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VARIATIONAL COMPLETENESS AND K-TRANSVERSAL DOMAINS

LAWRENCE CONLON

Introduction

Let M be a complete Riemannian manifold, and K a compact connected Lie group, and suppose that K acts differentiably and isometrically on M. Bott and Samelson [5] and R. Hermann [8], [9] have given classes of examples (all related to symmetric spaces) in which the following two properties are verified:

(A) There is a flat closed connected totally geodesic imbedded submanifold $T \subset M$ which meets every K-orbit and is orthogonal to the K-orbits at every point of intersection. (Such a submanifold T will be called a K-transversal domain.)

(B) The action of K on M is variationally complete in the sense of [5, p. 974].

These properties make possible some interesting applications of Morse theory to the spaces $\Omega = \Omega(M; N, q)$ of piecewise C^∞ paths from a K-orbit N to a point $q \in M$ (e.g., cf. [3], [4], [7], [5]). Here Ω is topologized as in [5]. In particular, if K has a fixed point $p \in M$ one can study in this way the loopspaces $\Omega(M) \simeq \Omega(M; p, q)$, while if M is contractible one obtains results on $N \simeq \Omega(M; N, q)$. (Here, as elsewhere in the paper, $\simeq$ denotes homotopy equivalence.)

We do not know whether (A) and (B) are equivalent, but we will prove the following result which does not seem to have been noticed before.

Theorem I. (A) *implies* (B).

As a consequence, a number of interesting properties (which were established for the symmetric space cases by use of root systems — cf. [1], [5], [7]) can be shown to follow from (A). In particular, we obtain characterizations of the singular set, the Weyl group, and the Bott-Samelson K-cycles as follows.

Theorem II. *Let $T \subset M$ be a K-transversal domain and assume $\pi_1(M)=0$. Then there is a finite collection $\{P_1, P_2, \cdots, P_r\}$ of closed connected flat submanifolds of codimension one in T, together with positive integers $m(i), i = 1, \cdots, r$, such that for each $x \in T$,*

$$\dim (N_x) = \dim (M) - \dim (T) - \sum_{i \in I_x} m(i) ,$$

where N_x is the K-orbit of x and $I_x = \{i \colon x \in P_i\}$.

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In what follows we continue to denote the K-orbit of a point $x \in M$ by N_x.

Definition. Each P_i as above is called a singular variety of multiplicity $m(i)$, and each connected component of the complement in T of the union of the singular varieties a Weyl domain in T. The Weyl group $W = W(K, T)$ is the group of transformations of T produced by those $a \in K$ such that $a \cdot T = T$.

Theorem III. *If T is a K-transversal domain and $\pi_1(M) = 0$, then the orthogonal reflection of T in each singular variety P_i exists, W is a finite group generated by all such reflections, and W permutes simply transitively the set of Weyl domains in T. If $x \in T$ lies on no singular variety, then W permutes simply transitively the set $N_x \cap T$.*

Definition. $K_i = \{a \in K\colon a \cdot q = q,$ all $q \in P_i\}$, $i = 1, \cdots, r$, and $K_T = \{a \in K\colon a \cdot q = q$, all $q \in T\}$.

Let $P = (P_{i_1}, \cdots, P_{i_k})$ be a sequence of singular varieties, repetitions allowed, and form the compact manifold

$$\Gamma_P = K_{i_1} \times_{K_T} K_{i_2} \times_{K_T} \cdots \times_{K_T} (K_{i_k}/K_T) \ .$$

If $x \in M$ and $q \in T$, there are topological imbeddings (well-defined up to homotopy)

$$f_P\colon \Gamma_P \to \Omega(M; N_x, q)$$

defined via geodesic polygons on M exactly as in [3, p. 40]. These provide mod 2 homology cycles in Ω of dimension $= \dim(\Gamma_P)$ (called the Bott-Samelson K-cycles) which are integral cycles if each Γ_P is orientable. Selecting q so as to lie on no singular variety, one obtains by use of the Morse theory a canonical homology basis consisting of certain of the K-cycles, one for each K-transversal geodesic in Ω (cf. [5, Theorem I] and the correction in [6]).

Γ_P is an iterated fiber bundle with successive fibers K_{i_j}/K_T and final base space K_{i_j}/K_T, and each of these fibrations admits a canonical cross section. The following theorem, generalizing results of Araki [1] and the author [7], completes the picture.

Theorem IV. *Let $\pi_1(M) = 0$. If there is a K-transversal domain $T \subset M$, and $P_i \subset T$ is a singular variety with multiplicity $m(i)$, then $K_i/K_T = S^{m(i)}$.*

These theorems may lead to the discovery of interesting new examples of (B). Indeed, they may enable us to classify all cases of (A), at least for simply connected M. Classification attempts should probably begin with the "infinitesimal" case $M = \boldsymbol{R}^n$ where the action of K is an orthogonal representation satisfying (A). Not only do these form the simplest interesting class of examples, but the general case gives rise to these infinitesimal examples at each $x \in M$, as will be described in Theorem (3.7).

