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RIGIDITY OF HYPERSURFACES OF CONSTANT SCALAR CURVATURE

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In [7] S. Kobayashi proved that the only compact homogeneous hypersurfaces of a Euclidean space are the spheres. This result was extended by T. Nagano and T. Takahashi [9] who proved that if a homogeneous Riemannian manifold has an isometric immersion in a Euclidean space of one dimension greater such that the rank of the second fundamental form is distinct from two at some point, then it is isometric to the Riemannian product of a sphere by a Euclidean space. The original purpose of this paper was to show that this fact remains true without the restriction on the second fundamental form.

In both [7] and [9], the concept of rigidity has an important role. In fact, if M^n is assumed to be rigid (see preliminaries), the theorem is an immediate consequence of results of E. Cartan [3] and K. Nomizu and B. Smyth [10].

For a homogeneous hypersurface of a Euclidean space, having non-zero constant scalar curvature, there are only two possibilities a priori; it is either rigid or contains no rigid open submanifold (see Corollary 1–8).

The main result of this paper (Theorem 3–1) is that a hypersurface of a Euclidean space, having non-zero constant scalar curvature and containing no open rigid submanifold, is isometric to the product of a two-dimensional sphere and a Euclidean space. This result with the remarks made above gives a proof of Nagano and Takahashi's theorem in the most general case.

The proofs contained in this paper rely heavily on methods developed by E. Cartan [2] and S. Dolbeaut Lemoine [5].

Finally using very similar arguments, the following is proved.

If M^n is a hypersurface of a space form $\tilde{M}^{n+1}(K), n \geq 4$, having constant scalar curvature and an isometric immersion with type number greater than one at all points, then M^n is rigid.

1. All manifolds and maps considered in this work will be assumed of class C^∞. Let M^n be an n-dimensional Riemannian manifold, and denote its tangent space at a point p by T_pM^n and the scalar product given by the Riemannian

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structure by $\langle\ ,\ \rangle$. Following [8], ∇ will be the covariant derivation of M^n.

An r-dimensional C^∞ distribution $\mathscr{H}$ on M^n is said to be parallel at a point $p \in M^n$ if for any vector field X belonging to $\mathscr{H}$ and any tangent vector $Y_p \in T_pM^n$, we have

$$(\nabla_{Y_p}X)_p \in \mathscr{H}_p\ .$$

If this holds at all points p, then $\mathscr{H}$ is said to be parallel on M^n.

On the other hand, there is the notion of parallel translation of a vector along a path (see [8]). The following proposition relates these two concepts.

1-1. Proposition. *An r-dimensional distribution $\mathscr{H}$ is parallel on M^n if and only if the parallel translate of a vector $Y_p \in \mathscr{H}_p$ along any path still belongs to $\mathscr{H}_p$.*

For details see [1] and [8].

By means of the operator ∇, the curvature tensor of M^n can be expressed as

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z\ ,$$

where X, Y, Z are vector fields on M^n. The sectional curvature of the subspace π of T_pM^n, spanned by the vectors X, Y, is

$$S(\pi) = \frac{\langle R(X,Y)Y, X\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}\ .$$

A Riemannian manifold M^n has constant curvature K, if and only if

$$R(X,Y)Z = K(\langle Y,Z\rangle X - \langle X,Z\rangle Y)\ ,$$

for all vectors X, Y, Z and at all points of M^n. If $X_1, \cdots, X_n$ form an orthonormal basis of T_pM^n, then the scalar curvature of M^n at p is given by (see [8])

$$\operatorname{scal}(M^n) = \sum_{i \neq j} S(\pi_{ij})\ ,$$

where π_{ij} denotes the plane in T_pM^n spanned by X_i, X_j.

1-2. Proposition. *Let $\mathscr{D}$ be an $(n - r)$-dimensional C^∞ involutive distribution on M^n such that each leaf has constant curvature with respect to the Riemannian metric induced by M^n. Then for each point $p \in M^n$ it is possible to find a coordinate system, $x^1, \cdots, x^n$, on M^n defined around p in such a way that the vectors $\partial/\partial x^j, j > r$, form a basis for $\mathscr{D}$ and furthermore*

$$\left\langle\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle = \left(1 + \frac{K}{4}\Big(\sum_{k>r}(x^r)^2\Big)\right)^2 \delta_{ij}\ , \qquad i, j > r\ . \tag{1.2.1}$$

Indication of the proof. Around p, there are coordinates $y^1, \cdots, y^n$ such that the vector fields $\partial/\partial y^{r+1}, \cdots, \partial/\partial y^n$ form a basis of $\mathscr{D}$ at each point. It may be assumed that $y^1(p) = \cdots = y^n(p) = 0$. These coordinates give a diffeomorphism of a neighborhood of p in M^n, onto an open subset of R^n containing the origin. If the first neighborhood is conveniently small, it may be assumed that the second is of the form $U^r \times U^{n-r}$, where U^r and U^{n-r} are open neighborhoods of the origin in R^r and R^{n-r} respectively.

Consider the functions

$$g_{ij}(y^1, \cdots, y^n) = \left\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right\rangle , \qquad i, j > r , \tag{1.2.2}$$

which define a Riemannian metric on U^{n-r} at each point $(y^1, \cdots, y^r)$ of U^r, and it follows from the assumption made on the leaves of $\mathscr{D}$ that this metric has constant curvature K. On the other hand, the metric given by

$$\bar{g}_{ij} = \left(1 + \frac{K}{4} \sum_{k>r} (y^k)^2\right)^2 \delta_{ij} , \qquad i, j > r , \tag{1.2.3}$$

has the same constant curvature. The functions defining this diffeomorphism are solutions of a system of first order differential equations whose coefficients involve the g_{ij} and their derivatives. Hence this solution depends differentiably on the $y^1 \cdots y^r$. With this in mind, the coordinates $y^{r+1}, \cdots, y^n$ can be replaced by new functions $x^{r+1}, \cdots, x^n$ such that (1.2.1) holds.

Remark. Let E_i be the vector fields,

$$E_i = \left(1 + \frac{K}{4} \sum_{k>r} (x^k)^2\right)^2 \frac{\partial}{\partial x^i} , \qquad i > r ,$$

and assume that the leaves of $\mathscr{D}$ are totally geodesic submanifolds of M^n. Then the vectors E_i form an orthonormal basis and

$$\nabla_{E_i} E_j = \frac{K}{2} \sum_{k>r} (\delta^i_j x^2 - \delta^k_j x^i) E_k , \qquad i, j > r .$$

The next fact is the local part of a de Rham's theorem and can be found in [8].

1-3. Proposition. *Let $\mathscr{H}$ be a non-trivial parallel distribution on M^n and $\mathscr{H}^1$ its orthogonal complement. Then any point p of M^n has an open neighborhood $V \times V'$, where V and V' are open submanifolds of the leaves of $\mathscr{H}$ and $\mathscr{H}'$ respectively, and the Riemannian metric on $V \times V'$ is the direct product of the metrics of V and V'.*

Isometric immersions. Let M^n and $\tilde{M}^{n+r}$ be Riemannian manifolds of dimensions n and $n + r$ respectively. A differentiable map $f: M^n \to M^{n+r}$ is an isometric immersion if for each $p \in M^n$, the differential f_* of f is a scalar product preserving isomorphism between T_pM^n and a subspace of $T_{f(p)}\tilde{M}^{n+r}$. Consider two vector fields X, Y defined in some neighborhood of a point $p \in M^n$.

Since f is locally a diffeomorphism, it is possible to consider the vector fields f_*X, f_*Y on some submanifold of $\tilde{M}^{n+r}$. If $\nabla, \tilde{\nabla}$ denote the covariant derivations of M^n and $\tilde{M}^{n+r}$ respectively, then

$$\tilde{\nabla}_{f_*X}(f_*Y) = f_*(\nabla_X Y) + \alpha(X, Y) ,$$

where $\alpha(X, Y)$ belongs to the orthogonal complement of $f_*(TM^n)$ in $T\tilde{M}^{n+r}$ (see [1], [8]). When α vanishes at a point p, the immersion f is said to be totally geodesic at this point. If this holds for all points, f is called totally geodesic.

In case $r = 1, M^n$ is usually called a hypersurface of $\tilde{M}^{n+1}$. Denote by ξ a local unit normal field of M^n in $\tilde{M}^{n+1}$. Then

$$\alpha(X, Y) = \langle AX, Y\rangle\xi ,$$

where A is the symmetric operator of TM^n given by

$$AX = -f_*^{-1}(\tilde{\nabla}_{f_*X}\xi) .$$

From now on the operator A will be called the second fundamental form of f with respect to ξ. The rank of A at a point p is called the type number of f at this point and is commonly denoted by $t(p)$.

1-4. Proposition. *If the type number of f is greater than one at a point p, then the kernel of A_p is given by*

$$\ker A_p = \{X \in T_pM^n \mid \tilde{R}(X, Y) = R(X, Y), \textit{ for all } Y \in T_pM^n\} ,$$

R and $\tilde{R}$ denoting the curvature tensors of M^n and $\tilde{M}^{n+1}$ respectively.

For a proof of this fact see [12].

The following equations are basic in the study of hypersurfaces:

$$R(X, Y)Z = \mathrm{proj}_{TM_p}(\tilde{R}(X, Y)Z) + \langle AY, Z\rangle AX - \langle AX, Z\rangle AY ,$$
$$\mathrm{proj}_{TM_p}(\tilde{R}(X, Y)\xi) = \nabla_X(AY) - \nabla_Y(AX) - A[X, Y] ,$$

where $R, \tilde{R}$ denote the curvature tensors of M^n and $\tilde{M}^{n+1}$ respectively, ξ being a local unit normal field and A the second fundamental form of f with respect to ξ. These relations are known as Gauss and Codazzi equations respectively. If $\tilde{M}^{n+1}$ has constant curvature, the Codazzi equation becomes

$$\nabla_X(AY) - \nabla_Y(AX) = A[X, Y] .$$

For details, see [1], [7], [10].

