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RIGIDITY THEOREMS IN RANK-1 SYMMETRIC SPACES

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0. Introduction

Recently, J. Simons proved a number of theorems which in various ways distinguished the geodesic subspheres of Euclidean spheres from all other minimal submanifolds of the sphere. One purpose here is to prove analogous theorems for the totally geodesic submanifolds of CP^k and QP^k-type in the complex and quaternionic projective spaces CP^n and QP^n. In particular, we prove an extrinsic pinching theorem for compact $2k$ (resp. $4k$) dimensional minimal submanifolds of CP^n (resp. QP^n) which leads to an intrinsic rigidity result for the standard imbedding of CP^k (resp. QP^k) within the class of minimal immersions.

We then turn attention to the (real) codimension-one case where, as follows from this work, there are no compact, totally geodesic submanifolds. There are however certain distinguished minimal hypersurfaces $M^C_{p,q}$, $M^Q_{p,q}$ of CP^n and QP^n, for $p, q \geq 0$ and $p + q = n - 1$, which naturally generalize the equatorial hypersurfaces of spheres. We show that there exist positive constants c_n and c'_n such that if M is any compact minimal hypersurface of CP^n (QP^n) over which either the length $\|B\|$ of the second fundamental form satisfies $\|B\| \leq c_n$ or, equivalently, the scalar curvature K satisfies $K \geq c'_n$, then equality holds identically and $M \cong M^C_{p,q}(M^Q_{p,q})$ for some p, q. Moreover, any (not necessarily compact) minimal hypersurface whose scalar curvature is identically equal to c'_n must be an open subset of $M^C_{p,q}$ ($M^Q_{p,q}$).

The method of proof, roughly speaking, is to use the standard fibrations to push known theorems on the sphere down to the spaces CP^n and QP^n.

The author would like to note that results similar to those of Theorem 2 have been found independently by Wu-Hsiung Huang, and also wishes to express gratitude to Wu-Yi Hsiang for many valuable conversations related to this work.

1. Riemannian fibre bundles

Let M' and M be Riemannian manifolds of dimensions m and $m + p$ respectively, and assume that there exists a fibration $\pi: M \to M'$ where:

a) The fibres are totally geodesic in M.

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b) At each point p of M the differential π_* carries the normal space to the fibre at p isometrically onto the tangent space of M' at $\pi(p)$.

The fundamental geometry of this situation has been discussed by Barrett O'Neill [4], and we shall recall the notation and results we need here.

Let X denote a tangent vector at $p \in M$. Then X decomposes as $\mathscr{V}X + \mathscr{H}X$, where $\mathscr{V}X$ is tangent to the fibre through p and $\mathscr{H}X$ is perpendicular to it. If $X = \mathscr{V}X$, it is called a *vertical* vector; and if $X = \mathscr{H}X$, it is called *horizontal*. Let ∇ and ∇' denote the Riemannian connections of M and M' respectively. To each tangent vector field X on M' there corresponds a unique horizontal vector field $\tilde{X}$ on M such that for all $p \in M$ we have $\pi_* \tilde{X}_p = X_{\pi(p)}$. If X and Y are any tangent vector fields on M', we have that

$$(1.1) \qquad \mathscr{H}\nabla_{\tilde{X}}\tilde{Y} = (\widetilde{\nabla'_X Y}) .$$

Furthermore, if V is a vertical vector field on M, then $\pi_*[\tilde{X}, V] = [\pi_*\tilde{X}, \pi_* V] = 0$. Thus we have

$$(1.2) \qquad \mathscr{H}(\nabla_{\tilde{X}} V) = \mathscr{H}(\nabla_V \tilde{X}) .$$

We define on M a tensor A of type (1, 2) called *the fundamental tensor of the fibration* as follows. Let X, Y be tangent vectors at $p \in M$ and extend Y to a local field. Then we set

$$(1.3) \qquad A_X Y = \mathscr{V}\nabla_{\mathscr{H}X}\mathscr{H}Y + \mathscr{H}\nabla_{\mathscr{H}X}\mathscr{V}Y .$$