The standard cases of orthogonal representations satisfying (A) are the isotropy representations coming from Riemannian symmetric spaces. In addition, any orthogonal representation of K transitive on the unit sphere of $\boldsymbol{R}^n$ trivially satisfies (A). The only other examples presently known to the author are a representation of Spin (8) on $\boldsymbol{R}^{16}$ and one of Spin (7) on $\boldsymbol{R}^{15}$, both having K-transversal domains of dimension two. In each case the singular varieties are a pair of mutually orthogonal lines, the multiplicities are $m(1) = m(2) = 7$ in the first case and $m(1) = 6$ and $m(2) = 7$ in the second case, and the respective principal orbit types are

$$K/K_T = \text{Spin}\,(8)/G_2 = S^7 \times S^7\ ,$$
$$K/K_T = \text{Spin}\,(7)/SU(3) = S^6 \times S^7\ .$$

Notational conventions. Throughout this paper $T \subset M$ will be a K-transversal domain. If $x, y \in M$, the Riemannian distance between these points will be denoted by $d(x, y)$. For $x \in M, K_x \subset K$ denotes the isotropy subgroup of x, hence $N_x = K/K_x$. V_x will denote the orthogonal complement of the tangent space $T_x(N_x)$ in $T_x(M)$.

If the orbit N_x is fixed throughout a certain discussion, we will denote $\Omega(M; N_x, q)$ by Ω_q. If q is also fixed, we will simply write Ω. If $a > 0$, the symbols $\Omega^a(M; N_x, q)$, Ω^a_q, Ω^a will denote respectively the subspaces of the above spaces consisting of paths of length $\leq a$. If u: $[a, b] \to M$ is a smooth path, $\dot{u}(t)$ denotes the tangent field along $u, a \leq t \leq b$.

The Lie algebras of groups K, K_T, K_i, K_x, etc. will be denoted by corresponding $\boldsymbol{k}, \boldsymbol{k}_T, \boldsymbol{k}_i, \boldsymbol{k}_x$, etc. The identity components of Lie groups K, K_T, etc. will be denoted by K^0, K^0_T, etc. Whenever homology theory is employed, the singular theory will be understood. In addition, notations already established in the introduction will be used without further explanation.

1. The proof of Theorem I

We assume (A) and seek to establish (B). This will require a number of propositions and lemmas, some of which are interesting in their own right.

(1.1) Proposition. *Let $N \subset M$ be a K-orbit or maximal dimension. Then* $\dim(N) = \dim(M) - \dim(T)$.

Proof. $\dim(N) \leq \dim(M) - \dim(T)$ is immediate from (A). We will prove the reverse inequality.

We may assume that N is a principal K-orbit. The union of all principal K-orbits is an open set $U \subset M$, [2], and $U \cap T$ is a nonempty open subset of T. Let $W \subset U \cap T$ be a convex open set of T. If $x, y \in W$, let s: $[0, 1] \to W$ be a geodesic with $s(0) = x, s(1) = y$. Then $K_x = K_{s(t)} = K_y$, $0 \leq t \leq 1$; this is by the slice property [2] together with the fact that the K-orbit of each $s(t)$ is principal. Since W is open in T and T is totally geodesic, $K_x = K_T$ for all $x \in W$ and hence for all $x \in U \cap T$. In particular, $N = K/K_T$.

Define

$$\phi\colon T \times (K/K_T) \to M$$

by

$$\phi(x, yK_T) = y \cdot x \ .$$

Clearly ϕ is smooth and well defined, and $\mathrm{Im}\,(\phi) = M$. Since $T \times (K/K_T)$ is second countable and ϕ is C^∞, we have (cf. [10, Théorème 3])

$$\dim (T \times N) \geq \dim (M) \ ,$$

hence

$$\dim (N) \geq \dim (M) - \dim (T) \ .$$

(1.2) Corollary. *Any two K-transversal domains in M are conjugate under the action of K.*

Proof. Let T and T' satisfy (A), and N be a maximal K-orbit. Choose $x \in N \cap T$ and $y \in N \cap T'$. There is $a \in K$ such that $a \cdot y = x$. Then $a \cdot T'$ and T are totally geodesic closed submanifolds of M, each containing x, orthonormal to N at x, and of complementary dimension to N. This implies $T = a \cdot T'$. q.e.d.

Let $x \in M$. By Corollary (1.2) no generality is lost by assuming $x \in T$. Let U be the open ε-ball in V_x with center 0. For $\varepsilon > 0$ sufficiently small, $S = \exp(U)$ is a K-slice at x and is invariant under K_x^0. For $b \in S$, let Q_b denote the K_x^0-orbit of b. Thus $Q_b \subset S$. Note that $T \cap S$ is an open convex set in T.