1-5. Proposition. *Let f be an isometric immersion of M^n in $\tilde{M}^{n+1}$ such that its type number is constant and greater than one. If $\tilde{M}^{n+1}$ has constant curvature, then the nullity distribution $\mathcal{N}$ of f is integrable and its leaves are totally geodesic both in M^n and $\tilde{M}^{n+1}$.*

Proof. The integrability of $\mathscr{N}$ follows from Proposition 1–4 and [6]. Next it will be shown that the restriction of f to each leaf of $\mathscr{N}$ is a totally geodesic immersion.

Let k be the dimension of $\mathscr{N}$, and consider an orthonormal frame field $\xi_1, \cdots, \xi_{n-k}, \xi_{n-k+1}$ in such a way that the first $n-k$ vectors are orthogonal to a given leaf $\mathscr{N}_p$ in M^n and ξ_{n-k+1} is orthogonal to M^n in $\tilde{M}^{n+1}$.

The bilinear form $\alpha(X, Y)$ defined by the immersion $f\colon W_p \to \tilde{M}^{n+1}$ can be written as

$$\alpha(X, Y) = \sum_{i=1}^{n-k+1} \langle H_i(X), Y\rangle \xi_i ,$$

where

$$H_i(X) = -f_*^{-1}(\mathrm{proj}_{T\mathscr{N}_p} \tilde{\nabla}_{f_*X}\xi_i)$$

for any $X \in T\mathscr{N}_p$. From the choice of the normal frame it follows that

$$\begin{aligned} H_i(X) &= -\mathrm{proj}_{T\mathscr{N}_p}(\nabla_X\xi_i) , \qquad 1 \leq i \leq n-k , \\ H_{n-k+1}(X) &= -f_*^{-1}(\mathrm{proj}_{f_*(T\mathscr{N}_p)}\tilde{\nabla}_{f_*X}\xi_{n-k+1}) \\ &= -\mathrm{proj}_{T\mathscr{N}_p}(f_*^{-1}(\mathrm{proj}_{f_*(TM^n)} \tilde{\nabla}_{f_*X})) \\ &= \mathrm{proj}_{T\mathscr{N}_p} A(X) = 0 . \end{aligned}$$

On the other hand, the vector fields $A\xi_1, \cdots, A\xi_{n-k}$ form a basis for $(T\mathscr{N})^\perp$ (in M^n). Let $Y \in T\mathscr{N}_p$ ($\Longleftrightarrow AY = 0$), then from the Codazzi equation it follows

$$\langle \nabla_X A\xi_i, Y\rangle = \langle A[X, \xi_i], Y\rangle = 0 ,$$

which gives

$$\mathrm{proj}_{T\mathscr{N}_p} \nabla_X A\xi_i = 0 , \qquad 1 \leq i \leq n-k ,$$

and from this it follows that

$$\mathrm{proj}_{T\mathscr{N}_p} \nabla_X \xi_i = 0 .$$

These relations prove that all the H_i vanish, or in other words, that f restricted to $\mathscr{N}_p$ is totally geodesic.

Rigidity. A Riemannian manifold M is said to be homogeneous if for any pair of points p, g there is an isometry ϕ of M such that $\phi(p) = q$. A simply connected Riemannian manifold of constant curvature K is called a space form and will be denoted by $\tilde{M}(K)$. It is well known that the space forms are homogeneous.

Let $\tilde{M}^{n+1}(K)$ denote an $(n+1)$-dimensional space form. A Riemannian manifold M^n is said to be rigid in $\tilde{M}^{n+1}(K)$ if for any pair of isometric immersions

$f, \bar{f}$ of M^n into $\bar{M}^{n+1}(K)$, there is an isometry ϕ of $\bar{M}^{n+1}(K)$ such that $\bar{f} = \phi \circ f$.

The following result is basic:

1-6. Proposition. *If the type number of an isometric immersion f of M^n in $\bar{M}^{n+1}(K)(n \geq 3)$ is ≥ 3 at all points, then M^n is rigid.*

A simple proof is given in [12].

1-7. Proposition. *Let f be an isometric immersion of M^n in $\bar{M}^{n+1}(K)$ such that M^n contains no open subset on which f is totally geodesic. If there are open submanifolds U_α which are rigid and form a covering of M^n, then M^n is rigid.*

Proof. Consider another isometric immersion $\bar{f}$ of M^n in $\bar{M}^{n+1}(K)$ and denote by $f_\alpha, \bar{f}_\alpha$ the ristrictions of $f, \bar{f}$ to U_α respectively. Since U_α is assumed to be rigid, there is an isometry ϕ_α of $\bar{M}^{n+1}(K)$ such that $\bar{f}_\alpha = \phi_\alpha \circ f_\alpha$; thus if α, β are such that $U_\alpha \cap U_\beta$ is non-void, then $\phi_\alpha \circ f_\alpha = \phi_\beta \circ f_\beta$ at all points of $U_\alpha \cap U_\beta$. This means that $f(U_\alpha \cap U_\beta)$ is kept pointwise fixed by the isometry $\phi_\alpha^{-1} \cdot \phi_\beta$.

Now it is easy to show that if $\phi_\alpha \neq \phi_\beta$, then $f(U_\alpha \cap U_\beta)$ is contained in a totally geodesic submanifold of $\bar{M}^{n+1}$, which is a contradiction. By the connectedness of M^n it follows that all ϕ_α must coincide with an isometry ϕ, which gives $\bar{f} = \phi \circ f$, thus proving the proposition.

1-8. Corollary. *If M^n is a homogeneous hypersurface of $M^{n+1}(K)$, with scalar curvature distinct from $n(n-1)K$, then it is either rigid or contains no rigid open submanifold.*

Proof. The assumption on the scalar curvature excludes the existence of points at which the given immersion is totally geodesic.

Complexification. The *complex tangent space* $T_x^c M$ of a manifold M is the complexification of the tangent space $T_x M$. A *complex vector field* (resp. *complex differential form*) is defined by assigning to each point x of M an element of $T_x^c M$ (resp. $T_x^{c*} M$). Any complex vector field Z can be written uniquely as $Z = Z' + iZ''$ where Z' and Z'' are real vector fields. By duality it follows that a complex differential form w can be expressed uniquely as $w = w' + iw''$, w' and w'' being real differential forms.

1-9. Proposition. *Let $\mathscr{H}$ be an $(n-2)$-dimensional integrable distribution on an n-dimensional manifold M^n, $n \geq 3$, and Z, W be two linearly independent complex vector fields satisfying:*

i) *Z, W and $\mathscr{H}^c$ span $T_x^c M^n$ at each point x.*

ii) *$[Z, \mathscr{H}^c] \subset (Z) \oplus \mathscr{H}^c, [W, \mathscr{H}^c] \subset (W) \oplus \mathscr{H}^c$.*

Then there are locally defined, non-zero complex valued functions p, q such that:

$$[pZ, \mathscr{H}^c] \subset \mathscr{H}^c, \qquad [qW, \mathscr{H}^c] \subset \mathscr{H}^c.$$

The proof is straightforward and therefore omitted.

If M^n is a Riemannian manifold, then the scalar product $\langle\ ,\ \rangle$ and the Riemannian connection can be extended to complex vector fields by linearity.

The same notations will be used for these extensions. Let P_x denote a two-dimensional subspace of T_xM^n, and Z, W be a basis for P_x^c. The sectional curvature of P_x is given by

$$S(P_x) = \frac{\langle R(Z,W)W, Z\rangle}{\langle Z,Z\rangle\langle W,W\rangle - \langle Z,W\rangle^2},$$

as it can be easily verified. For an isometric immersion f of M^n in $\tilde{M}^{n+1}$ with second fundamental form A, the Gauss and Codazzi equations are valid for complex vector fields, provided A is extended to T^cM^n by linearity.

2. An n-dimensional Riemannian manifold M^n, isometrically immersed in the $(n+1)$-dimensional space form $\tilde{M}^{n+1}(K)$ is said to be *deformable* in $\tilde{M}^{n+1}(K)$ if it contains no open rigid submanifold. If each point $x \in M^n$ has a deformable neighborhood, then M^n is said to be *locally deformable* in $\tilde{M}^{n+1}(K)$. It should be noticed that deformability implies local deformability but the converse is not true in general. The following fact is basic and will be used without further mention.

If M^n is locally deformable in $\tilde{M}^{n+1}(K)$ with $n \geq 3$, and the scalar curvature of M^n is distinct from $n(n-1)K$ at each point, then the type number of any isometric immersion of M^n in $\tilde{M}^{n+1}(K)$ equals two at all points.

In fact, since M^n contains no rigid submanifold, in view of Proposition 1–6 the type number of any isometric immersion of M^n in $\tilde{M}^{n+1}(K)$ is at most two at all points. Let $\lambda_1, \cdots, \lambda_n$ denote the eigenvalues (not necessarily distinct) of the second fundamental form of a given isometric immersion. From the Gauss equation and the definition of the scalar curvature it follows that

$$\operatorname{scal}(M^n) = Kn(n-1) + \sum_{i \neq j} \lambda_i \lambda_j ,$$

which shows that the type number has to be exactly 2.

The main objective in this section is to prove the following results.

