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MANIFOLDS ADMITTING NO METRIC OF CONSTANT NEGATIVE CURVATURE

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Let M be a compact n-dimensional Riemannian manifold of strictly negative sectional curvature, $K(\pi) < 0$ for all 2-planes π. If M admits a Riemannian metric of constant negative curvature, its Pontryagin classes are zero and consequently, if M is orientable and of dimension $4k$, its index in the sense of Hirzebruch is zero. For every positive integer k, there exist compact complex analytic manifolds of real dimension $4k$, arbitrarily large index and sectional curvature $-4 \leq K(\pi) \leq -1$. Such manifolds can admit no Riemannian metric of constant negative curvature.

This paper answers affirmatively a fundamental question: is the class of manifolds admitting Riemannian metrics of strictly negative sectional curvature larger than the class of manifolds admitting Riemannian metrics of constant negative sectional curvature ([7, p. 801])? In [6] Calabi asked a related question: Let M be a compact n-dimensional Riemannian manifold of strictly negative sectional curvature $K(\pi) < 0$. Can we find $\delta > 0$ sufficiently small so that if $-1 - \delta \leq K(\pi) \leq -1$ then M admits a Riemannian metric of constant negative sectional curvature? This paper does not resolve the question of Calabi but shows that $\delta < 3$ is necessary if the conjecture is true. For the rest of this paper see [2].

Definition 1. Let M be a connected, simply connected Riemannian manifold. A Clifford-Klein form of M is a Riemannian manifold M' whose simply connected Riemannian covering space is M.

The bounded symmetric domains M in C^n endowed with the Bergman metric are Riemannian symmetric spaces, and the group of complex analytic homeomorphisms of M contains the identity component of the isometries of M.

Definition 2. For a bounded symmetric domain M in C^n, a Clifford-Klein form M' of M is said to be complex analytic if it is a complex analytic manifold and if the natural map of M to M' is analytic. In [2] A. Borel proved the following:

Theorem 1. *A bounded symmetric domain always has a compact complex analytic Clifford-Klein form, and any such form has a regular finite Galois covering.*

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In [5] Hirzebruch defined the index of a compact oriented $4k$-manifold. The index of a manifold may be represented as a linear combination of Pontryagin numbers and consequently is zero if all Pontryagin classes of the manifold are zero. In [3] Chern proved that if M is a Riemannian manifold (not necessarily compact) of constant negative sectional curvature, then its Pontryagin classes are zero, and hence if M is compact and orientable, then its index is zero.

There is a one to one correspondence between irreducible bounded symmetric domains M of C^n and compact hermitian symmetric spaces N, [2], [4]. In [4] Hirzebruch proved:

Theorem 2. *Let M' be a compact, complex analytic Clifford-Klein form of an irreducible bounded symmetric domain M in C^n with compact counterpart N. Then M' is an algebraic manifold and index (M') = index $(N) \times$ algebraic genus of M', where the genus is positive if n is even, and negative if n is odd.*

Let $B^{2r} \subseteq C^{2r}$ be the open unit ball. With the Bergman metric (normalized), B^{2r} has sectional curvature $-4 \leq K(\pi) \leq -1$, [1]. In fact, B^{2r} has constant holomorphic curvature -4. The compact counterpart $N = U(2r+1)/U(2r) \times U(1)$ and index $(N) = 1$. If M' is any compact complex analytic Clifford-Klein form of B^{2r}, then index (M') = algebraic genus of $M' \geq 1$ by Theorem 2. If M'' is an r-sheeted covering space of M', then index $(M'') = r \times$ index (M'). Combining these facts with Theorems 1 and 2 we obtain:

Theorem 3. *Let r, s be any two positive integers. Then there exists a compact complex analytic manifold M' of real dimension $4r$ such that index $(M') > s$ and $-4 \leq K(\pi) \leq -1$ for all 2-planes π; such a manifold admits no metric of constant negative sectional curvature.*

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