(1.3) Lemma. *$T \cap S$ meets Q_b for all $b \in S$, and these manifolds are orthogonal at each point of intersection. If Q_b has maximal dimension, then* $\dim (Q_b) = \dim (S) - \dim (T)$.

Proof. Orthogonality is immediate from (A). First asssume $b \in S \cap T$ and suppose $\dim (Q_b) \geq \dim (Q_c)$ for all $c \in S \cap T$. Remark that

$$\begin{aligned} \dim (Q_b) &= \dim (K_x) - \dim (K_b) \\ &= (\dim (K) - \dim (K_b)) - (\dim (K) - \dim (K_x)) \\ &= \dim (N_b) - \dim (N_x) \\ &= \mathrm{codim}\,(N_x) - \mathrm{codim}\,(N_b) \\ &= \dim (S) - \mathrm{codim}\,(N_b) \ . \end{aligned}$$

Since $T \cap S$ meets all K-orbits sufficiently near N_x and the maximal K-orbits unite to form a dense subset of M [2], N_b must be a maximal K-orbit. Thus, by Proposition (1.1),

$$\dim (Q_b) = \dim (S) - \dim (T) \ .$$

Now let $c \in S$. If σ: $[0, 1] \to S$ is a minimal path from Q_b to Q_c, we may suppose $\sigma(0) = b$. Then $\dot{\sigma}(0) \perp T_b(Q_b)$. By the above formula and the fact that $T \cap S$ is totally geodesic in M and hence in S, we conclude that σ lies on $T \cap S$. Thus $\sigma(1) \in Q_c \cap S \cap T \neq \emptyset$, and all assertions follow.

(1.4) Corollary. *If T and T' are K-transversal domains and $x \in T \cap T'$, then there is $a \in K_x^0$ such that $a \cdot T = T'$.*

(1.5) Corollary. *Every $v \in V_x$ is tangent to some K-transversal domain.*

These corollaries are easy consequences of Lemma (1.3).

Let $N \subset M$ be any K-orbit, let $q \in M$, and choose $s \in \Omega(M; N, q)$ a critical path for the energy functional (i.e., s is a K-transversal geodesic as in [5, p. 967]). For $r > 0$ sufficiently small, let

$$\pi: \Sigma_r \rightarrow N$$

be the normal sphere bundle of radius r for $\Sigma_r \subset M$, and t_0 be the smallest positive value such that $s(t_0) \in \Sigma_r$ (hence the length of $s|_{[0,t_0]}$ is r). Let $s(t_0) = x$ and remark that $N_x \subset \Sigma_r$.

Definition. J_s^N is the set of Jacobi fields along s produced by the variations

$$V: [-\varepsilon, \varepsilon] \times [0, 1] \rightarrow M$$

such that, for each $\mu \in [-\varepsilon, \varepsilon]$, s_μ defined by $s_\mu(t) = V(\mu, t)$ is a geodesic satisfying

1) $s_\mu(0) \in N$,
2) length of s_μ = length of s ,
3) $\dot{s}_\mu(0) \perp T_{s\mu(0)}(N)$,

$-\varepsilon \leq \mu \leq \varepsilon$.

It is well known that J_s^N is a linear space of dimension $= \dim(M) - 1$. Indeed, the map

$$J_s^N \rightarrow T_x(\Sigma_r)$$

defined by

$$U \mapsto U(t_0)$$

is a linear isomorphism.

Definition. $\Lambda_s(N) = \{U \in J_s^N : U(1) = 0\}$.

Property (B) in the introduction means that for every choice of q, N, and s as above each $U \in \Lambda_s(N)$ is produced by a variation

$$V(\mu, t) = \sigma(\mu) \cdot s(t) ,$$

where σ is a one-parameter subgroup of K_q^0.

If $U \in J_s^V$, write $U = U' + U''$, U' and $U'' \in J_s^N$ with $U'(t_0) \in T_x(N_x)$ and $U''(t_0) \in T_x(\Sigma_r)$ such that $U''(t_0) \perp T_x(N_x)$. Evidently this is a unique decomposition.

(1.6) Lemma. *If $U \in J_s^N$, then U' as above is produced by a variation*

$$V(\mu, t) = \sigma(\mu)\cdot s(t) ,$$

where σ is a one parameter subgroup of K.

Proof. Given $Y \in T_x(N_x)$, a suitable choice of σ produces $Z \in J_s^N$ with $Z(t_0) = Y$. Any $Z \in J_s^N$ is uniquely determined by $Z(t_0)$. Thus choose $Y = U'(t_0)$ and obtain $Z = U'$.

(1.7) Lemma. $\dot{s}(t_0) \in T_x(T)$ *for every K-transversal domain $T \subset M$ containing x.*

Proof. $\dot{s}(t_0) \perp T_x(N_x)$, hence is tangent to some T by (1.5). Let T' be another K-transversal domain through x. By Corollary (1.4) there is $a \in K_x^0$ such that $a\cdot T = T'$. Then $s' = (a\cdot s)|_{[0,t_0]}$ is a geodesic of length r meeting N orthogonally and with $s'(t_0) = x$. Therefore $s' = s|_{[0,t_0]}$, so $\dot{s}(t_0)$ is also tangent to T' at x.