2-1. Theorem. *Let $\bar{M}^n$ be an n-dimensional Riemannian manifold with $n \geq 3$, having non-zero constant scalar curvature and being deformable in the Euclidean space E^{n+1}, and $\bar{f}$ be an isometric immersion of $\bar{M}^n$ in the Euclidean space E^{n+1}. Then the relative nullity distribution $\bar{\mathscr{N}}$ of $\bar{f}$ is parallel on $\bar{M}^n$.*

2-2. Theorem. *Let $\bar{M}^n$ be an n-dimensional Riemannian manifold with $n \geq 4$, and f an isometric immersion of $\bar{M}^n$ in the space form $\tilde{M}^{n+1}(K)$, $K \neq 0$. Assume further that the scalar curvature of $\bar{M}^n$ is constant and distinct from $n(n-1)K$. Then $\bar{M}^n$ is not deformable in $\tilde{M}^{n+1}(K)$,* i.e., *$\bar{M}^n$ contains an open rigid submanifold.*

The proofs of these theorems will depend on several lemmas. In order to simplify the statements of these lemmas the following definition is useful.

Throughout this section it will be assumed that $n \geq 3$.

2-3. Let M^n be a Riemannian manifold. A *local isometric immersion* of

M^n in $\tilde{M}^{n+1}(K)$ is a triple (h, H, U), where U is an open orientable submanifold, h is an isometric immersion of U in $\tilde{M}^{n+1}(K)$, and H is the second fundamental form operator of h.

2-4. Lemma. *Let M' be an n-dimensional orientable Riemannian manifold, and f an isometric immersion of M' in the space form $\tilde{M}^{n+1}(K)$, with second fundamental form A' and nullity distribution $\mathscr{H}'$. Assume that there is an orthonormal frame $X, Y, E_3, \cdots, E_n$ defined on M' in such a way that the $E_i, \cdots, E_n, i \geq 3$, form a basis for $\mathscr{N}'$, and that for any local isometric immersion (h', H', U) of M' in $\tilde{M}^{n+1}(K)$ (see Definition 2–3) the equation*

$$\langle H'(X), X\rangle = 0 \tag{2.4.1}$$

holds at all points of U. Assume further that M' is deformable in $\tilde{M}^{n+1}(K)$. Then the following equations hold on M':

$$\langle \nabla_{E_i}X, Y\rangle = 0 , \qquad \textit{for all } i \geq 3 . \tag{2.4.2}$$

$$\langle \nabla_X E_i, Y\rangle = 0 , \qquad \textit{for all } i \geq 3 . \tag{2.4.3}$$

$$\langle \nabla_X X, Y\rangle = 0 , \qquad \textit{for all } i \geq 3 . \tag{2.4.4}$$

Proof. The proofs of (2.4.2), (2.4.3), (2.4.4) follow the same pattern. They consist in showing that if some of these equations are not verified at a point of M', then this point is contained in an open rigid submanifold; this contradicts the deformability of M'.

Assume $\langle \nabla_{E_i}X, Y\rangle$ to be non-zero at a point p of M' for some index i. Thus it will be non-zero at all points of an open orientable submanifold U'. Let h' be an isometric immersion of U' in $\tilde{M}^{n+1}(K)$, and denote by H' its second fundamental form. Then

$$\langle H'Y, \nabla_{E_i}X\rangle = \nabla_{E_i}\langle H'Y, X\rangle - \langle \nabla_{E_i}H'Y, X\rangle .$$

Since $\langle H'X, X\rangle$ and $H'E_i$ are zero at all points, the above relation can be written as

$$\langle H'Y, \nabla_{E_i}X\rangle = \nabla_{E_i}\langle H'Y, X\rangle - \langle [E_i, Y], Y\rangle\langle Y, H'X\rangle . \tag{2.4.5}$$

A similar relation holds for the restriction of A' to U'.

From the Gauss equation it follows that

$$\langle H'X, X\rangle\langle H'Y, Y\rangle - \langle H'X, Y\rangle^2 = \langle A'X, X\rangle\langle A'Y, Y\rangle - \langle A'X, Y\rangle^2 ,$$

which together with (2.4.1) gives

$$\langle H'X, Y\rangle = e\langle A'X, Y\rangle , \tag{2.4.6}$$

where e is a constant, either $+1$ or -1. From (2.4.5) and (2.4.6), we thus have

$$\langle (H' - eA')Y, \nabla_{E_i}X \rangle = 0 ,$$

or

$$\langle (H' - eA')Y, Y \rangle \langle Y, \nabla_{E_i}X \rangle = 0 ,$$

or

$$\langle H'Y, Y \rangle = e \langle A'Y, Y \rangle . \tag{2.4.7}$$

Now (2.4.1), (2.4.6) and (2.4.7) show that $H' = eA'$ and therefore that U' is rigid. Since M' is assumed to be deformable, this is a contradiction. Thus (2.4.7) is proved.

By (2.4.1) it results that

$$0 = \nabla_{E_i} \langle A'X, X \rangle = \langle \nabla_{E_i} A'X, X \rangle + \langle A'X, \nabla_{E_i} X \rangle .$$

Using (2.4.1), (2.4.2) and noting that $A'E_i$ vanishes, this relation becomes

$$\begin{aligned} 0 &= \langle A'[E_i, X], X \rangle = \langle [E_i, X], A'X \rangle \\ &= \langle \nabla_{E_i} X, Y \rangle \langle Y, A'X \rangle - \langle \nabla_X E_i, Y \rangle \langle Y, A'X \rangle . \end{aligned}$$

Using (2.4.2) again it follows that

$$\langle \nabla_X E_i, Y \rangle \langle A'X, Y \rangle = 0 , \qquad \text{for all } i \geq 3 .$$

By assumption, $\mathcal{N}'$ is an $(n-2)$-dimensional distribution which means that $\langle A'X, Y \rangle$ is never zero. Thus,

$$\langle \nabla_X E_i, Y \rangle = 0 , \qquad \text{for all } i \geq 3 ,$$

and (2.4.3) is proved.

The relation (2.4.4) is proved in a way similar to the proof of (2.4.2) by replacing E_i by X. It is sufficient to start with

$$\langle A'Y, \nabla_X X \rangle = \nabla_X \langle A'Y, X \rangle - \langle \nabla_X A'X, Y \rangle ,$$

and to show that

$$\langle (H' - eA')Y, Y \rangle \langle \nabla_X X, Y \rangle = 0 .$$

If the term $\langle \nabla_X X, Y \rangle$ does not vanish, U' must be rigid which again is a contradiction. q.e.d.

Relations (2.4.2), (2.4.3) and (2.4.4) have a geometrical interpretation which will be stated next.

2-5. Corollary. *Let M' and f' be as in Lemma 2–4. Then the $(n-1)$-dimensional distribution $\mathcal{N}' \oplus X$ is integrable, its leaves are totally geodesic* *submanifolds of M' and they are mapped by f' into totally geodesic submanifolds of $\tilde{M}^{n+1}(K)$.*

Proof. Relations (2.4.2) and (2.4.3) show that

$$\langle [E_i, X], Y\rangle = 0 , \qquad i = 1, \cdots, n ,$$

on M', which mean that $[E_i, X]$ belongs to $\mathscr{N}' \oplus X$. Since $[E_i, E_j]$ belongs to $\mathscr{N}'$, it also belongs to $\mathscr{N}' \oplus X$, thus showing the integrability.

Next consider a leaf $\mathscr{F}_0$. It will be shown that the inclusion map $i\colon \mathscr{F}_0 \to M'$, considered as an isometric immersion is totally geodesic. The vector field Y may be viewed as a unit normal vector field of $\mathscr{F}_0$ in M'. Thus it suffices to show that the covariant derivatives of Y with respect to tangent vectors of $\mathscr{F}_0$ are orthogonal to $\mathscr{F}_0$. In fact, $\langle \nabla_X Y, E_i\rangle$ and $\langle \nabla_X Y, X\rangle$ vanish by (2.4.3) and (2.4.4) respectively. On the other hand, $\langle \nabla_{E_j} Y, E_i\rangle$ vanishes because $\mathscr{N}'$ is totally geodesic (see Proposition 1–5) and $\langle \nabla_{E_j} Y, X\rangle$ is zero by (2.4.2). To show the last part it has to be proved that the product of the isometric immersions f' and i is totally geodesic.

Let ξ be a unit normal field of M' with respect to the immersion f'. After suitable identifications, Y and ξ may be viewed as normal vectors of $\mathscr{F}_0$ with respect to the immersion $f' \circ i$. With this in mind, the fact that $f' \circ i$ is totally geodesic is equivalent to the fact that the covariant derivatives of Y and ξ with respect to the tangent vectors of $\mathscr{F}_0$ are orthogonal to $\mathscr{F}_0$ in $\tilde{M}^{n+1}(K)$.

In fact,

$$\langle \tilde{\nabla}_X, \xi\rangle = -\langle A'X, X\rangle = 0 ,$$

by using (2.4.1). On the other hand,

$$\langle \tilde{\nabla}_X \xi, E_i\rangle = -\langle A'X, E_i\rangle = -\langle A'E_i, X\rangle = 0 ,$$

because $A'E_i = 0$. Furthermore, by (2.4.4),

$$\langle \tilde{\nabla}_X Y, X\rangle = \langle \nabla_X Y, X\rangle = 0 .$$

From (2.4.3) it follows that

$$\langle \tilde{\nabla}_X Y, E_i\rangle = \langle \nabla_X Y, E_i\rangle = 0 .$$

Since $\mathscr{N}'$ is totally geodesic (see Proposition 1–5), we have

$$\begin{aligned} \langle \tilde{\nabla}_{E_j} \xi, E_i\rangle &= -\langle A'E_j, E_i\rangle = 0 , \\ \langle \tilde{\nabla}_{E_j} \xi, X\rangle &= -\langle A'E_j, X\rangle = 0 , \\ \langle \tilde{\nabla}_{E_j} Y, E_i\rangle &= \langle \nabla_{E_j} Y, E_i\rangle = -\langle Y, \nabla_{E_j} E_i\rangle = 0 . \end{aligned}$$

Finally, by (2.4.2),

$$\langle \tilde{\nabla}_{E_j} Y, X \rangle = \langle \nabla_{E_j} Y, X \rangle = 0 ,$$

and the proof of Corollary 2–5 is complete.