This definition is independent of the extension of Y. If X is vertical, $A_X Y = 0$. If X and Y are horizontal fields, then $A_X Y = -A_Y X = \frac{1}{2}\mathscr{V}[X, Y]$. Let R and R' denote the curvature tensors of M and M' with the sign chosen so that $S(X, Y) = \langle R_{X,Y}X, Y\rangle$ (and $S'(X, Y) = \langle R'_{X,Y}X, Y\rangle$) equals the sectional curvature when $|X \wedge Y| = 1$. (The bracket $\langle \cdot, \cdot \rangle$ will denote the metric on either manifold.) It follows from O'Neill that for horizontal vectors X, Y and vertical vector V we have

$$(1.4) \qquad S'(\pi_* X, \pi_* Y) = S(X, Y) + 3|A_X Y|^2 ,$$

$$(1.5) \qquad S(X, V) = |A_X V|^2 .$$

2. Fundamental lemmas

We now consider four real-valued functions on M defined as follows. For $p \in M$ let $K(p)$ denote the (unnormalized) scalar curvature of M at p, $K'(p)$ the scalar curvature of M' at $\pi(p)$, and $\rho(p)$ the scalar curvature of the fibre at p. Finally we define the *twisting curvature of the fibration* $\tau(p)$ at p by

$$\tau(p) = \sum_{j=1}^{m} \sum_{k=1}^{p} S(e_j, \nu_k) ,$$

where $e_1, \cdots, e_m$ and $\nu_1, \cdots, \nu_p$ are orthonormal bases for the horizontal and vertical spaces at p. From (1.5) we see that $\tau \geq 0$. Moreover, $\tau \equiv 0$ if and only if the bundle splits locally as a Riemannian direct product.

Lemma 1. $K' = K + \tau - \rho$.

Proof. Let $e_1, \cdots, e_m$ $(\nu_1, \cdots, \nu_p)$ be as above, and extend these vectors to local, orthonormal, horizontal (vertical) fields $\mathscr{E}_1, \cdots, \mathscr{E}_m$ $(\mathscr{N}_1, \cdots, \mathscr{N}_p)$. From (1.4) we have

$$\begin{aligned} K' &= \sum_{i,j=1}^{m} S'(\pi_* e_i, \pi_* e_j) \\ &= \sum_{i,j=1}^{m} \{S(e_i, e_j) + 3|A_{e_i} e_j|^2\} . \end{aligned}$$

However, using (1.5) and the fact that $\langle \mathscr{E}_i, \mathscr{N}_k \rangle \equiv 0$ we have

$$\begin{aligned} \sum_{i,j=1}^{m} |A_{e_i} e_j|^2 &= \sum_{i,j=1}^{m} \sum_{k=1}^{p} \langle \nabla_{\mathscr{E}_i} \mathscr{E}_j, \mathscr{N}_k \rangle_p^2 = \sum_{i,j} \sum_{k} \langle \mathscr{E}_j, \nabla_{\mathscr{E}_i} \mathscr{N}_k \rangle_p^2 \\ &= \sum_{i} \sum_{k} |\mathscr{H} \nabla_{\mathscr{E}_i} \mathscr{N}_k|_p^2 = \sum_{i} \sum_{k} |A_{e_i} \nu_k|^2 = \sum_{i} \sum_{k} S(e_i, \nu_k) . \end{aligned}$$

Hence,

$$\begin{aligned} K' &= \left\{ \sum_{i,j=1}^{m} S(e_i, e_j) + 2 \sum_{i=1}^{m} \sum_{k=1}^{p} S(e_i, \nu_k) + \sum_{k,l=1}^{p} S(\nu_k, \nu_l) \right\} \\ &\quad + \sum_{i=1}^{m} \sum_{k=1}^{p} S(e_i, \nu_k) - \rho \\ &= K + \tau - \rho . \end{aligned}$$

Suppose now that N is an n-dimensional submanifold of M which respects the fibration π. That is, suppose there is a fibration $\pi \colon N \to N'$ where N' is a submanifold of M' such that the diagram

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, "\pi"'] & M \arrow[d, "\pi"] \\ N' \arrow[r, "f'"] & M' \end{tikzcd}$$

commutes and the immersion f is a diffeomorphism on the fibres.

