end{pmatrix},$$

$$\xi = (0, 0, 2), \qquad 2\eta = (-y, 0, 1),$$

$$4g = \begin{pmatrix} 1 + y^2 & 0 & -y \\ 0 & 1 & 0 \\ -y & 0 & 1 \end{pmatrix}.$$

Then (ϕ, ξ, η, g) is a Sasakian structure (cf. [3]). Let β be a non-constant positive function of x and y, i.e., $\beta(x, y) > 0$, and define

$$g^* = \beta g + (1 - \beta)\eta \otimes \eta .$$

Then (ϕ, ξ, η, g^*) is a normal almost contact Riemannian structure. In this case,

$$\Phi^* = \beta\Phi = \frac{1}{2}\beta d\eta = \frac{1}{4}\beta dx \wedge dy .$$

Since β is a function of x and y, we have $d\Phi^* = 0$, and therefore $E^3(\phi, \xi, \eta, g^*)$ is a quasi-Sasakian manifold of rank 3, which is not Sasakian.

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