2-6. Corollary. *Let M' and f' verify the conditions of Lemma 2–4 for $K = 0$ (i.e., M' is assumed to be deformable in the $(n+1)$-dimensional Euclidean space). Then the scalar curvature of M' is not constant.*

Proof. Since the dimension of $\mathcal{N}'$ is assumed to be $n-2$, the scalar curvature of M' has to be nonzero at each point. It will be shown that the assumption of constancy of the scalar curvature contradicts this fact.

By (2.4.1) it follows that

$$\operatorname{scal} M^n = -2\langle A'X, Y \rangle^2 ,$$

which shows that if $\operatorname{scal} M^n$ is constant, so is $\langle A'X, Y \rangle$. Hence

$$0 = \nabla_{E_i} \langle A'Y, X \rangle = \langle \nabla_{E_i} A'Y, X \rangle + \langle A'Y, \nabla_{E_i} X \rangle , \tag{2.6.1}$$

By (2.4.2),

$$\langle A'Y, \nabla_{E_i} X \rangle = \langle A'Y, Y \rangle \langle \nabla_{E_i} X, Y \rangle = 0 . \tag{2.6.2}$$

On the other hand,

$$\begin{aligned} \langle \nabla_{E_i} A'Y, X \rangle &= \langle \nabla_Y (A'E_i) + A[E_i, Y], X \rangle \\ &= \langle [E_i, Y], A'X \rangle = \langle [E_i, Y], Y \rangle \langle A'X, Y \rangle , \end{aligned}$$

since $\langle A'X, X \rangle$ vanishes by (2.4.1). Therefore

$$\langle \nabla_{E_i} A'Y, X \rangle = -\langle \nabla_Y E_i, Y \rangle \langle A'X, Y \rangle . \tag{2.6.3}$$

Relations (2.6.1), (2.6.2), (2.6.3) give

$$\langle \nabla_Y E_i, Y \rangle = 0 , \qquad i = 3, \cdots, n . \tag{2.6.4}$$

Since $\langle A'X, Y \rangle$ is constant, it follows that

$$0 = \nabla_X \langle A'Y, X \rangle = \langle \nabla_X A'Y, X \rangle + \langle A'Y, \nabla_X X \rangle . \tag{2.6.5}$$

Making use of (2.4.4) one obtains

$$\langle A'Y, \nabla_X X \rangle = \langle A'Y, Y \rangle \langle Y, \nabla_X X \rangle = 0 ,$$

hence (2.6.5) gives

$$\langle \nabla_X A'Y, X \rangle = 0 . \tag{2.6.6}$$

On the other hand,

$$\begin{aligned}\langle \nabla_X A'Y, X\rangle &= \langle \nabla_Y A'X + A'[X,Y], X\rangle \\ &= \langle \nabla_Y A'X, X\rangle + \langle [X,Y], A'X\rangle \\ &= \langle \nabla_Y A'X, X\rangle - \langle \nabla_Y X, A'X\rangle + \langle \nabla_X Y, A'X\rangle .\end{aligned}$$

Since $\langle A'X, X\rangle$ vanishes, the above relation can be written as

$$\begin{aligned}\langle \nabla_X A'Y, X\rangle &= -2\langle \nabla_Y X, A'X\rangle + \langle \nabla_X Y, A'X\rangle \\ &= -2\langle \nabla_Y X, Y\rangle\langle A'X, Y\rangle .\end{aligned} \tag{2.6.7}$$

Equations (2.6.6) and (2.6.7) give

$$\langle \nabla_Y X, Y\rangle = 0 . \tag{2.6.8}$$

The parallelism of the distribution $\mathscr{N}' \oplus X$ is a consequence of Lemma 2–4 and relations (2.6.4) and (2.6.8). Now, it follows from Proposition 1–3 and Corollary 2–5 that M' is locally flat; hence its scalar curvature is zero, which is the desired contradiction. This ends the proof of Corollary 2–6.

Remark. The proof of this corollary shows also that relations (2.6.4) and (2.6.8) hold whether K is zero or not, thus the following is true.

2-7. Corollary. *Let M', f' verify the conditions of Lemma* (2–4) *for $K \neq 0$. In addition, assume the scalar curvature of M' to be constant. Then the distribution $\mathscr{N}' \oplus X$ is parallel on M'.*

Proof. See remark above.

2-8. Corollary. *Let M', f' verify the assumptions of Lemma* 2–4 *for $K \neq 0$, and assume further that the relations*

$$\langle \nabla_X E_i, X\rangle = \langle \nabla_Y E_i, Y\rangle , \qquad i = 3, \cdots, n , \tag{2.8.1}$$

hold at all points of M'. Then the scalar curvature of M' is not constant.

Proof. Assume the scalar curvature of M' to be constant. Using Corollary 2–7 one obtains

$$\langle \nabla_Y E_i, Y\rangle = 0 ,$$

and from (2.8.1) it follows that

$$\langle \nabla_X E_i, X\rangle = 0 , \qquad i = 3, \cdots, n . \tag{2.8.2}$$

The Gauss equation gives

$$\langle R(X, E_i)E_i, X\rangle = K + \langle A'E_i, E_i\rangle X - \langle A'X, E_i\rangle E_i ,$$

which yields, since $A'E_i$ vanishes,

$$\langle R(X, E_i)E_i, X\rangle = K . \tag{2.8.3}$$

On the other hand,

$$(2.8.4)\quad \begin{aligned}&\langle R(X, E_i)E_i, X\rangle \\ &= \langle \nabla_X \nabla_{E_i} E_i, X\rangle - \langle \nabla_{E_i} \nabla_X E_i, X\rangle - \langle \nabla_{[X, E_i]} E_i, X\rangle .\end{aligned}$$

The distributions $\mathscr{N}'$ and $\mathscr{N}' \oplus X$ are totally geodesic (see Proposition 1–5 and Corollary 2–5, respectively). This fact together with (2.8.3) and (2.8.4) gives

$$\langle R(X, E_i)E_i, X\rangle = 0 ,$$

which by (2.8.3) implies $K = 0$. This is a contradiction since K is assumed to be non-zero, and therefore the scalar curvature of M' is not constant.

2-9. Lemma. *Let M'' be an n-dimensional orientable Riemannian manifold, and f'' an isometric immersion of M'' in the space form $\tilde{M}^{n+1}(K)$ with second fundamental form A'' and relative nullity distribution $\mathscr{N}''$. Assume that $E_1, \cdots, E_n$ form an orthonormal frame defined on M'' such that the vector fields $E_3, \cdots, E_n$ form a basis for $\mathscr{N}''$. Suppose that there are two complex vector fields Z and W belonging to the complexification of the vector space spanned by E_1, E_2 such that for any loacl isometric immersion (h'', H'', U) of M'' in $\tilde{M}^{n+1}(K)$ (see § 2–3), the equation*

$$(2.9.1)\qquad \langle H''Z, W\rangle = 0$$

holds at all points of U''. Finally assume M'' to be deformable in $\tilde{M}^{n+1}(K)$. Then the complex vector fields $\nabla_{E_i} Z, \nabla_Z E_i$ (respectively $\nabla_{E_i} W, \nabla_W E_i$) have no W-component (respectively Z-component) for all $i \geq 3$.

Proof. Denote by $(\nabla_Z E_i)_{(W)}$ (resp. $(\nabla_W E_i)_{(Z)}$) the W-component (resp. Z-component) of $\nabla_Z E_i$ (resp. $\nabla_W E_i$). Let p be a point of M'', and assume

$$(\nabla_Z E_i)_{(W)} \neq 0 , \qquad \text{for some } i \geq 3 ,$$

at all points of an open orientable manifold $M''(p)$ containing p.

Consider a local isometric immersion $(h'', H'', M''(p))$ of M'' in $\tilde{M}^{n+1}(K)$ (see Definition 2–3). Since $H''(E_i)$ vanish for all $i \geq 3$, it follows that

$$\langle E_i, H''W\rangle = 0 , \qquad i = 3, \cdots, n .$$

By covariant derivation with respect to Z, this relation yields

$$(2.9.2)\qquad \langle \nabla_Z E_i, H''W\rangle + \langle E_i, \nabla_Z H''W\rangle = 0 .$$

In view of (2.9.1) the first term of the left-hand side of (2.9.2) can be written as

$$(2.9.3)\qquad \langle \nabla_Z E_i, H''W\rangle = (\nabla_Z E_i)_{(W)} \langle W, H''W\rangle ,$$

while the second term as

$$\begin{aligned}\langle E_i, \nabla_Z H''W\rangle &= \langle E_i, \nabla_W H''Z + H''[Z, W]\rangle \\ &= \langle E_i, \nabla_W E''Z\rangle = -\langle \nabla_W E_i, E''Z\rangle .\end{aligned}$$

Again by (2.9.1) this equation becomes

$$\langle E_i, \nabla_Z H''W\rangle = -(\nabla_W E_i)_{(Z)}\langle Z, H''Z\rangle . \tag{2.9.4}$$

From equations (2.9.2), (2.9.3), and (2.9.4) follows immediately

$$(\nabla_Z E_i)_{(W)}\langle W, H''W\rangle = (\nabla_W E_i)_{(Z)}\langle Z, H''Z\rangle . \tag{2.9.5}$$