(1.8) Lemma. *If $U \in \Lambda_s(N)$, then $U'' = 0$.*

Proof. $U''(t_0) \perp T_x(N_x)$, hence is tangent to some T containing x by Corollary (1.5). $\dot{s}(t_0)$ is tangent to this same T by Lemma (1.7), so s lies on T. In particular, $s(0) \in T$ and $\Sigma_r \cap T$ is a sphere in T with center $s(0)$ and of radius r. $U''(t_0)$ is tangent to this sphere.

Let u be a smooth curve in $T \cap \Sigma_r$ with $u(0) = x, \dot{u}(0) = U''(t_0)$. Consider the geodesic variation $V(\mu, t) = s_\mu(t)$ where s_μ is the unique geodesic with $s_\mu(0) = s(0), s_\mu(t_0) = u(\mu)$, and length of $s_\mu|_{[0,t_0]} = r$, $-\varepsilon \le \mu \le \varepsilon$. Clearly $\mathrm{Im}(V) \subset T$ and V produces $Z \in J_s^N$. $Z(t_0) = U''(t_0)$, so $Z = U''$.

But T is flat and U'' is also a Jacobi field along s in T, $U''(0) = 0$. It follows that U'' is identically zero or $U''(t) \neq 0$ for all $t \neq 0$. In this latter case, $0 \neq U''(1) \in T_{s(1)}(T) \perp T_{s(1)}(N_{s(1)})$. Since $U = U' + U''$ where U' is produced by a one-parameter group of K, we see that $U(1) \neq 0$ unless U'' is identically zero. But $U \in \Lambda_s(N)$ implies $U(1) = 0$. q.e.d.

The proof of Theorem I is now easy. We have seen, under assumption (A), that any $U \in \Lambda_s(N)$ is produced by a variation

$$V(\mu, t) = \sigma(\mu)\cdot s(t) ,$$

where σ is a one-parameter subgroup of K. Since $U(1) = 0$, σ must be contained in K_q^0; this proves (B).

2. The singular set and the Weyl group

We assume (A), hence (B), and we further assume $\pi_1(M) = 0$. The latter assumption assures us of the following crucial fact.

(2.1) Lemma. *Let $q \in T$ such that N_q is an orbit of maximal dimension, and let $N \subset M$ be any K-orbit. Then there is one and only one K-transversal geodesic of index zero joining N to q.*

Proof. Let $\Omega = \Omega(M; N, q)$. N is path connected since K is. Since also $\pi_1(M) = 0$, the exact sequence

$$\pi_1(M) \to \pi_0(\Omega) \to \pi_0(N)$$

implies that Ω is path connected, hence $H_0(\Omega; \boldsymbol{Z}_2) = \boldsymbol{Z}_2$. Since (B) holds and the orbit of q is maximal, Theorem I of [5] applies and gives the desired conclusion. q.e.d.

We will repeatedly reduce undesirable possibilities to contradictions of Lemma (2.1). This approach is basically due to Bott and Samelson (e.g., cf. [5, Proposition 2.14, pp. 1022–1023]).

(2.2) Proposition. *Under the above assumptions, every K-orbit in M of maximal dimension is principal.*

Proof. Let $y \in T$ and suppose that N_y has maximal dimension. A sufficiently small ball neighborhood U of y in T is a K-slice at y. Thus, if $x \in U$, then $K_x \subset K_y$. By the density of the principal orbits there is $x_0 \in U$ such that N_{x_0} is principal. We must prove $K_{x_0} = K_y$. It not, let $a \in K_y - K_{x_0}$, and let s be a geodesic from x_0 to y in U. Using (B) and the fact that all orbits crossed by s have the same dimension (indeed, all K-orbits of points of U have maximal dimension), we see that s has Morse index zero as a critical point in $\Omega = \Omega(M; N_{x_0}, y)$. Then $a \cdot s \in \Omega$ is also a K-transversal geodesic having index zero. $s \neq a \cdot s$ since $a \notin K_{x_0}$, and this contradicts (2.1).

(2.3) Lemma. *If $y \in T$, then each component of $P_y = \{x \in T: K_y \subset K_x\}$ is a closed flat totally geodesic submanifold of T.*

Proof. P_y is closed in T, hence so is each component. Since T is flat, it will be enough to show that each component of P_y is a submanifold of T geodesically immersed at each point. Let $x \in P_y$, and $F_x = \{v \in T_x(T): a \cdot v = v,$ each $a \in K_y\}$. Clearly F_x is a linear subspace of $T_x(T)$, $\exp(F_x) \subset P_y$, and for a sufficiently small ball neighborhood U of x in T, $U \cap P_y = U \cap \exp(F_x)$. Thus all assertions follow. q.e.d.