We shall now relate the second fundamental forms of the submanifolds N and N'. The discussion will be local, and so for convenience we shall consider $N(N')$ imbedded in $M(M')$ with the usual identification of tangent spaces. If X is a tangent vector of M at $p \in N \subset M$, we denote by X^T and X^N respectively the projections of X on the tangent and normal spaces of N at p. Note that the normal space is always horizontal. Let X, Y be tangent vector fields along $N(N')$. Then the Riemannian connection and second fundamental form of

$N(N')$ are given respectively by

$$\bar{\nabla}_X Y = (\nabla_X Y)^T \qquad (\bar{\nabla}'_X Y = (\nabla'_X Y)^T) ,$$
$$B_{X,Y} = (\nabla_X Y)^N \qquad (B'_{X,Y} = (\nabla'_X Y)^N) .$$

A submanifold is called *minimal* if the trace of its second fundamental form is identically zero.

Lemma 2. *N is a minimal submanifold if and only if N' is a minimal submanifold.*

Proof. Fix a point $p \in N$ and choose a set $\mathscr{N}_1, \cdots, \mathscr{N}_p$ of local, orthonormal vertical fields about p. Let $\mathscr{E}_1, \cdots, \mathscr{E}_n$ be local, orthonormal tangent fields on N' about $\pi(p)$, and let $\tilde{\mathscr{E}}_1, \cdots, \tilde{\mathscr{E}}_n$ be their horizontal lifts. Then N is minimal if and only if

$$\begin{aligned} 0 &= \sum_{k=1}^{p} (\nabla_{\mathscr{N}_k} \mathscr{N}_k)^N + \sum_{j=1}^{n} (\nabla_{\tilde{\mathscr{E}}_j} \tilde{\mathscr{E}}_j)^N \\ &= \sum_{j=1}^{n} (\nabla_{\tilde{\mathscr{E}}_j} \tilde{\mathscr{E}}_j)^N = \sum_j (\widetilde{\nabla'_{\mathscr{E}_j} \mathscr{E}_j})^N . \end{aligned}$$

This is equivalent to trace $(B') = 0$, and the lemma is proved.

Let τ_M and τ_N be the twisting curvatures of the manifolds M and N respectively. Then we have

Lemma 3. $\|B\|^2 \leq \|B'\|^2 + 2(\tau_M - \tau_N)$.

Proof. Choose fields $\mathscr{N}_k$ and $\mathscr{E}_j$ as above. Using (1.3), (1.5) and the fact that the fibres are totally geodesic we see that

$$\begin{aligned} \|B\|^2 &= \sum_{i,j=1}^{n} |\nabla_{\tilde{\mathscr{E}}_i} \tilde{\mathscr{E}}_j^N|^2 + \sum_{i=1}^{n} \sum_{k=1}^{p} |\nabla_{\tilde{\mathscr{E}}_i} \mathscr{N}_k^N|^2 \\ &\quad + \sum_{k=1}^{p} \sum_{i=1}^{n} |\nabla_{\mathscr{N}_k} \tilde{\mathscr{E}}_i^N|^2 + \sum_{k,l=1}^{p} |\nabla_{\mathscr{N}_k} \mathscr{N}_l^N|^2 \\ &= \sum_{i,j} |\nabla_{\tilde{\mathscr{E}}_i} \tilde{\mathscr{E}}_j^N|^2 + 2 \sum_i \sum_k |\nabla_{\tilde{\mathscr{E}}_i} \mathscr{N}_k^N|^2 \\ &= \sum_{i,j} |\widetilde{\nabla'_{\mathscr{E}_i} \mathscr{E}_j^N}|^2 + 2 \sum_i \sum_k (|\mathscr{H} \nabla_{\tilde{\mathscr{E}}_i} \mathscr{N}_k|^2 - |\mathscr{H} \nabla_{\tilde{\mathscr{E}}_i} \mathscr{N}_k^T|^2) \\ &\leq \|B'\|^2 + 2(\tau_M - \tau_N). \end{aligned}$$