On the other hand, the extension of the Gauss equation to complex vector fields gives

$$\langle H''Z, Z\rangle\langle H''W, W\rangle - \langle Z, H''W\rangle^2 = \langle A''Z, Z\rangle\langle A''W, W\rangle - \langle Z, A''W\rangle^2 ,$$

which implies, due to (2.9.1),

$$\langle H''Z, Z\rangle\langle H''W, W\rangle = \langle A''Z, Z\rangle\langle A''W, W\rangle . \tag{2.9.6}$$

From (2.9.5) we obtain

$$\begin{aligned}(\nabla_Z E_i)_{(W)}\langle W, H''W\rangle^2 &= (\nabla_Z E_i)_{(W)}\langle Z, H''Z\rangle\langle W, H''W\rangle , \\ (\nabla_Z E_i)_{(W)}\langle W, A''W\rangle^2 &= (\nabla_Z E_i)_{(W)}\langle Z, A''Z\rangle\langle W, A''W\rangle ,\end{aligned} \tag{2.9.7}$$

which together with (6) yields

$$(\nabla_Z E_i)_{(W)}(\langle W, H''W\rangle^2 - \langle W, A''W\rangle^2) = 0 . \tag{2.9.8}$$

Since $(\nabla_Z E_i)_{(W)}$ is assumed to be non-zero, it follows from (2.9.8) that

$$\langle W, H''W\rangle^2 - \langle W, A''W\rangle^2 = 0$$

at all points of $M''(p)$, which means that

$$\langle H''W, W\rangle = e\langle A''W, W\rangle , \tag{2.9.9}$$

where e is a constant either $+1$ or -1. From (2.9.6) and (2.9.9) we obtain

$$\langle H''Z, Z\rangle = e\langle A''Z, Z\rangle . \tag{2.9.10}$$

Finally, from (2.9.1), (2.9.9) and (2.9.10) it follows that

$$H'' = eA'' . \tag{2.9.11}$$

Since (11) holds for any local immersion, it follows that $M''(p)$ is rigid in

$\tilde{M}^{n+1}(K)$, which contradicts the deformability of M''. Thus $(\Delta_Z E_i)_{(W)}$ vanishes at p. Since the above proof is symmetric in Z, W it results that $(\nabla_W E_i)_{(Z)}$ also vanishes on M''.

Next, denote by $(\nabla_{E_i} Z)_{(W)}$ (Resp. $(\nabla_{E_i} W)_{(Z)}$) the W-component (resp. Z-component) of $\nabla_{E_i} Z$ (resp. $\nabla_{E_i} W$). Let p be a point of M'' and suppose $(\nabla_{E_i} Z)_{(W)} \neq 0$ for some $i \geq 3$ and at all points of an open orientable submanifold $M''(p)$ containing p.

Consider a local isometric immersion $(h'', H'', M''(p))$. By covariant derivation, with respect to E_i, of both sides of the relation (2.9.1), we have

$$\langle \nabla_{E_i} Z, H''W \rangle + \langle Z, \nabla_{E_i} H''W \rangle = 0 . \tag{2.9.12}$$

In view of (2.9.1), the first term of the left-hand side of (2.9.12) becomes

$$\langle \nabla_{E_i} Z, H''W \rangle = (\nabla_{E_i} Z)_{(W)} \langle W, H''W \rangle . \tag{2.9.13}$$

By (2.9.1) and the fact that $H''(E_i)$ is zero, the second term of (2.9.12) takes the form

$$\begin{aligned} &\langle Z, \nabla_{E_i} H''W \rangle = \langle Z, H''[E_i, W] \rangle , \\ &\langle H''Z, [E_i, W] \rangle = \langle H''Z, Z \rangle ((\nabla_{E_i} W)_{(Z)} - (\nabla_W E_i)_{(Z)}) . \end{aligned} \tag{2.9.14}$$

Since $(\nabla_W E_i)_{(Z)}$ vanishes as it was shown above, the relation (2.9.14) is simplified to

$$\langle Z, \nabla_{E_i} H''W \rangle = (\nabla_{E_i} W)_{(Z)} \langle H''Z, Z \rangle , \tag{2.9.15}$$

which, together with (2.9.12) and (2.9.13), implies

$$(\nabla_{E_i} Z)_{(W)} \langle H''W, W \rangle + (\nabla_{E_i} W)_{(Z)} \langle H''Z, Z \rangle = 0 , \tag{2.9.16}$$

and of course

$$(\nabla_{E_i} Z)_{(W)} \langle A''W, W \rangle + (\nabla_{E_i} W)_{(Z)} \langle A''Z, Z \rangle = 0 . \tag{2.9.17}$$

By the same argument used before, it can be concluded from (2.9.17) that $M''(p)$ is rigid, which contradicts the deformability of M''. Hence the proof of Lemma 2–9 is complete.

2-10. Corollary. *Assume the manifold M'', the immersion f'' and the vector fields Z, W satisfy the conditions of Lemma 2–9. Then the following conclusions hold*:

(a) *for $K \neq 0$ and $n \geq 4$, the scalar curvature of M'' cannot be constant*,

(b) *for $K = 0$ and M'' with constant scalar curvature, the relative nullity distribution $\mathcal{N}''$ of f'' is parallel on M''.*

Proof. From Lemma 2–9 it follows that

$$(2.10.1)\qquad [Z, E_i]_{(W)} = [W, E_i]_{(Z)} = 0\ , \qquad i = 3, \cdots, n\ .$$

Consider a point $p \in M''$. From (2.10.1) and Proposition 1–9 there are two non-vanishing complex valued functions α, β defined in a neighborhood of p in such a way that the new complex vector fields Z', W' defined by

$$(2.10.2)\qquad Z' = \alpha Z\ , \qquad W' = \beta W\ ,$$

have the propriety

$$(2.10.3)\qquad \begin{aligned} [Z', E_i] &= \sum_{k\geq 3} a_i^k E_k\ , \\ [W', E_i] &= \sum_{k\geq 3} b_i^k E_k\ , \end{aligned}$$

where a_i^k, b_i^k are complex valued functions defined in a neighborhood of p.

Furthermore, by Propositions 1–5 and 1–9 the frame $E_1, \cdots, E_n$ may be assumed to verify

$$(2.10.4)\qquad \nabla_{E_i} E_j = \frac{K}{2} \sum_{k=3}^{n} (\delta_j^i x^k - \delta_j^k x^i) E_k\ ,$$

where the functions x^k are part of suitable coordinate system of M'' at p. Finally, it is possible to assume the existence of an open orientable submanifold $M''(p)$ containing p such that (2.10.3) and (2.10.4) hold at all of its points.

It follows from (2.10.2) that $M''(p)$ and the vector field Z', W' verify the assumption of Lemma (2–9). Hence

$$(2.10.5)\qquad \begin{aligned} (\nabla_{Z'} E_i)_{(W')} &= (\nabla_{E_i} Z')_{(W')} = 0\ , \\ (\nabla_{W'} E_i)_{(Z')} &= (\nabla_{E_i} W')_{(Z')} = 0\ . \end{aligned}$$

On the other hand, since $\nabla_{E_i} E_j$ belong to $\mathcal{N}''$ and $\langle Z', E_j \rangle$ vanish, it follows that

$$(2.10.6)\qquad \begin{aligned} \nabla_{E_i} Z' &= (\nabla_{E_i} Z')_{(Z')} Z'\ , \\ \nabla_{E_i} W' &= (\nabla_{E_i} W')_{(W')} W'\ , \qquad i = 3, \cdots, n\ , \end{aligned}$$

at each point of $M''(P)$.

Recalling that A'' denotes the second fundamental form of f'', it may be written:

$$(2.10.7)\qquad \nabla_{E_i} \langle A'' Z', Z' \rangle = \langle \nabla_{E_i} A'' Z', Z' \rangle + \langle A'' Z', \nabla_{E_i} Z' \rangle\ ,$$

for all $i \geq 3$ on $M''(p)$. Next it will be shown that the first term of the right-hand side of (2.10.7) vanishes. In fact,

$$\langle \nabla_{E_i} A''Z', Z' \rangle = \langle \nabla_{Z'} A''E_i + A''[E_i, Z'], Z' \rangle = 0 \ ,$$

as a consequence of (2.10.3). For the second term, relations (2.10.6) yield

$$\langle A''Z', \nabla_{E_i} Z' \rangle = (\nabla_{E_i} Z')_{(Z')} \langle A''Z', Z' \rangle \ . \tag{2.10.8}$$

Combining (2.10.7) and (2.10.8) we obtain

$$\nabla_{E_i} \langle A''Z', Z' \rangle = (\nabla_{E_i} Z')_{(Z')} \langle A''Z', Z' \rangle \ , \tag{2.10.9}$$

and, similarly for W',

$$\nabla_{E_i} \langle A''W', W' \rangle = (\nabla_{E_i} W')_{(W')} \langle A''W', W' \rangle \ . \tag{2.10.10}$$

Furthermore, the relations below are also a consequence of (2.10.6):

$$\begin{aligned} \nabla_{E_i} \langle Z', Z' \rangle &= 2(\nabla_{E_i} Z')_{(Z')} \langle Z', Z' \rangle \ , \\ \nabla_{E_i} \langle W', W' \rangle &= 2(\nabla_{E_i} W')_{(W')} \langle W', W' \rangle \ , \\ \nabla_{E_i} \langle Z', W' \rangle &= ((\nabla_{E_i} Z')_{(Z')} + (\nabla_{E_i} W')_{(W')}) \langle Z', W' \rangle \ , \end{aligned} \tag{2.10.11}$$

for all $i \geq 3$ at all points of $M''(p)$.