By the proof of Proposition (1.1), the set of $y \in T$ such that N_y is principal is exactly the set of $y \in T$ such that $K_y = K_T$. The complement of this set is called the singular set $S^* \subset T$. By Proposition (2.2), S^* is exactly the set $\{y \in T: \dim(K_y) > \dim(K_T)\}$ and can also be identified as $\{y \in T: P_y \neq T\}$.

Definition. $P \subset T$ is called a singular variety if P is a component of P_y for some $y \in S^*$, but is not properly contained in a component of P_z for any $z \in S^*$.

Remark that S^* is the union of the singular varieties. The proof of Theorem II will show that this notion of singular variety agrees with the one already formulated, which depends on Theorem II.

(2.4) Lemma. *Let $P \subset T$ be a singular variety. Then there are $y \in P$ and an ε-neighborhood U of y in T such that $P \cap U = S^* \cap U$ and U is symmetric about $P \cap U$.*

Proof. The symmetry will be an obvious consequence of Lemma (2.3) and the fact that the metric in T is locally Euclidean. Choose any $z_0 \in P$ and an ε_0-neighborhood U_0 of z_0 in T, which is a subset of a K-slice at z_0. Thus $z \in U_0$ implies $K_z \subset K_{z_0}$. If $K_z = K_{z_0}$ for every $z \in P \cap U_0$ choose $y = z_0$, but if not choose $z_1 \in P \cap U_0$ with $K_{z_1} \neq K_{z_0}$. Then let U_1 be an ε_1-neighborhood of z_1, which is a subset of a K-slice at z_1. Iterate this procedure until obtaining $z_n \in P$ and an ε_n-neighborhood U_n of z_n in T such that $z \in P \cap U_n$ implies $K_z = K_{z_n}$. Let $y = z_n$, $\varepsilon = \varepsilon_n$, $U = U_n$. Note that $P \subset P_y$. If $x \in S^* \cap U$, then $K_x \subset K_y$ and the shortest geodesic from x to y is contained in $P_x \cap U \subset S^* \cap U$. Thus if $x \notin P \cap U$, P is properly contained in a component of P_x, contradicting the definition of P. Thus $S^* \cap U \subset P \cap U$ and the reverse inclusion is obvious.

(2.5) Proposition. *If $P \subset T$ is a singular variety, then P has codimension one in T.*

Proof. Let $y \in P$ and U be as in Lemma (2.4). Let $x \in U - P$ and let $x' \in U$ be the reflection of x in P. Let s be the minimal geodesic in T (hence in U) from x' to x. s cannot be minimal from $N_{x'}$ to x since it properly crosses P and hence has Morse index greater than zero.

Assume without loss of generality that

$$d(x, x') < \text{radius}\,(U) - d(y, x) .$$

If $\sigma \in \Omega(M; N_{x'}, x)$ is a minimal geodesic (hence, since $x \notin S^*$, σ lies on T), then the point $x'' = \sigma(0) \in N_{x'}$ must lie in U. By the above remarks, $x'' \neq x'$.

If codim $(P) > 1$, then there is $z \in U - P$ which can be joined to x' and to x'' by geodesics in $U - P$. This implies that there are two geodesics of index zero in $\Omega(M; N_{x'}, z)$, contradicting Lemma (2.1). q.e.d.

In order to prove that there are only finitely many distinct singular varieties (these will then constitute the set $\{P_1, \cdots, P_r\}$ of Theorem II) we will investigate certain properties of the Weyl group W.

(2.6) Lemma. *Let N be a principal K-orbit. Then W permutes the set $N \cap T$ simply transitively.*

Proof. Clearly W permutes $N \cap T$. If $x, y \in N \cap T$, there is $a \in K$ with $a \cdot x = y$. Since T is totally geodesic, meets N orthogonally at x and y, and $\dim (T) = \dim (M) - \dim (N)$, we conclude that $a \cdot T = T$. Thus a defines $w \in W$ such that $w(x) = y$. Finally, if $w(x) = x$, the same argument which was used in Proposition (2.2) shows the existence of a neighborhood U of x in T such that $w|_U =$ identity. It follows that w is the identity of W.

(2.7) Corollary. *W is a finite group.*

Proof. By Lemma (2.6), the order $|W|$ is the cardinality of $N \cap T$. Since N is compact and orthogonal to T at each point of $N \cap T$, one concludes that $N \cap T$ is finite.

(2.8) Proposition. *If $P \subset T$ is a singular variety, the orthogonal reflection of T in P is well defined and is an element of W.*

Proof. Let U, y, x, x' all be as in the proof of Proposition (2.5), the radius of U being denoted by r. Arguing as in Proposition (2.5), we produce $x'' \in N_{x'} \cap U$ on the same side of $P \cap U$ as x.