Note that if M has constant sectional curvature 1, then

$$\|B\|^2 = \|B'\|^2 + 2(pn - \tau_N) \quad \text{and} \quad 0 \leq \tau_N \leq pn .$$

3. Rigidity of the linear subspaces of a projective space

Let CP^n and QP^n denote the complex and quaternionic projective spaces (of real dimensions $2n$ and $4n$) equipped with the standard symmetric space metrics normalized so that the maximum sectional curvature is four. The standard fibrations

$$S^1 \to S^{2n+1} \to CP^n \ , \tag{3.1}$$

$$S^3 \to S^{4n+3} \to QP^n \ , \tag{3.2}$$

where S^k denotes the Euclidean sphere of curvature 1, satisfy conditions a) and b) above. The idea is now to use the elementary observations of § 2 to transfer theorems for the sphere in [3] and [5] to these symmetric spaces.

Let N' be a $2n$-dimensional minimal submanifold of CP^{n+m} with scalar curvature K' and second fundamental form B'. Let $S^C(X, Y) = \langle R^C_{X,Y}X, Y\rangle$ where R^C is the curvature tensor of CP^{n+m}, and let $S'(X, Y)$ be the corresponding object on N'. From the *Gauss curvature equation*

$$\langle B'_{X,Y}, B'_{X,Y}\rangle - \langle B'_{X,X}, B'_{Y,Y}\rangle = S^C(X, Y) - S'(X, Y) \tag{3.3}$$

and the minimality of N' we have

$$0 \leq \|B'\|^2 = \sum_{i,j=1}^{2n} S^C(e_i, e_j) - K' \leq K^C_n - K' \ , \tag{3.4}$$

where $K^C_n = 4n(n + 1)$ is the scalar curvature of the linear subspace CP^n, and $e_1, \cdots, e_{2n}$ are orthonormal tangent vectors on N'. Similarly, if N' is a $4n$-dimensional minimal submanifold of QP^{n+m}, then

$$0 \leq K^Q_n - K' \ , \tag{3.5}$$

where $K^Q_n = 16n(n + 2)$ is the scalar curvature of QP^n. Letting $q = 2 - \dfrac{1}{2m}$ $\left(q' = 2 - \dfrac{1}{4m}\right)$ we now have the following extrinsic pinching theorem.

Theorem 1. *Let N' be a compact $2n$ $(4n)$-dimensional minimal submanifold of CP^{n+m} (QP^{n+m}) with scalar curvature K'. If K' satisfies the inequality*

$$K^C_n - K' \leq \frac{2n + 1}{q} \left(K^Q_n - K' \leq \frac{4n + 3}{q'}\right)$$

over N', then N' is a totally geodesic CP^n (QP^n).

Corollary 1 (*Intrinsic rigidity*). *Let g be the standard metric on CP^n (QP^n) normalized with maximum sectional curvature 4. Then there exists a neighborhood U of g in the space of inequivalent metrics on CP^n (QP^n) with the C^2-topology such that no other metric in U can be realized from a minimal immersion into CP^{n+m} (QP^{n+m}).*

Hence, the totally geodesic imbedding of CP^n into CP^{n+m} (QP^n into QP^{n+m}) is isolated in the C^3-topology from all other inequivalent minimal immersions.

(Two immersions are equivalent if they differ by a diffeomorphism of CP^n and an isometry of CP^{n+m}.)