The scalar curvature of $M''(p)$ at each point is given by

$$\begin{aligned} &\mathrm{scal}\,(M''(p)) \\ &\quad = n(N-1)K + 2\frac{\langle A''Z', Z' \rangle \langle A''W', W \rangle}{\langle Z', Z' \rangle \langle W', W' \rangle - \langle Z', W' \rangle^2} \ . \end{aligned} \tag{2.10.12}$$

Since scal $(M''(p))$ is constant, it follows, from (2.10.12),

$$\begin{aligned} &(\langle Z', Z' \rangle \langle W', W' \rangle - \langle Z', W' \rangle^2) \nabla_{E_i} (\langle A''Z', Z' \rangle \langle A''W', W' \rangle) \\ &\quad = \nabla_{E_i} (\langle Z', Z' \rangle \langle W', W' \rangle - \langle Z', W' \rangle^2)(\langle A''Z', Z' \rangle \langle A''W', W' \rangle) \ . \end{aligned} \tag{2.10.13}$$

From (2.10.9), (2.10.10), (2.10.11), (2.10.13), we obtain

$$\begin{aligned} &((\nabla_{E_i} Z')_{(Z')} + (\nabla_{E_i} W')_{(W')})(\langle Z', Z' \rangle \langle W', W' \rangle \\ &\qquad\qquad - \langle Z', W' \rangle^2) \langle A''Z', Z' \rangle \langle A''W', W' \rangle \\ &\quad = 2((\nabla_{E_i} Z')_{(Z')} + (\nabla_{E_i} W')_{(W')}) \\ &\qquad \cdot (\langle Z', Z' \rangle \langle W', W' \rangle - \langle Z', W' \rangle^2) \langle A''Z', Z' \rangle \langle A''W', W' \rangle \ , \end{aligned} \tag{2.10.14}$$

which gives

$$(\nabla_{E_i} Z')_{(Z')} + (\nabla_{E_i} W')_{(W')} = 0 \ , \tag{2.10.15}$$

for all $i \geq 3$ at all points of $M''(p)$.

On the other hand, since $A''E_i$ vanishes for all $i \geq 3$, the Gauss equation implies

$$R(Z', E_i)E_i = \tilde{R}(Z', E_i)E_i = KZ' . \tag{2.10.16}$$

By the definition of curvature we have

$$R(Z', E_i)E_i = \nabla_{Z'}(\nabla_{E_i}E_i) - \nabla_{E_i}(\nabla_{Z'}E_i) - \nabla_{E_i}([Z', E_i]) , \tag{2.10.17}$$

which may be written, in consequence of (2.10.4),

$$\nabla_{E_i}E_i = \sum_{j \neq i} \lambda^j E_j , \tag{2.10.18}$$

and the covariant differentiation of (2.10.18) with respect to Z' gives

$$\begin{aligned} \nabla_{Z'}(\nabla_{E_i}E_i) &= \sum_{j \neq i} (\lambda^j \nabla_{Z'}E_j + Z'(\lambda^j)E_j) \\ &= \sum_{j \neq i} (\lambda^j(\nabla_{E_j}Z') + \lambda^j[Z', E_j] + Z'(\lambda^j)E_j) . \end{aligned} \tag{2.10.19}$$

From (2.10.3), (2.10.6) and (2.10.19) it follows that

$$[\nabla_{Z'}(\nabla_{E_i}E_i)]_{(Z')} = \sum_{j \neq i} \lambda^j(\nabla_{E_j}Z')_{(Z')} , \tag{2.10.20}$$

and also

$$[\nabla_{W'}(\nabla_{E_i}E_i)]_{(W')} = \sum_{j \neq i} \lambda^j(\nabla_{E_j}W')_{(W')} . \tag{2.10.21}$$

The Z'-component of $\nabla_{E_i}(\nabla_{Z'}E_i)$ is given by

$$[(\nabla_{E_i}(\nabla_{Z'}E_i)]_{(Z')} = E_i[(\nabla_{E_i}Z')_{(Z')}] + ((\nabla_{E_i}Z')_{(Z')})^2 . \tag{2.10.22}$$

In fact, covariantly differentiating

$$\nabla_{Z'}E_i = \nabla_{E_i}Z' + [Z', E_i] , \tag{2.10.23}$$

and using (2,10.3) we obtain

$$(\nabla_{E_i}(\nabla_{Z'}E_i))_{(Z')} = (\nabla_{E_i}(\nabla_{E_i}Z'))_{(Z')} , \tag{2.10.24}$$

the right-hand side of which may be written, in consequence of (2.10.6),

$$\begin{aligned} \nabla_{E_i}(\nabla_{E_i}Z')_{(Z')} &= \nabla_{E_i}((\nabla_{E_i}Z')_{(Z')}Z')_{(Z')} \\ &= E_i((\nabla_{E_i}Z')_{(Z')}) + ((\nabla_{E_i}Z')_{(Z')})^2 , \end{aligned} \tag{2.10.25}$$

proving (2.10.22).

The same relation holds for the W'-components.

Taking the Z'-components in (2.10.16) and using (2.10.17), (2.10.20) and (2.10.22) we have

$$\sum_{j\neq i}\lambda^j(\nabla_{E_j}Z')_{(Z')} - E_i((\nabla_{E_i}Z')_{(Z')}) - ((\nabla_{E_i}Z')_{(Z')})^2 = K. \tag{2.10.26}$$

Adding (2.10.26) to its analog for W', and using (2.10.15), we obtain

$$(\nabla_{E_i}Z')^2_{(Z')} + (\nabla_{E_i}W')^2_{(W')} = -2K\,, \tag{2.10.27}$$

and, again by (2.10.15),

$$(\nabla_{E_i}Z')^2_{(Z')} = (\nabla_{E_i}W')^2_{(W')} = -K\,, \qquad i = 3,\cdots,n\,. \tag{2.10.28}$$

From (2.10.26) and (2.10.28) it follows

$$\sum_{j\neq i}\lambda^j(\nabla_{E_j}Z')_{(Z')} = 0\,, \qquad i,j \geq 3\,, \tag{2.10.29}$$

and from (2.10.4), (2.10.18) and (2.10.28) we get, respectively,

$$\lambda^j = \frac{K}{2}x^j\,, \qquad (\nabla_{E_j}Z')_{(Z')} = \sqrt{-K}\,, \qquad j \geq 3\,. \tag{2.10.30}$$

The relations (2.10.29) and (2.10.30) show that if $n \geq 4$, then $K = 0$, hence proving the statement (a) by contradiction. In the case $K = 0$, (2.10.3) and (2.10.28) prove (b).

2-11. Proof of Theorem 2-1. Let p_0 be a given point in $\overline{M}^n$, and consider an open neighborhood U_0 of p_0, on which there is an orthonormal frame E_1, $E_2, \cdots, E_n$, in such a way that $E_3, \cdots, E_n$ is a basis for $\overline{\mathcal{N}}$ and

$$\nabla_{E_i}E_j = 0\,, \qquad i,j \geq 3\,. \tag{2.11.1}$$

This is possible in view of Propositions 1–2, 1–5 and the fact that $\overline{M}^n$ is isometrically immersed in the Euclidean space E^{n+1}.

For any local isometric immersion (h, H, U_0) of U_0, we have

$$H[E_1, E_2], E_i = 0\,, \qquad i = 3,\cdots,n\,, \tag{2.11.2}$$

on U_0. This relation and the Codazzi equations yield

$$\begin{aligned}\langle\nabla_{E_1}E_i, E_2\rangle\langle HE_2, E_2\rangle &+ [\langle\nabla_{E_1}E_i, E_1\rangle \\ &- \langle\nabla_{E_2}E_i, E_2\rangle]\langle HE_1, E_2\rangle - \langle\nabla_{E_2}E_i, E_1\rangle\langle HE_1, E_1\rangle = 0\,,\end{aligned} \tag{2.11.3}$$

for all $i \geq 3$ at all points of U_0. Equation (2.11.2) will be used to define locally vector fields satisfying either the conditions of Lemma 2–4 or 2–9. Since this involves several discussions, it is convenient to consider the following subset of U_0, P: set of the points q of U_0, such that

$$\text{(2.11.4)}\qquad \begin{aligned} &\langle \nabla_{E_1}E_i, E_2\rangle_q = \langle \nabla_{E_2}E_i, E_1\rangle_q = 0\ , \\ &\langle \nabla_{E_1}E_i, E_1\rangle_q = \langle \nabla_{E_2}E_i, E_2\rangle_q\ , \qquad i = 3, \cdots, n\ . \end{aligned}$$

This set P has the following propriety:

(2.11.5) *The relative nullity distribution $\bar{\mathscr{N}}$ is parallel at any point of the interior of P.*

In fact, consider a point $q \in \operatorname{Int} P$. Locally it is possible to replace E_1, E_2 by unit vector fields X, Y such that

$$\text{(2.11.6)}\qquad \langle X, Y\rangle = 0\ , \qquad \langle \bar{A}X, Y\rangle = 0\ ,$$

in a neighborhood of q, provided the non-zero eigenvalues of $\bar{A}_q$ are distinct. A direct computation gives:

$$\text{(2.11.7)}\qquad \begin{aligned} &\langle \nabla_X E_i, Y\rangle = \langle \nabla_Y E_i, X\rangle = 0\ , \\ &\langle \nabla_X E_i, X\rangle = \langle \nabla_Y E_i, Y\rangle\ , \end{aligned}$$

for all $i \geq 3$ and at all points of a neighborhood of q_0.