By Lemma (2.6) let $w_0 \in W$ such that $w_0(x') = x''$. Let $y' = w_0(y)$ and remark that

$$d(y, y') \leq d(y, x'') + d(x'', y') = d(y, x'') + d(x', y) < 2r .$$

Since W is finite and $r > 0$ was arbitrarily small, we can assume $d(y, y')$ less than the minimal positive value of $d(y, w(y))$ as w ranges over W, that is, $w_0(y) = y$; hence $w_0(U) = U$ and $w_0(P) = P$. Also, w_0 interchanges the two sides of $P \cap U$ in U.

Suppose $w_0^2 \neq$ identity. Then there is $z \in U - P$ such that $w_0^2(z) \neq z$. But $w_0^2(z)$ is on the same side of $P \cap U$ in U as is z, leading to our usual contradiction. Thus $w_0^2 =$ identity.

Suppose $w_0|_P \neq$ identity. Then there is $z \in P \cap U$ with $w_0(z) \neq z$. But then there are K-transversal geodesics of Morse index zero in $\Omega(M; N_z, x)$, one from z to x in U and one from $w_0(z)$ to x in U. As usual, this contradicts Lemma (2.1), hence proves $w_0|_P =$ identity.

Since w_0 is an isometry of T, it follows from the above that w_0 must be the orthogonal reflection of T in P.

(2.9) Corollary. *There are only finitely many distinct singular varieties.*

Proof. Given a singular variety P, let $w_0 \in W$ be the orthogonal reflection of T in P. Suppose w_0 is also the orthogonal reflection of T in a singular variety $P' \neq P$. Clearly $P' \cap P = \emptyset$. If $p \colon \boldsymbol{R}^n \to T$ is the universal covering, $p^{-1}(P) \cup p^{-1}(P')$ is a family of parallel hyperplanes. If s is a geodesic on T, $s(0) \in P'$ and $\dot{s}(0) \perp T_{s(0)}(P')$, then s lifts to a straight line on $\boldsymbol{R}^n$ perpendicular to the family of hyperplanes. Thus s meets P. If σ is a minimal segment of s from P' to P, then $\sigma + w_0(\sigma)$ is clearly a closed geodesic on T and hence describes a geodesic circle $S^1 \subset T$. The above reasoning then shows that S^1 meets any singular veriety P'' such that w_0 is the orthogonal reflection in P''. Clearly S^1 is invariant under w_0, w_0 is nontrivial on S^1, and w_0 has at least two fixed points on S^1. Since w_0 is an isometric involution of S^1, it follows that these are the only fixed points of w_0 on S^1, and hence that P and P' are the only singular varieties relative to which w_0 is the orthogonal reflection. Since W is finite, our assertion follows. q.e.d.

We now let $\{P_1, P_2, \cdots, P_r\}$ denote the set of distinct singular varieties, and $w_i \in W$ be the orthogonal reflection of T in P_i, $i = 1, \cdots, r$.

(2.10) Corollary. *The reflections w_i, $i = 1, \cdots, r$, generate W, which is a simply transitive group on the set of Weyl domains in T.*

Proof. Since W leaves the singular set S^* invariant, it is clear that W permutes the set of Weyl domains. If $w \in W$, $D \subset T$ is a Weyl domain, and $w(D) = D$, then our standard argument using Lemma (2.1) shows that $w|_D$ is the identity, and hence that w is the identity. Given Weyl domains D_1 and D_2, let s be a geodesic on T with $s(0) \in D_1$ and $s(1) \in D_2$. By slightly moving s we can insure that it crosses the singular varieties P_i singly. Let $P_{i_1}, P_{i_2}, \cdots, P_{i_q}$ be the singular varieties crossed by s in order. Then $w_{i_1}w_{i_2}\cdots w_{i_q} = w \in W$ satisfies $w(D_2) = D_1$, and all assertions follow. q.e.d.

We have verified every assertion of Theorem III and identified the singular varieties of Theorem II. The proof of Theorem II will be completed in the following section.

3. Definition and properties of $m(i)$

Let $\boldsymbol{k}_i$ and $\boldsymbol{k}_T$ denote the respective Lie algebras of K_i and K_T, and write

$$\boldsymbol{k}_i = \boldsymbol{k}_T \oplus \boldsymbol{m}_i \ ,$$

where $\boldsymbol{m}_i$ is the orthogonal complement of $\boldsymbol{k}_T$ in $\boldsymbol{k}_i$.

Definition. $m(i) = \dim (\boldsymbol{m}_i)$ is called the multiplicity of P_i.

(3.1) Lemma. *The set $A_i = \{y \in P_i \colon K_y = K_i\}$ is an open dense subset of P_i.*

Proof. Let $x \in P_i$. Given a sufficiently small neighborhood U of x in P_i, standard application of the slice property (cf. proof of Lemma (2.4)) shows that there are $y \in U$ and an open neighborhood V of y in U such that every $z \in V$ satisfies $K_z \subset K_i$. The reverse inclusion is evident and the lemma follows.