Remark. This rigidity applies only to the *totally geodesic* imbeddings of CP^n or, equivalently, to the *relatively normalized* standard metrics on CP^n. There exist non-totally geodesic imbeddings of CP^n, with the standard metric, into CP^{n+m} of the form

$$(3.6) \qquad (Z_0, \cdots, Z_n) \mapsto \left(Z_0^k, \sqrt{k} Z_0^{k-1} Z_1, \cdots, \sqrt{\frac{k!}{\alpha!}} Z^\alpha, \cdots, Z_n^k, 0, \cdots, 0\right),$$

where $\alpha = (\alpha_0, \cdots, \alpha_n)$, $\sum \alpha_i = k$, and the imbedding is represented in homogeneous coordinates. The metric induced on CP^n has constant holomorphic curvature $4/k$. (In fact, these represent even locally all holomorphic immersions of the standard metric.) Since the imbedding (3.6) is holomorphic, it is minimal, and clearly by perturbing the coefficients above one obtains a great number of deformations of (3.6) through non-singular holomorphic imbeddings with non-standard metrics.

Proof of Theorem 1. Let N be the minimal submanifold of the sphere with fibration $\pi: N \to N'$ compatible with (3.1) (or (3.2)) as in § 2. Locally N is just the inverse image of N'. Let K, K', τ and ρ be functions on N defined as in § 1. Then we have

$$\rho \equiv 0 \quad \text{and} \quad 0 \le \tau \le 2n \qquad \text{for the } CP^{n+m} \text{ case ,}$$
$$\rho \equiv 6 \quad \text{and} \quad 0 \le \tau \le 12n \qquad \text{for the } QP^{n+m} \text{ case .}$$

We first consider the CP^{n+m} case. From Lemma 1 we have $2n(2n+1) - K = 2n(2n+1) - K' + \tau \le K_n^C - K'$. By the results in [5] and [1] the condition $2n(2n+1) - K \le (2n+1)/q$ implies that $K \equiv 2n(2n+1)$ and that N is totally geodesic. Therefore N is a linear variety invariant under multiplication by $e^{i\theta}$. Hence, N is a complex linear variety and $N' = CP^n$.

In the QP^{n+m} case we have that $(4n+2)(4n+3) - K = (4n+2)(4n+3) - K' + \tau - 6 \le K_n^Q - K'$, and the proof proceeds analogously.

Remark. It should be noted that Theorem 1 does not follow from a direct application of Simons' methods for the sphere to the spaces CP^n and QP^n. The difficulty comes from the lack of complete symmetry of the curvature tensors and the existence in these spaces of compact, totally geodesic submanifolds which are not homeomorphic to CP^k or QP^k. The idea here was to concentrate on the intrinsic scalar curvature function instead of the function $\|B\|$ considered in [5].

4. Rigidity of the equatorial hypersurfaces

It is natural to ask whether there are any (real) hypersurfaces of CP^n and QP^n which somehow generalize the equators of spheres. In general there are not totally geodesic hypersurfaces, as seen below. However, there is a set of likely candidates in the class of minimal hypersurfaces. In S^{n+1} we have the family of generalized Clifford surfaces

$$M_{p,q} = S^p\left(\sqrt{\frac{p}{n}}\right) \times S^q\left(\sqrt{\frac{q}{n}}\right),$$

where $p + q = n$. These are the only algebraic minimal hypersurfaces of degree ≤ 2 in S^{n+1}, and admit in the class of minimal hypersurfaces certain intrinsic characterizations [3]. By choosing the spheres to lie in complex or quaternionic subspaces we get fibrations

$$S^1 \longrightarrow M_{2p+1,2q+1} \longrightarrow M^{\boldsymbol{C}}_{p,q} ,$$
$$S^3 \longrightarrow M_{4p+3,4q+3} \longrightarrow M^{\boldsymbol{Q}}_{p,q} ,$$

compatible with (3.1) and (3.2) where $p + q = n - 1$. The surfaces $M^{\boldsymbol{C}}_{p,q}$ and $M^{\boldsymbol{Q}}_{p,q}$ which we shall refer to as "generalized equators" are discussed in detail in [2]. In the special case $p = q = 0$, both $M^{\boldsymbol{C}}_{0,0}$ and $M^{\boldsymbol{Q}}_{0,0}$ are in fact the totally geodesic hyperspheres, and whenever $p = 0$ these surfaces are homogeneous, positively curved manifolds diffeomorphic to the sphere. We shall show here that these manifolds are distinguished in the class of minimal hypersurfaces.