From the constancy of the scalar curvature it follows

$$\text{(2.11.8)}\qquad \nabla_{E_i}[\langle \bar{A}X, X\rangle\langle \bar{A}Y, Y\rangle] = 0\ , \qquad i = 3, \cdots, n\ ,$$

or

$$[\nabla_{E_i}\langle \bar{A}X, X\rangle]\langle \bar{A}Y, Y\rangle + \langle \bar{A}X, X\rangle[\nabla_{E_i}\langle \bar{A}Y, Y\rangle] = 0\ .$$

On the other hand, we have

$$\text{(2.11.9)}\qquad \begin{aligned} \nabla_{E_i}\langle \bar{A}X, X\rangle &= \langle \nabla_{E_i}\bar{A}X, X\rangle + \langle \bar{A}X, \nabla_{E_i}X\rangle \\ &= \langle \bar{A}[E_i, X], X\rangle + \langle \bar{A}X, \nabla_{E_i}X\rangle \\ &= \langle [E_i, X], X\rangle\langle \bar{A}X, X\rangle = -\langle \nabla_X E_i, X\rangle\langle \bar{A}X, X\rangle\ . \end{aligned}$$

Similarly,

$$\text{(2.11.10)}\qquad \nabla_{E_i}\langle \bar{A}Y, Y\rangle = -\langle \nabla_Y E_i, Y\rangle\langle \bar{A}Y, Y\rangle\ .$$

The relations (2.11.8), (2.11.9) and (2.11.10) give

$$\text{(2.11.11)}\qquad [\langle \nabla_X E_i, X\rangle + \langle \nabla_Y E_i, Y\rangle]\langle \bar{A}X, X\rangle\langle \bar{A}Y, Y\rangle = 0\ ,$$

which implies, in cosequence of (2.11.7),

$$\text{(2.11.12)}\qquad \langle \nabla_X E_i, X\rangle = \langle \nabla_Y E_i, Y\rangle = 0\ , \qquad \text{for all } i \geq 3\ .$$

Next assume that the non-zero eigenvalues of $\bar{A}_q$ coincide. If they coincide in a neighborhood of q, it is possible to find vector fields X, Y satisfying (2.11.6) and therefore to show that $\bar{\mathscr{N}}$ is parallel at q.

Finally assume that the non-zero eigenvalues of $\bar{A}_q$ coincide at q, but each neighborhood of q contains a point at which they are distinct. A simple continuity argument shows that in this case $\bar{\mathscr{N}}$ is also parallel at q.

On the other hand,

(2.11.13) *The distribution $\bar{\mathscr{N}}$ is parallel at any point of $U_0 - P$.*

To show this, consider a point $q \in U - P_0$; this means that for some index $i_0 \geq 3$ the numbers

$$(2.11.14) \quad \langle \nabla_{E_1} E_{i_0}, E_2 \rangle_q, \langle \nabla_{E_1} E_{i_0}, E_1 \rangle_q - \langle \nabla_{E_2} E_{i_0}, E_2 \rangle_q, \langle \nabla_{E_2} E_i, E_1 \rangle_q$$

are not simultaneously zoro. For any $i \geq 3$, let Δ^i denote the function on U_0:

$$(2.11.15) \quad \Delta^i = [\langle \nabla_{E_1} E_i, E_1 \rangle - \langle \nabla_{E_2} E_i, E_2 \rangle]^2 + 4\langle \nabla_{E_1} E_i, E_2 \rangle \langle \nabla_{E_2} E_i, E_1 \rangle .$$

The distinct cases to be discussed can be indicated in the following way:

a)

$$\begin{array}{lll} \langle \nabla_{E_1} E_{i_0}, E_2 \rangle_q \neq 0 & \left\{ \begin{array}{l} \Delta^{i_0}(q) \neq 0 . \\ \Delta^{i_0}(q) = 0 \end{array} \right. & \left\{ \begin{array}{l} \Delta^{i_0} = 0 \text{ in a neighborhood of } q. \\ \text{Any neighborhood of } q \text{ has a point} \\ \text{at which } \Delta^{i_0} \text{ is non-zero.} \end{array} \right. \\ (\text{or } \langle \nabla_{E_2} E_{i_0}, E_1 \rangle_q \neq 0) & & \end{array}$$

b)

$$\begin{array}{lll} \langle \nabla_{E_1} E_{i_0}, E_2 \rangle_q = 0 & \left\{ \begin{array}{l} \Delta^{i_0}_q = 0 . \\ \Delta^{i_0}_q \neq 0 \end{array} \right. & \left\{ \begin{array}{l} \text{The functions } \langle \nabla_{E_1} E_{i_0}, E_2 \rangle, \langle \nabla_{E_2} E_{i_0}, E_1 \rangle \\ \text{both vanish on a neighborhood of } q. \\ \text{Any neighborhood of } q \text{ contains a point} \\ \text{at which either one of } \langle \nabla_{E_1} E_{i_0}, E_2 \rangle, \\ \langle \nabla_{E_2} E_{i_0}, E_1 \rangle \text{ is non-zero at this point.} \end{array} \right. \\ \langle \nabla_{E_2} E_{i_0}, E_1 \rangle_q = 0 & & \end{array}$$

The proof of (2.11.13) consists in showing the parallelism of $\bar{\mathscr{N}}$ in each of the above cases.

a) Assume $\langle \nabla_{E_1} E_{i_0}, E_2 \rangle$, Δ^{i_0} to be non-zero at all points of a neighborhood V_q of q.

In view of the assumption made above, the quadratic equation

$$(2.11.16) \quad \begin{aligned} &\langle \nabla_{E_1} E_{i_0}, E_2 \rangle x^2 - [\langle \nabla_{E_1} E_{i_0}, E_1 \rangle - \langle \nabla_{E_2} E_{i_0}, E_2 \rangle] x \\ &\qquad - \langle \nabla_{E_2} E_{i_0}, E_1 \rangle = 0 \end{aligned}$$

defines two complex valued C^∞-functions α, β such that

$$(2.11.17) \qquad \alpha\beta = -\frac{\langle \nabla_{E_2} E_{i_0}, E_1\rangle}{\langle \nabla_{E_1} E_{i_0}, E_2\rangle} ,$$

$$(2.11.18) \qquad \alpha + \beta = \frac{[\langle \nabla_{E_1} E_{i_0}, E_1\rangle - \langle \nabla_{E_2} E_{i_0}, E_2\rangle]}{\langle \nabla_{E_1} E_{i_0}, E_2\rangle} .$$

Consider the complex vector fields Z, W on V_q defined by

$$Z = \alpha E_1 + E_2 ,$$
$$W = \beta E_1 + E_2 ,$$

which are linearly independent at each point since α and β are distinct (at each point). Let (h, H, U) be any local isometric immersion of V_q in E^{n+1}. Then

$$\begin{aligned} \langle HZ, W\rangle &= \alpha\beta\langle HE_1, E_1\rangle + (\alpha + \beta)\langle HE_1, E_2\rangle + \langle HE_2, E_2\rangle \\ &= \frac{1}{\langle \nabla_{E_1} E_{i_0}, E_2\rangle}[\langle \nabla_{E_1} E_{i_0}, E_2\rangle\langle HE_2, E_2\rangle + (\langle \nabla_{E_1} E_{i_0}, E_1\rangle \\ &\quad - \langle \nabla_{E_2} E_{i_0}, E_2\rangle)\langle HE_1, E_2\rangle - \langle \nabla_{E_2} E_{i_0}, E_1\rangle\langle HE_1, E_1\rangle] , \end{aligned}$$

which is zero due to (2.11.3). Thus Z, W satisfy the conditions of Lemma 2–9, and therefore by part b) of Corollary (2–10) the distribution $\overline{\mathcal{N}}$ is parallel in the neighborhood V_q.

The next case to be analyzed is that of

$$(2.11.19) \qquad \langle \nabla_{E_1} E_{i_0}, E_2\rangle \neq 0 ,$$

in a neighborhood V_q of q and Δ^{i_0} vanishing at all points of V_q. In this case the functions α, β coincide at each point, and the vector field

$$X = \frac{\alpha E_1 + E_2}{\| \alpha E_1 + E_2 \|}$$

satisfies the conditions of Lemma 2–4. Therefore Corollary 2–6 shows that this case cannot occur.

The parallelsim of $\overline{\mathcal{N}}$ in the last subcase of a) is proved by using the reasoning of the proof of the first subcase and a simple continuity argument.

b) The first subcase cannot occur, for otherwise all functions listed in (2.11.14) would vanish at q. Hence to study the next case it may be assumed that in a neighborhood V_q of q, the functions $\langle \nabla_{E_1} E_{i_0}, E_2\rangle, \langle \nabla_{E_2} E_{i_0}, E_1\rangle$ vanish, while Δ^{i_0} is never zero. Using again (2.11.3) we obtain

$$[\langle \nabla_{E_1} E_{i_0}, E_1\rangle - \langle \nabla_{E_2} E_{i_0}, E_2\rangle]\langle HE_1, E_2\rangle = 0 ,$$

which shows that the vector fields E_1, E_2 satisfy the conditions of Lemma 2–9, and again, by Corollary 2–10, the parallelsim of $\overline{\mathcal{N}}$ is established in V_q.

Finally, the last case of b) can be related to the first case of a), and as before, a continuity argument proves the parallelsim of $\overline{\mathcal{N}}$ at q, in this case.

From (2.11.5) and (2.11.13) it follows that $\overline{\mathcal{N}}$ is parallel at all points of U_0 and particularly at p_0. Since this point can be arbitrarily chosen, $\overline{\mathcal{N}}$ is parallel everywhere. Thus Theorem 2–1 is proved.

2-12. Proof of Theorem 2-2. This proof presents a great analogy with the one given in § 2–11. As before, let p_0 be an arbitrarily chosen point of $\bar{M}^n$, and U_0 be a neighborhood of p_0 in which there are

(2.12.1) a coordinate system $x^1, \cdots, x^n$,

and

(2.12.2) an orthonormal frame $E_1, \cdots, E_n$, such that the vectors $E_3, \cdots, E_n$, form a basis of the relative nullity distribution $\overline{\mathcal{N}}$, and the relation

$$\nabla_{E_i} E_j = \frac{K}{2} \sum_{k=3}^{n} (\delta^i_j x^k - \delta^k_j x^i) E_k \tag{2.12.3}$$

holds.