(3.2) Lemma. *If $x \in T$ lies on one and only one P_i, then $K_x^0 = K_i^0$. Consequently,* $\dim (N_x) = \dim (M) - \dim (T) - m(i)$.

Proof. By Corollary (2.9) we can find an ε-neighborhood U of x in T such that $P_i \cap U = S^* \cap U$. Suppose $K_x^0 \neq K_i^0$ so that $\dim (K_x) > \dim (K_i)$. Choose $z \in U \cap A_i$ and let $z' \in P_i \cap U$ be the reflection of z in x. The minimal geodesic s from z' to z does not have index zero in $\Omega(M; N_{z'}, z)$ since it properly crosses x and $\dim (K_x) > \dim (K_i)$. Thus if z was chosen sufficiently near x (possible by Lemma (3.1)), there is $z'' \in N_{z'} \cap U \cap P_i$ such that the minimal geodesic from z'' to z does have Morse index zero. Thus z', z'' are distinct and both lie in $P_i \cap U \cap N_{z'}$. Let $q \in U - P_i$. Then the minimal geodesics from z' to q and from z'' to q both have index zero in $\Omega(M; N_{z'}, q)$, contradicting Lemma (2.1). q.e.d.

Let $x \in T$ and let $I_x \subset \{1, 2, \cdots, r\}$ be as in Theorem II. If $I_x = \emptyset$, then Proposition (1.1) gives the final assertion of Theorem II, while if I_x is a singleton then Lemma (3.2) gives the result. Without loss of generality assume $I_x = \{1, 2, \cdots, n\}$, $n \leq r$.

For a sufficiently small ε-neighborhood U of x in T we have $U \cap P_j = \emptyset$ for $j > n$, and $U \cap P_i$ is a disk of codimension one in U dividing $U - P_i$ into two components, $1 \leq i \leq n$. Let $z \in U - S^*$, and choose $a > 0$ such that $d(x, z) < a < \varepsilon$.

(3.3) Lemma. $\Omega^a(M; N_z, x) \simeq K_x/K_T$.

Proof. Let $s \in \Omega^a$ be a K-transversal geodesic. By Corollary (1.4) and Corollary (1.5), there is $b \in K_x$ such that $b \cdot s$ lies on T and hence on $T \cap U$. Each of $w_1, \cdots, w_n$ leaves x and hence U invariant, and so each component of $U - P_1 \cup P_2 \cup \cdots \cup P_n$ contains one (and only one) element of $N_z \cap U$. These points are equidistant from x, and $b \cdot s$ is the minimal geodesic from some point of $N_z \cap U$ to x. Using $w_1, \cdots, w_n$, we see that

$$K_x/K_T = K_x \cdot s \subset \Omega^a$$

is the subset of all K-transversal geodesics in Ω^a. Since these geodesics are of index zero, [4, Theorem III] implies $\Omega^a \simeq K_x/K_T$. q.e.d.

Choose $x' \in U - S^*$ such that $c = d(x, x')$ satisfies $a + 3c < \varepsilon$. Keeping the endmanifold N_z fixed, we consider Ω^a_x and $\Omega^a_{x'}$.

(3.4) Lemma. $\Omega^a_x \simeq \Omega^a_{x'}$.

Proof. $\Omega^a_x \subset \Omega^{a+c}_x \subset \Omega^{a+2c}_x$ are deformation retracts since the three spaces contain exactly the same K-transversal geodesics. The same holds when x is replaced by x'. Let σ be the minimal geodesic from x to x', and define

$$\phi_\sigma \colon \Omega^a_x \to \Omega^{a+c}_{x'}$$

by

$$\phi_\sigma(u) = u + \sigma .$$

Similarly use σ^{-1} to define a map

$$\Omega^{a+c}_{x'} \to \Omega^{a+2c}_x$$

Using the above deformation retractions we may interpret these maps as

$$\phi_1 \colon \Omega^a_x \to \Omega^a_{x'} ,$$
$$\phi_2 \colon \Omega^a_{x'} \to \Omega^a_x ,$$

which are readily verified to be mutual homotopy inverses.

(3.5) Corollary. $\dim(K_x/K_T) = \sum_{i=1}^{n} m(i)$.

Proof. By (3.3) and (3.4),

$$H_*(\Omega^a_{x'}; \mathbf{Z}_2) = H_*(K_x/K_T; \mathbf{Z}_2) .$$

Without loss of generality, we can assume that each geodesic in U from $N_z \cap U$ to x' crosses the singular varieties singly. These are the only K-transversal geodesics in $\Omega^a_{x'}$, and correspond one-one to a basis in $H_*(\Omega^a_{x'}; \mathbf{Z}_2)$ by [5, Theorem I], the dimension of the homology class being the index of the geodesic. By Lemma (3.2) and [5, Proposition 9.2] each such geodesic has index given by the sum of the multiplicities $m(i)$ of the singular varieties P_i which it crosses. Thus the highest dimensional homology class in $H_*(K_x/K_T; \boldsymbol{Z}_2)$ is of dimension equal to the sum of the $m(i)$, $i = 1, \cdots, n$. Since K_x/K_T is a compact manifold, the desired conclusion is clear. q.e.d.