Let N' be a (small, imbedded) minimal hypersurface of $\boldsymbol{C}P^m$ whith unit normal field η'. It is possible to choose a set of orthonormal tangent vector fields on N' of the form: $\mathscr{E}_1, J'\mathscr{E}_1, \cdots, \mathscr{E}_{m-1}, J'\mathscr{E}_{m-1}, J'\eta'$ where J' is the almost complex structure of $\boldsymbol{C}P^m$. Let K' and B' be the scalar curvature and second fundamental form of N'. Using the Gauss curvature equation, the minimality of N', and the fact that the sectional curvature of $\boldsymbol{C}P^m$ has the form $S^{\boldsymbol{C}}(e, f) = 1 + 3\langle e, J'f\rangle^2$ (where $|e \wedge f| = 1$) we easily get that

$$4(m^2 - 1) - K' = \|B'\|^2 . \tag{4.1}$$

Let N be the inverse image of N' in S^{2m+1} with corresponding objects η, K and B, and let $\mathscr{N}$ denote the field of unit tangent vectors along the fibres in S^{2m+1}. We can compute the twisting curvature of N by

$$\tau_N = \sum_{j=1}^{m-1} \{|A_{\tilde{\mathscr{E}}_j}\mathscr{N}|^2 + |A_{\widetilde{J'\mathscr{E}_j}}\mathscr{N}|^2\} + |A_{\widetilde{J'\eta'}}\mathscr{N}|^2 ,$$

where A is the fundamental tensor of $\pi\colon N \to N'$. Letting J denote the almost complex structure on $\boldsymbol{C}^{m+1}$, we have that $\widetilde{J'X} = J\tilde{X}$ for any tangent vector field X on N'. Moreover, if ∇ denotes the connection on S^{2m+1}, then $A_{\tilde{X}}\mathscr{N} = \mathscr{H}(\nabla_{\tilde{X}}\mathscr{N})^T = \mathscr{H}(\nabla_{\mathscr{N}}\tilde{X})^T = (J\tilde{X})^T$. It follows that $\tau_N \equiv 2(m - 1)$, and thus

$$K' = K + 2(m - 1) , \tag{4.2}$$

$$\|B'\|^2 = \|B\|^2 - 2 . \tag{4.3}$$

For clarity of statement we set $n = 2m - 1$ ($= \dim N'$) and note that equation (4.1) becomes

$$(n + 3)(n - 1) - K' = \|B'\|^2 .$$

Relations (4.2) and (4.3) then lead immediately to

Theorem 2. *Let N' be a compact (n-dimensional) minimal hypersurface of $CP^{(n+1)/2}$ over which either of the following equivalent inequalities*

a) $$K' \geq (n + 2)(n - 1) ,$$

b) $$\|B'\|^2 \leq n - 1$$

holds. Then $K' \equiv (n + 2)(n - 1)$, $\|B'\|^2 \equiv n - 1$, and up to isometries of $CP^{(n+1)/2}$, $N' = M^C_{p,q}$ for some p, q.

Proof. Using (4.2) and (4.3) we lift a) and b) to similar conditions on N. These conditions imply by Theorem 3 in [3] that up to rotations $M = M_{p',q'}$ where $p' + q' = n + 1$. The rest is straightforward.

We now consider the QP^m case. The fibration (3.2) can be considered as coming from the free action of $SU(2) = S^3$ on S^{4m+3} where QP^m is identified as the orbit space. Let $\{\nu_1, \nu_2, \nu_3\}$ be an orthonormal basis for the tangent space at the identity of $SU(2)$. Each ν_k gives rise to a skew-symmetric linear endomorphism of R^{4m+4} by defining $\nu_k(x) = \frac{d}{dt}[\exp(t\nu_k)x]_{t=0}$. The tangent vector fields $\mathscr{N}_1, \mathscr{N}_2, \mathscr{N}_3$ on S^{4m+3}, where $\mathscr{N}_k(x) = \nu_k(x)$ under the usual identification of tangent spaces, represent globally defined, orthonormal, vertical vector fields which when restricted to the fibres $SU(2)$ are right invariant. Moreover for any $x \in R^{4m+4}$ both the linear span of $x, \nu_1(x), \nu_2(x), \nu_3(x)$ and its orthocompliment are ν_k-invariant. (Up to sign ν_k simply permutes the vectors $x, \cdots, \nu_3(x)$.) Hence, at any point p of S^{4m+3}, ν_k maps the horizontal space into itself; and if e, f are horizontal vectors at p with $|e \wedge f| = 1$, the corresponding sectional curvature of QP^m at $\pi(p)$ can be written