Again, Proposition 1–2 allows us to make these assumptions.

Since the space $\tilde{M}^{n+1}(K)$ has constant curvature, equation (2.10.3) holds in the present situation. Further, the set P is defined exactly in the same way, but its proprieties are drastically different in face of the assumption $K \neq 0$. In this case the following holds:

(2.12.4) *The set P has no interior points.*

First it should be noted that by the same argument used to prove (2.11.5), it turns out that $\overline{\mathcal{N}}$ is parallel and

$$\langle R(X, E_i)E_i, X\rangle = K \, , \tag{2.12.5}$$

where X is a unit vector field orthogonal to the E_i. On the other hand, the parallelsim of $\overline{\mathcal{N}}$ and the fact that its leaves are totally geodesic imply that

$$\langle R(X, E_i)E_i, X\rangle = 0 \, , \tag{2.12.6}$$

which contradicts (2.12.5). Hence (2.12.4) is proved.

Next it will be shown that:

(2.12.7) *$U_0 - P$ has no interior points.*

(2.12.7) can be proved by a series of discussions; we follow the same pattern of a) and b) of § 2–11, and use the first part of the conclusions of Corollary 2–10.

In order to eliminate the case where

$$\langle \nabla_{E_1} E_{i_0}, E_2 \rangle_q \neq 0 \,, \tag{2.12.8}$$

and $\Delta^{i_0} = 0$ in a neighborhood of q, we need some further information. As in (2.11.19), there is an orthonormal frame X, Y defined in a neighborhood V_q of q, which is orthogonal to the distribution $\overline{\mathcal{N}}$ and satisfies

$$\langle HX, X \rangle = 0 \,, \tag{2.12.9}$$

for any local isometric immersion (h, H, U) of V_q in $\tilde{M}^{n+1}(K)$. From (2.12.9) and

$$\begin{aligned}\langle \nabla_X E_{i_0}, Y \rangle \langle HY, Y \rangle + [\langle \nabla_X E_{i_0}, X \rangle - \langle \nabla_Y E_{i_0}, Y \rangle] \langle HX, Y \rangle \\ - \langle \nabla_Y E_{i_0}, X \rangle \langle HX, X \rangle = 0 \,,\end{aligned}$$

it follows

$$\begin{aligned}\langle \nabla_X E_{i_0}, Y \rangle \langle HY, Y \rangle + [\langle \nabla_X E_{i_0}, X \rangle \\ - \langle \nabla_Y E_{i_0}, Y \rangle] \langle HX, Y \rangle = 0 \,.\end{aligned} \tag{2.12.10}$$

If $\langle \nabla_X E_{i_0}, Y \rangle$ is non-zero at all points of some open subset $V'' \subset V_q$, the relations (2.12.9) and (2.12.10), together with the Gauss equation, imply the rigidity of V'', which is a contradiction. Thus

$$\langle \nabla_X E_{i_0}, Y \rangle = 0 \,, \tag{2.12.11}$$

at all points of V_q.

In order to complete the discussion, Corollary 2–8 will be applied, but it requires that

$$\langle \nabla_X E_i, X \rangle - \langle \nabla_Y E_i, Y \rangle = 0 \tag{2.12.12}$$

for all indices $i \geq 3$ and all points of V_q. The last relations are a consequence of

$$\langle \nabla_X E_i, Y \rangle = 0 \,, \qquad \Delta^i = 0 \,, \tag{2.12.13}$$

for all $i \geq 3$ on V_q. In fact, consider the linear system

$$\langle \nabla_X E_i, Y \rangle a_1 + [\langle \nabla_X E_i, X \rangle - \langle \nabla_Y E_i, Y \rangle] a_2 = 0 \,. \tag{2.12.14}$$

For any local isometric immersion (h, H, U), it is known that

$$a_1 = \langle HY, Y \rangle \,, \qquad a_2 = \langle HX, Y \rangle$$

is a solution of (2.12.14) (see § 2–11).

If at some point of V_q the matrix of (2.12.14) had rank greater than one, this system together with (2.12.9) and the Gauss equation would imply the existence of an open rigid submanifold of V_q, therefore contradicting the deformability of $\bar{M}^n$. Thus *the rank of the matrix of* (2.12.14) *is less than* 2 *at every point*, and (2.12.13) follows from this fact and (2.12.8), (2.12.11). Therefore Corollary 2–8 leads to another contradiction, showing that this case cannot occur. Hence (7) is proved.

The conclusions (2.12.4) and (2.12.7) are obviously incompatible, and hence $\bar{M}^n$ is not deformable $\tilde{M}^{n+1}(K)$.

3. Theorem 2–1 in spite of being local, has an interesting global consequence.

3-1. Theorem. *Let M^n, $n \geq 3$, be a complete Riemannian manifold with non-zero constant scalar curvature and being locally deformable in E^{n+1}. Then M^n is isometric to the Riemannian product $S^2 \times E^{n-2}$ of a two-dimensional sphere by an $(n-2)$-dimensional Euclidean space.*

Proof. Let f be an isometric immersion of M^n in E^{n+1}. From the local deformability of M^n and the fact that the scalar curvature of M^n is non-zero, it follows that the type number of f is two everywhere. Let $\mathscr{N}$ be the relative nullity distribution of f. Then by Theorem 2–1, $\mathscr{N}$ is *parallel* on M^n (since it is parallel on a neighborhood of each point), and therefore the universal covering $\hat{M}^n$ of M^n has the decomposition

$$\hat{M}^n = \hat{M}^2 \times E^{n-2} \tag{3.1.1}$$

by de Rham's theorem. Since M^n is non-flat, $\hat{M}^2$ is necessarily irreducible. Under these circumstances, a result of S. Alexander [1] shows that M^n itself is isometric to a Riemannian product

$$M^n = M^2 \times E^{n-2}\,, \tag{3.1.2}$$

and f immerses M^2 isometrically in a 3-dimensional Euclidean space. Thus from (3.1.2) it follows that the curvature of M^2 equals the scalar curvature of M^n and is therefore constant, and that M^2 is complete. By a well-known theorem of Hilbert, the curvature of M^2 is positive, and therefore isometric to a sphere (see [7]).

3-2. Theorem. *Let M^n be a homogeneous Riemannian manifold, having an isometric immersion f in the Euclidean space E^{n+1}, such that its type number is two everywhere. Then M^n is isometric to the Riemannian product of a 2-sphere by an $(n-2)$-plane.*

Proof. For $n = 2, M^n$ is compact and the proof for this case is given in [7]. Next assume $n \geq 3$. Since M^n is homogeneous, by Proposition 1–7 the only possibilities to be discussed are

(3.2.1) M^n is locally deformable,

(3.2.2) M^n is rigid.

From Theorem 3–1 it follows that in case (3.2.1), M^n is isometric to $S^2 \times E^{n-2}$. To prove (3.2.2) the following result of K. Nomizu and B. Smyth [10] is used:

Let M be a complete Riemannian manifold of dimension n with nonnegative sectional curvature, and $\phi\colon M \to E^{n+1}$ be an isometric immersion with constant mean curvature. If the trace of A^2 is constant, then M^n is isometric to $S^2 \times E^{n-2}$

Next it will be shown that M^n and f satisfy the conditions of this theorem. Let U be an open orientable submanifold of M^n, and denote by A a second fundamental form of f on U. It is well-known (see [12]) that there are n continuous functions on U:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n ,$$

such that at each point $p \in U$, $\lambda_1(p), \cdots, \lambda_n(p)$ are the eigenvalues of A_p.

Since M^n is rigid, these functions are constant, and by a theorem of E. Cartan [3] at most two of them can be distinct at each point. On the other hand, these functions are two non-zero eigenvalues λ_1, λ_2 and $n-2$ (≥ 1) zero ones. Thus λ_1 and λ_2 must coincide at each point. This means that

$$\operatorname{tr} A^2 = 2\lambda_1^2 = \text{constant} ,$$
$$\operatorname{tr} A = 2\lambda_1 = \text{constant} ,$$

and it is also clear that all sectional curvatures are nonnegative. Thus the above mentioned result gives the proof for case (3.2.2). q.e.d.

Regarding hypersurfaces of spaces of constant curvature, Theorem 2–2 gives

3-3. Theorem. *A hypersurface of $\tilde{M}^{n+1}(K)$, $K \neq 0$, having constant scalar curvature distinct from $n(n-1)K$, is rigid, provided that $n \geq 4$.*

Proof. Let M^n be such a hypersurface, and consider two isometric immersions $f, \bar{f}$ with second fundamental forms $A, \bar{A}$ deffned on some orientable open submanifold of M^n. Then the assumption on the scalar curvature implies that A and $\bar{A}$ have rank ≥ 2 everywhere.

Let U be the subset of M^n consisting of those points which are contained in some rigid open neighborhood (this neighborhood may depend on the point). It follows from Theorem 2–2 that $M^n - U$ has no interior points, i.e., that U is dense in M^n. Since U is covered by open rigid submanifolds, each connected component of U is rigid. Let p be a point of M^n, and V an orientable neighborhood of p. It will be shown that there is a function $e(q)$ defined on V, assuming only the values $+1$ or -1, and such that

$$\bar{A}_q = e(q)A_q , \qquad \text{for all } q \in V . \tag{3.3.1}$$

In fact, if $q \in U$, this follows from the rigidity of each component. On the other hand, if $q \notin U$, it can be approximated by points at which (3.3.1) holds, and by cotinuity (3.3.1) holds at q. Again the continuity of A and $\bar{A}$ gives the continuity of e. Since V is assumed connected, e must be constant, and therefore V is rigid. This argument shows that M^n can be covered by rigid neighborhoods, and hence is rigid.

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