Theorem II follows immediately.

(3.6) Corollary. $\boldsymbol{k}_x = \boldsymbol{k}_T \oplus \sum_{i=1}^{n} \boldsymbol{m}_i$, *a direct sum.*

Proof. Suppose $1 \leq i < j \leq n$. Clearly $K_T \subset K_i \cap K_j$. The reverse inclusion also holds. Indeed, if $a \in K_i \cap K_j$, a leaves $T_x(P_i)$ and $T_x(P_j)$ pointwise fixed, hence a leaves $T_x(T)$ pointwise fixed, so $a \in K_T$. Thus $K_T = K_i \cap K_j$ and $\boldsymbol{m}_i \cap \boldsymbol{m}_j = 0$. This shows that $\boldsymbol{k}_T \oplus \sum_{i=1}^{n} \boldsymbol{m}_i$ is contained as a direct sum in $\boldsymbol{k}_x$. By Corollary (3.5) these vector spaces have the same dimension, so equality holds. q.e.d.

For $x \in M$, the isotropy representation of K_x on $T_x(M)$ restricts to

$$K_x^0 \times V_x \to V_x .$$

(3.7) Theorem. $K_x^0 \times V_x \to V_x$ *satisfies* (A).

Proof. Without loss of generality suppose $x \in T$. Let $\boldsymbol{T} = T_x(T) \subset V_x$ and $\boldsymbol{P}_i = T_x(P_i) \subset V_x$, for all $i \in I_x$. Again we assume $I_x = \{1, \cdots, n\}$; hence

$$\boldsymbol{k}_x = \boldsymbol{k}_T \oplus \sum_{i=1}^{n} \boldsymbol{m}_i .$$

The representation of K_x^0 on V_x induces a representation of $\boldsymbol{k}_x$ on V_x. If $X \in V_x$, then the tangent space at X to the K_x^0-orbit of X is identified with $\{A(X): A \in \boldsymbol{k}_x\}$. Furthermore, the representation of $\boldsymbol{k}_x$ is skew symmetric, so $A(X) \perp X$. Suppose $X \in \boldsymbol{T}$. If $A \in \boldsymbol{k}_T$, then $A(X) = 0$. If $A \in \boldsymbol{m}_i$ for some $i = 1, \cdots, n$, then $A(X) \perp \boldsymbol{P}_i$ and $A(X) \perp X$. If also $X \in \boldsymbol{P}_i$, then $A(X) = 0$. Thus in all cases $A(X) \perp \boldsymbol{T}$, so the K_x^0-orbit of any $X \in \boldsymbol{T}$ meets $\boldsymbol{T}$ orthogonally at X.

Since the map

$$\exp: V_x \to M$$

commutes with the action of K_x^0, for $X \in \boldsymbol{T} - \bigcup_{i=1}^{n} \boldsymbol{P}_i$ sufficiently near 0 the K_x^0-orbit of X is of maximal dimension among all the K_x^0-orbits in V_x. By Lemma (1.3) this dimension is exactly the codimension of $\boldsymbol{T}$ in V_x. By standard theory, any two K_x^0-orbits in V_x can be joined by a straight line orthogonal to each orbit at the respective endpoints. Hence $\boldsymbol{T}$ must meet every K_x^0-orbit, and we have proven that $\boldsymbol{T} \subset V_x$ is a K_x^0-transversal domain. q.e.d.

Thus, as remarked in the introduction, the linear orthogonal representations satisfying (A) play a special role in the general case of (A).

4. The K-cycles

We prove Theorem IV. Given the singular variety $P_i \subset T$, one chooses $x \in P_i$ such that $K_x = K_i$ (by Lemma (3.1)). Write

$$T_x(M) = T_x(N_x) \oplus T_x(P_i) \oplus L_x ,$$

where L_x is the orthogonal complement of the first two summands. K_x leaves L_x invariant. Let $Z \in L_x \cap T_x(T)$, $\|Z\| = 1$. If $a \in K_x = K_i$ satisfies $a \cdot Z = Z$, then $a \in K_T$. Conversely, $a \in K_T$ implies $a \cdot Z = Z$. Therefore K_i/K_T is identified with the K_x-orbit of Z in L_x and hence is a closed submanifold of the unit sphere in L_x.

Remark that

$$\begin{aligned} \dim(L_x) &= \dim(M) - \dim(N_x) - \dim(P_i) \\ &= \dim(T) + m(i) - \dim(T) + 1 \\ &= m(i) + 1 \ . \end{aligned}$$

Therefore, since $\dim(K_i/K_T) = m(i)$, K_i/K_T must be diffeomorphic to the unit sphere $S^{m(i)}$ in L_x; this is the assertion of Theorem IV.

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