$$S(\pi_* e, \pi_* f) = 1 + 3 \sum_{k=1}^{3} \langle e, \nu_k(f) \rangle^2 . \tag{4.4}$$

Let N' be a (small, imbedded) minimal hypersurface of QP^m, N be its inverse image in S^{4m+3}, and $\eta, \eta', K, K', B, B'$ be as above. It is possible to choose orthonormal tangent fields $\mathscr{E}_1, \cdots, \mathscr{E}_{m-1}$ on N' so that the horizontal fields $\tilde{\mathscr{E}}_1, \nu_1\tilde{\mathscr{E}}_1, \nu_2\tilde{\mathscr{E}}_1, \nu_3\tilde{\mathscr{E}}_1, \tilde{\mathscr{E}}_2, \nu_1\tilde{\mathscr{E}}_2, \cdots, \nu_3\tilde{\mathscr{E}}_{m-1}, \nu_1\eta, \nu_2\eta, \nu_3\eta$ are orthonormal on N. Clearly the image under π_* of these vector fields at any point p of N forms an orthonormal basis for the tangent space of N' at $\pi(p)$. Using the Gauss curvature equation, minimality and equation (4.4) we can then show that

$$(4m + 8)(4m - 2) - K' = \|B'\|^2 . \tag{4.5}$$

Let A be the fundamental tensior for the fibration $\pi\colon N \to N'$. If X is a tangent vector field on N', then we have $A_{\tilde{X}}\mathscr{N}_k = \mathscr{H}\nabla_{\tilde{X}}\mathscr{N}_k^T = \mathscr{H}\nabla_{\mathscr{N}_k}\tilde{X}^T = \nu_k\tilde{X}^T$ where ∇ is the Riemannian connection on S^{4m+3}. Similarly, we can show that $A_{\nu_j}\tilde{X}\mathscr{N}_k = \pm\nu_k\nu_j\tilde{X}^T$. Using these facts we proceed as above to find that

$$K' = K + 12(m-1)\,, \tag{4.6}$$

$$\|B'\|^2 = \|B\|^2 - 6\,. \tag{4.7}$$

Again we set $n = 4m - 1$ ($= \dim N'$). Then

$$(n+9)(n-1) - K' = \|B'\|^2\,,$$

and by the above arguments we have

Theorem 3. *Let N' be a compact (n-dimensional) minimal hypersurface of $QP^{(n+1)/4}$ over which either of the equivalent inequalities*

a) $$K' \geq n^2 + 7n - 6\,,$$

b) $$\|B'\|^2 \leq n - 3$$

holds. Then $K' \equiv (n^2 + 7n - 6)$, $\|B'\|^2 \equiv n - 3$, and up to isometries of $QP^{(n+1)/4}$, $N' = M^{Q}_{p,q}$ for some p, q.

Using equations (4.2) and (4.6) and Theorem 1 of [3] we immediately obtain

Theorem 4 (*Local rigidity*). *Let N be any minimal hypersurface of $CP^{(n+1)/2}$ ($QP^{(n+1)/4}$) having scalar curvature $\equiv (n+2)(n-1)$ ($\equiv n^2 + 7n - 6$). Then up to isometries of $CP^{(n+1)/2}$ ($QP^{(n+1)/4}$) N is an open submanifold of $M^{C}_{p,q}$ ($M^{Q}_{p,q}$) for some p, q.*

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